# Chapter 02

# Computation of Eigenvalues and Eigenvectors

# Introduction

Given an $n \times n$ matrix $A$, the eigenvalue problem is to find scalars $\lambda$ and nonzero vectors $v$ such that :

$$Av = \lambda v. \tag{1}$$

Here, $\lambda$ are called *eigenvalues*, and $v \neq 0$ are the corresponding *eigenvectors*. This document presents numerical techniques for computing eigenvalues efficiently.

# Direct Methods for Computing Eigenvalues

Eigenvalues of a matrix $A$ are values $\lambda$ that satisfy the characteristic equation :

$$\det(A - \lambda I) = 0$$

where $I$ is the identity matrix. Solving this equation directly gives the eigenvalues. For small matrices, this can be done algebraically, but for larger matrices, numerical methods are preferred.

## Example :

Consider the matrix :

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

Find its eigenvalues by solving :

$$\det\left(\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} - \lambda I\right) = 0$$

Expanding :

$$\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(3-\lambda) - (1 \cdot 2) = 0$$

$$12 - 4\lambda - 3\lambda + \lambda^2 - 2 = 0$$

$$\lambda^2 - 7\lambda + 10 = 0$$

Solving $(\lambda - 5)(\lambda - 2) = 0$, we get :

$$\lambda_1 = 5, \quad \lambda_2 = 2$$

**Eigenvectors :**

To find the eigenvectors, solve $(A - \lambda I)\vec{v} = 0$ for each eigenvalue.
**For** $\lambda_1 = 5$ :

$$(A - 5I) = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

Solving :

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -x + y = 0 \Rightarrow y = x$$

An eigenvector is :

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

**For** $\lambda_2 = 2$ :

$$(A - 2I) = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

Solving :

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2x + y = 0 \Rightarrow y = -2x$$

An eigenvector is :

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

# Power method : computing the eigenvalue of largest modulus of a matrix A

In many engineering problems, it is often required to compute the numerically largest eigenvalue and its corresponding eigenvector. In such cases, the following iterative method, known as **Rayleigh's Power Method**, is quite convenient and well-suited for machine computations.

Let $X_1, X_2, \ldots, X_n$ be the eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then, an arbitrary column vector $X$ can be expressed as :

$$X = k_1X_1 + k_2X_2 + \cdots + k_nX_n$$

Multiplying by matrix $A$, we get :

$$AX = k_1AX_1 + k_2AX_2 + \cdots + k_nAX_n = k_1\lambda_1X_1 + k_2\lambda_2X_2 + \cdots + k_n\lambda_nX_n$$

Similarly,

$$A^2X = k_1\lambda_1^2X_1 + k_2\lambda_2^2X_2 + \cdots + k_n\lambda_n^2X_n$$

and in general,

$$A^rX = k_1\lambda_1^rX_1 + k_2\lambda_2^rX_2 + \cdots + k_n\lambda_n^rX_n$$

If $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$, then $\lambda_1$ is the dominant (largest in magnitude) eigenvalue. As $r$ increases, the contribution of the term $k_1\lambda_1^rX_1$ dominates, and thus, each multiplication by $A$ brings the vector closer to the eigenvector $X_1$.

To avoid the scalar multiplier $k_1$, we normalize the resulting vector at each step, typically by making the largest component equal to one.

Thus, we begin with an initial guess $X^{(0)}$, and evaluate :

$$AX^{(0)} = \lambda^{(1)} X^{(1)}$$

where $\lambda^{(1)}$ and $X^{(1)}$ are the first approximations to the dominant eigenvalue and eigenvector, respectively, after normalization.

We then repeat :

$$AX^{(1)} = \lambda^{(2)} X^{(2)}, \quad AX^{(2)} = \lambda^{(3)} X^{(3)}, \quad \dots$$

This iterative process continues until the difference $\|X^{(r)} - X^{(r-1)}\|$ becomes negligible. At this point, $\lambda^{(r)}$ approximates the largest eigenvalue, and $X^{(r)}$ the corresponding eigenvector.

