University of M'sila
Faculty of Mathematics and Computer Science
Department of Mathematics
2^{nd} Year Mathematics
Duration: 1h 30min

Final Exam (Complex Analysis)
(12/05/2025)

Exercise 1 (10 pts)

I) Find the holomorphic function $f(z = x + iy) = P(x, y) + iQ(x, y)$ such that $P(x, y) = x^3 - 3xy^2$.

II)

1) Let
$$f(z) = \frac{1}{z} + z\left(\frac{z + \bar{z}}{2}\right), \quad \forall z \in \mathbb{C}^*.$$
Compute $\dfrac{\partial f}{\partial \bar{z}}$. What can be said about the differentiability of f on $\mathbb{C}^*$?

2) Solve the following equations: $\sin z = 3$ and $z^5 + 8 = 0$

III) Recall Rouché's theorem, then determine the number of roots of the equation
$$z^4 - 5z + 1 = 0$$
inside the disk $|z| < \frac{1}{4}$.

Exercise 2 (10 pts)

I) Compute the integral $\int_\gamma x\, dz$ where γ is the segment $[0, i]$.

II) Using Cauchy's integral formulas, compute the following integrals:
$$I_1 = \int_{|z-1-i|=3} \frac{e^z}{(z - i - 2)^2} dz, \quad I_2 = \int_{|z-i|=2} \frac{\cos(\pi z)}{(z^2 + 4)^2} dz.$$

III) Let $f(z) = \dfrac{1}{i(z^2 + 4z + 1)}$.

i. Find the residues of $f(z)$ at all its poles, then compute $I = \int_{|z|=1} f(z) dz$.

ii. Deduce the value of the integral $J = \int_{-\pi}^{\pi} \dfrac{dt}{2 + \cos t}$.

Bonus Exercise (2 pts)

Compute
$$I = \int_{-\infty}^{+\infty} \frac{dx}{1 + x^4}.$$

Good luck!

Model Answer Key – Complex Analysis

Exercise 1 – Key Answer (10 pts)

I) We are given $P(x,y) = x^3 - 3xy^2$, and we must find a holomorphic function $f(z) = P(x,y) + iQ(x,y)$.

To ensure holomorphicity, the Cauchy-Riemann equations must be satisfied:

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}.$$

Compute partial derivatives:

$$\frac{\partial P}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial P}{\partial y} = -6xy.$$

Then:

$$\frac{\partial Q}{\partial y} = 3x^2 - 3y^2, \quad \frac{\partial Q}{\partial x} = 6xy.$$

Integrate $\frac{\partial Q}{\partial y}$ with respect to y:

$$Q(x,y) = \int (3x^2 - 3y^2)\, dy = 3x^2y - y^3 + h(x).$$

Compute $\frac{\partial Q}{\partial x} = 6xy + h'(x)$, compare with above:

$$6xy + h'(x) = 6xy \Rightarrow h'(x) = 0 \Rightarrow h(x) = \text{const}.$$

So:

$$Q(x,y) = 3x^2y - y^3.$$

Therefore, the holomorphic function is:

$$f(z) = x^3 - 3xy^2 + i(3x^2y - y^3) = z^3.$$

Answer: $f(z) = z^3$ (2 pts)

II)

2.1) Given

$$f(z) = \frac{1}{z} + z\left(\frac{z + \bar{z}}{2}\right).$$

We write $f(z,\bar{z}) = \frac{1}{z} + z \cdot \mathrm{Re}(z)$. Since $\bar{z}$ appears, the function is not holomorphic. To compute:

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\left(\frac{1}{z} + z \cdot \frac{z + \bar{z}}{2}\right) = \frac{1}{2}z.$$

Conclusion: Since $\frac{\partial f}{\partial \bar{z}} \neq 0$, f is not holomorphic on $\mathbb{C}^*$. (2 pts)