### Note

Rewriting the eigenvalue equation $AX = \lambda X$ as :

$$A^{-1}AX = \lambda A^{-1}X \quad \Rightarrow \quad X = \lambda A^{-1}X$$

or

$$A^{-1}X = \frac{1}{\lambda}X$$

we see that applying the same power method to the matrix $A^{-1}$ yields the smallest eigenvalue of $A$, since the dominant eigenvalue of $A^{-1}$ is $\frac{1}{\lambda_{\min}}$.

## Example

Determine the largest eigenvalue and the corresponding eigenvector of the matrix

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

### Solution :

Let the initial approximation to the eigenvector corresponding to the largest eigenvalue of $A$ be

$$X^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then

$$AX^{(0)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \Rightarrow X^{(1)} = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}, \quad \lambda^{(1)} = 5$$

Now,

$$AX^{(1)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5 + 0.8 \\ 1 + 0.4 \end{bmatrix} = \begin{bmatrix} 5.8 \\ 1.4 \end{bmatrix} \Rightarrow X^{(2)} = \begin{bmatrix} 1 \\ 0.241 \end{bmatrix}, \quad \lambda^{(2)} = 5.8$$

Repeating the process :

$$AX^{(2)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = \begin{bmatrix} 5 + 0.964 \\ 1 + 0.482 \end{bmatrix} = \begin{bmatrix} 5.964 \\ 1.482 \end{bmatrix} \Rightarrow X^{(3)} = \begin{bmatrix} 1 \\ 0.248 \end{bmatrix}, \quad \lambda^{(3)} = 5.966$$

$$AX^{(3)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.248 \end{bmatrix} = \begin{bmatrix} 5 + 0.992 \\ 1 + 0.496 \end{bmatrix} = \begin{bmatrix} 5.992 \\ 1.496 \end{bmatrix} \Rightarrow X^{(4)} = \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}, \quad \lambda^{(4)} = 5.996$$

$$AX^{(4)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 5 + 1 \\ 1 + 0.5 \end{bmatrix} = \begin{bmatrix} 6 \\ 1.5 \end{bmatrix} \Rightarrow X^{(5)} = \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}, \quad \lambda^{(5)} = 6$$

$$AX^{(5)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 6 \\ 1.5 \end{bmatrix} \Rightarrow X^{(6)} = \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}, \quad \lambda^{(6)} = 6$$

Clearly, $\lambda^{(5)} = \lambda^{(6)}$ and $X^{(5)} = X^{(6)}$ up to three decimal places. Hence, the largest eigenvalue is

$$\boxed{\lambda = 6}$$

and the corresponding eigenvector is

$$\boxed{X = \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}}$$

To find the smallest eigenvalue in modulus using the Power Method, we apply the Power Method to the inverse matrix $A^{-1}$. The dominant eigenvalue of $A^{-1}$ corresponds to $\frac{1}{\lambda_{\min}}$, where $\lambda_{\min}$ is the smallest eigenvalue of $A$.

We continue from the previous example with :

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

## Step 1 : Compute $A^{-1}$

The inverse of a $2 \times 2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is given by :

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So for our matrix $A$, we compute the determinant :

$$\det(A) = (5)(2) - (4)(1) = 10 - 4 = 6$$

Thus,

$$A^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

### Step 2 : Apply the Power Method to $A^{-1}$

Let :

$$X^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

**Iteration 1 :**

$$A^{-1}X^{(0)} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{6} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{6} \end{bmatrix} \Rightarrow X^{(1)} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}, \quad \mu^{(1)} = \frac{1}{3}$$

**Iteration 2 :**

$$A^{-1}X^{(1)} = \begin{bmatrix} \frac{1}{3}(1) + (-\frac{2}{3})(-0.5) \\ -\frac{1}{6}(1) + \frac{5}{6}(-0.5) \end{bmatrix} = \begin{bmatrix} \frac{1}{3} + \frac{1}{3} \\ -\frac{1}{6} - \frac{5}{12} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{13}{24} \end{bmatrix} \Rightarrow X^{(2)} = \begin{bmatrix} 1 \\ -0.8125 \end{bmatrix}, \quad \mu^{(2)} \approx 0.6667$$

**Iteration 3 :**

$$A^{-1}X^{(2)} = \begin{bmatrix} \frac{1}{3}(1) + (-\frac{2}{3})(-0.8125) \\ -\frac{1}{6}(1) + \frac{5}{6}(-0.8125) \end{bmatrix} = \begin{bmatrix} \frac{1}{3} + 0.5417 \\ -\frac{1}{6} - 0.6771 \end{bmatrix} = \begin{bmatrix} 0.875 \\ -0.84375 \end{bmatrix} \Rightarrow X^{(3)} \approx \begin{bmatrix} 1 \\ -0.964 \end{bmatrix}, \quad \mu^{(3)} \approx 0.875$$

After a few more iterations,

$$\mu \to \frac{1}{\lambda_{\text{min}}} \quad \Rightarrow \quad \lambda_{\text{min}} \to \frac{1}{\mu}$$

### Final Result

Since the Power Method applied to $A^{-1}$ converges to :

$$\mu = \frac{1}{\lambda_{\text{min}}} \approx 1 \quad \Rightarrow \quad \lambda_{\text{min}} = 1$$

and the corresponding eigenvector (approximately) is :

$$X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

## Householder's Method

Consider an $n$-th order real symmetric matrix $A = [a_{ij}]$. This method consists of pre- and post-multiplying $A$ by a symmetric orthogonal matrix $P$ such that $PAP$ reduces to tridiagonal form.

Let the matrix $P$ be of the form :

$$P = I - 2ww^\top \tag{1}$$

where

$$w^\top w = w_1^2 + w_2^2 + \cdots + w_n^2 = 1 \tag{2}$$

Then :

$$P^\top = (I - 2ww^\top)^\top = I - 2ww^\top = P$$

and

$$P^\top P = (I - 2ww^\top)^\top (I - 2ww^\top) = I - 4ww^\top + 4ww^\top ww^\top = I \quad \text{(by (2))}$$

Hence, $P$ is a symmetric orthogonal matrix.

Now, let $w$ have the first $(k-1)$ zero components :

$$w_k^\top = [0, 0, \ldots, 0, x_k, x_{k+1}, \ldots, x_n] \tag{3}$$

with $x_k^2 + x_{k+1}^2 + \cdots + x_n^2 = 1$.

Then :

$$A_k = P_k A_{k-1} P_k^\top = P_k A_{k-1} P_k$$

We now perform the transformations :

$$A_k = P_k A_{k-1} P_k, \quad k = 2, 3, \ldots, n-1$$

After $(n-2)$ transformations, we arrive at a tridiagonal matrix.

# Example

Given the symmetric matrix :

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

we will reduce it to tridiagonal form using Householder transformations.

## Step 1 : First Householder Transformation

We aim to eliminate the elements $a_{13}$ and $a_{31}$. Consider the vector :

$$\mathbf{v} = \begin{bmatrix} 0 \\ a_{12} \\ a_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$$

Compute the Householder vector $\mathbf{w}$ :

$$\alpha = -\text{sign}(a_{12})\|\mathbf{v}\| = -5$$

$$\mathbf{w} = \frac{\mathbf{v} - \alpha \mathbf{e}_2}{\|\mathbf{v} - \alpha \mathbf{e}_2\|} = \frac{1}{\sqrt{(4-(-5))^2 + 3^2}} \begin{bmatrix} 0 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$$

The Householder matrix is :

$$P = I - 2\mathbf{w}\mathbf{w}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

### Step 2 : Compute the Tridiagonal Matrix

Applying the transformation :

$$PAP = \begin{bmatrix} 1 & -5 & 0 \\ -5 & \frac{73}{25} & -\frac{14}{25} \\ 0 & -\frac{14}{25} & -\frac{23}{25} \end{bmatrix}$$

In decimal form :

$$PAP \approx \begin{bmatrix} 1.0000 & -5.0000 & 0 \\ -5.0000 & 2.9200 & -0.5600 \\ 0 & -0.5600 & -0.9200 \end{bmatrix}$$