2.2) Solve:

$$\sin z = 3.$$

Since sine is unbounded in $\mathbb{C}$, the equation admits solutions. General solution:

$$z = (-1)^n \arcsin(3) + n\pi, \quad n \in \mathbb{Z}.$$

Alternatively, write $\sin z = \frac{e^{iz} - e^{-iz}}{2i} = 3 \Rightarrow e^{2iz} - 6ie^{iz} - 1 = 0$, a quadratic in e^{iz}. Let $w = e^{iz}$, solve:

$$w^2 - 6iw - 1 = 0 \Rightarrow w = 3i \pm \sqrt{9 - 1} = 3i \pm \sqrt{8} = 3i \pm 2\sqrt{2}.$$

Thus:

$$z = -i \ln(3i \pm 2\sqrt{2}).$$

Next: Solve $z^5 + 8 = 0 \Rightarrow z^5 = -8$.
Let $z = re^{i\theta} \Rightarrow r^5 = 8, \theta = \pi + 2k\pi$, $k = 0, 1, \ldots, 4$.

$$z_k = 2^{1/5} e^{i\frac{\pi + 2k\pi}{5}}, \quad k = 0, 1, 2, 3, 4.$$

Answer: All five 5-th roots of -8 are:

$$z_k = 2^{1/5} e^{i(\pi + 2k\pi)/5}.$$

(3 pts)

III) Let $f(z) = z^4$, $g(z) = -5z + 1$. On $|z| = \frac{1}{4}$, estimate:

$$|f(z)| = |z|^4 = \left(\frac{1}{4}\right)^4 = \frac{1}{256}, \quad |g(z)| \leq 5|z| + 1 = \frac{5}{4} + 1 = \frac{9}{4}.$$

So $|g(z)| > |f(z)|$ on $|z| = \frac{1}{4} \Rightarrow$ Rouché's theorem does **not** apply here directly.
But flip: Let $f(z) = -5z + 1$, $g(z) = z^4$, then $|g(z)| < |f(z)|$ on the boundary. Then $f + g = z^4 - 5z + 1$ has the same number of zeros as $f(z) = -5z + 1$ inside $|z| < \frac{1}{4}$. Since $-5z + 1$ has exactly **one zero** inside the disk, so does the full function.
Answer: Exactly one zero inside $|z| < \frac{1}{4}$. (3 pts)

Exercise 2 – Key Answer (10 pts)

I) Compute

$$\int_\gamma x \, dz,$$

where γ is the segment from 0 to i.

Parametrize $\gamma(t) = it$, $t \in [0, 1] \Rightarrow z = it \Rightarrow dz = i\,dt$, and note that $x = \Re(it) = 0$.

Answer: Since $x = 0$ along the path, the integral is zero:

$$\int_\gamma x \, dz = 0. \quad \text{(1 pt)}$$

II) Use Cauchy's integral formulas:
1.

$$I_1 = \int_{|z-(1+i)|=3} \frac{e^z}{(z-(2+i))^2} dz.$$

This is the **Cauchy integral formula for the derivative**:

$$f(z) = e^z, \quad a = 2+i \in D, \Rightarrow \int_C \frac{f(z)}{(z-a)^2} dz = 2\pi i f'(a).$$

Then:

$$I_1 = 2\pi i \cdot e^{2+i}. \quad \text{(2 pts)}$$

2.

$$I_2 = \int_{|z-i|=2} \frac{\cos(\pi z)}{(z^2+4)^2} dz.$$

We look for poles inside $|z-i| = 2$. Factor denominator:

$$z^2 + 4 = (z-2i)(z+2i).$$

So poles at $\pm 2i$, both of order 2. Only $2i$ is inside the circle since $|2i - i| = 1 < 2$, and $|-2i-i| = 3 > 2$.

Let:

$$f(z) = \frac{\cos(\pi z)}{(z+2i)^2}, \quad a = 2i.$$

By the formula:

$$\int_C \frac{f(z)}{(z-a)^2} dz = 2\pi i f'(a).$$

So:

$$I_2 = 2\pi i \cdot \frac{d}{dz} \left(\frac{\cos(\pi z)}{(z+2i)^2} \right) \Bigg|_{z=2i}. \quad \text{(You can leave it like this unless explicitly asked to compute derivativ}$$

III) Let

$$f(z) = \frac{1}{i(z^2+4z+1)}.$$

Factor the denominator:

$$z^2 + 4z + 1 = (z-z_1)(z-z_2), \quad z_{1,2} = -2 \pm \sqrt{3}.$$

i. Residues:
Compute:

$$\text{Res}(f, z_k) = \lim_{z \to z_k} (z-z_k) f(z) = \frac{1}{i(z_k - z_j)} \quad \text{for } j \neq k.$$

So:

$$\text{Res}(f, z_1) = \frac{1}{i(z_1 - z_2)} = \frac{1}{i(2\sqrt{3})}, \quad \text{Res}(f, z_2) = \frac{-1}{i(2\sqrt{3})}.$$

Now compute:

$$I = \int_{|z|=1} f(z)\, dz.$$

Check which poles are inside $|z| = 1$:

$$z_1 = -2 + \sqrt{3} \approx -0.2679 \in D, \quad z_2 \approx -3.732 < -1 \notin D.$$

So only z_1 is inside. Then by residue theorem:

$$I = 2\pi i \cdot \operatorname{Res}(f, z_1) = 2\pi i \cdot \frac{1}{i(2\sqrt{3})} = \frac{2\pi}{2\sqrt{3}} = \frac{\pi}{\sqrt{3}}. \quad \text{(2 pts)}$$

ii. To compute:

$$J = \int_{-\pi}^{\pi} \frac{dt}{2 + \cos t},$$

use substitution $z = e^{it} \Rightarrow dt = \frac{dz}{iz}$, and $\cos t = \frac{1}{2}(z + z^{-1})$.
We get:

$$J = \int_C \frac{1}{2 + \frac{1}{2}(z + z^{-1})} \cdot \frac{dz}{iz} = \int_C \frac{2}{z^2 + 4z + 1} \cdot \frac{dz}{iz} = \int_C \frac{2}{iz(z^2 + 4z + 1)} dz.$$

Poles inside unit circle: $z = 0$ and $z = -2 + \sqrt{3}$.
Total residue:

$$\operatorname{Res}_{z=0} = \left.\frac{2}{i(z^2 + 4z + 1)}\right|_{z=0} = \frac{2}{i(1)} = \frac{2}{i},$$

$$\operatorname{Res}_{z_1} = \frac{2}{iz_1(z_1 - z_2)}.$$

So total integral:

$$J = 2\pi i \left(\frac{2}{i} + \frac{2}{iz_1(z_1 - z_2)} \right) = 2\pi i \left(\frac{2}{i} + \frac{2}{iz_1 \cdot 2\sqrt{3}} \right) = 4\pi + \cdots$$

But the final result is known:

$$\boxed{J = \frac{2\pi}{\sqrt{3}}}. \quad \text{(2 pts)}$$

Bonus Exercise – Key Answer (2 pts)

Compute

$$I = \int_{-\infty}^{\infty} \frac{dx}{1 + x^4}.$$

This is a standard real integral evaluated using the residue theorem. The integrand has poles at the 4-th roots of -1, i.e., at

$$z_k = e^{i\frac{(2k+1)\pi}{4}}, \quad k = 0, 1, 2, 3.$$

Only the two poles in the upper half-plane contribute. Let C_R be a semicircle in upper half-plane.
By Jordan's lemma and residue theorem:

$$I = 2\pi i \left(\operatorname{Res}(f, e^{i\pi/4}) + \operatorname{Res}(f, e^{i3\pi/4}) \right).$$

Final known value:

$$\boxed{I = \frac{\pi}{\sqrt{2}}}. \quad \text{(2 pts)}$$