University of M'sila
Faculty of Mathematics and Computer Science
Department of Mathematics
2^{nd} Year Mathematics
Duration: 1h 30min

Final Exam (Complex Analysis)
13/05/2026

Exercise 1: Course Questions (4 pts)

1. State the Cauchy-Riemann equations in Cartesian coordinates.
2. Define an isolated singular point.
3. State the Residue Theorem.
4. What is a harmonic function? Give an example.

Exercise 2 (8 pts)
I) Find the holomorphic function $f(z = x + iy) = P(x, y) + iQ(x, y)$ such that $P(x, y) = x^3 - 3xy^2$.
II)

1) Let $f(z) = x + y, \quad \forall z = x + iy \in \mathbb{C}$. Compute $\frac{\partial f}{\partial \bar{z}}$.
   What can be said about the differentiability of f on $\mathbb{C}^*$?

2) Solve the following equations:
$$\cos z = 2, \qquad z^6 + i = 0.$$

III) Recall Rouch's theorem, then determine the number of roots of $z^8 - 4z^5 + z^2 - 1 = 0$ in $D = \{z \in \mathbb{C} : \ |z| < 1\}$.
IV) Find the Laurent series expansion of
$$f(z) = \frac{1}{(z-2)(z-3)}$$
in the region $2 < |z| < 3$. Deduce
$$I = \int_{|z|=1} f(z)\, dz$$
using the residue theorem.

Exercise 3 (8 pts)
I) Compute the integral
$$\int_{\gamma} (\bar{z} + z)\, dz$$
where γ is the segment $[0, i]$.
II) Using Cauchy's integral formulas, compute:
$$I_1 = \int_{|z|=2} \frac{z^2}{(z+4)(z+i)^2}\, dz$$
$$I_2 = \int_{|z|=1} \frac{e^z}{z^{2026}}\, dz$$
III) Let
$$f(z) = \frac{2}{z^2 + 4iz - 1}.$$

i. Find the residues at all poles, then compute
$$I = \int_{|z|=1} f(z)\, dz.$$

ii. Deduce the value of

$$J = \int_0^{2\pi} \frac{dx}{2 + \sin x}.$$

Bonus Exercise (2 pts)
Compute

$$I = \int_{-\infty}^{+\infty} \frac{dx}{1 + x^6}.$$

Good luck!

Model Answer Key – Complex Analysis

Exercise 1

1. The Cauchy–Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.(1pt)$$

2. A point $a \in \mathbb{C}$ is called an isolated singularity of f if f is holomorphic in

$$0 < |z - a| < r$$

for some $r > 0$, but not holomorphic at a.(1pt)

3. Residue theorem:

$$\int_C f(z)\, dz = 2\pi i \sum \text{Res}(f, a_k),$$

where a_k are the poles inside C.(1pt)

4. A harmonic function satisfies

$$u_{xx} + u_{yy} = 0.$$

Example:

$$u(x, y) = x^2 - y^2.(1pt)$$

Exercise 2
I) We are given

$$P(x, y) = x^3 - 3xy^2.$$

Compute:

$$P_x = 3x^2 - 3y^2, \qquad P_y = -6xy.$$

By the Cauchy–Riemann equations:

$$Q_y = 3x^2 - 3y^2, \qquad Q_x = 6xy.(0.5pt)$$

Integrating:

$$Q(x, y) = 3x^2y - y^3 + C.$$

Hence

$$f(z) = x^3 - 3xy^2 + i(3x^2y - y^3) = z^3.$$

Therefore

$$\boxed{f(z) = z^3}.(1pt)$$

II)

1)

$$f(z) = x + y.$$

We use

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Since

$$f_x = 1, \qquad f_y = 1,$$

we obtain

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(1+i).$$

Since

$$\frac{\partial f}{\partial \bar{z}} \neq 0,$$

the function is not holomorphic on $\mathbb{C}^*$. (1pt)

2) Solve

$$\cos z = 2.$$

Using

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$

let $w = e^{iz}$. Then

$$w + \frac{1}{w} = 4.$$

Hence

$$w^2 - 4w + 1 = 0.$$

Thus

$$w = 2 \pm \sqrt{3}.$$

Therefore

$$z = -i\ln(2 \pm \sqrt{3}) + 2k\pi, \qquad k \in \mathbb{Z}.(1pt)$$

Now solve

$$z^6 + i = 0 \iff z^6 = e^{-i\pi/2}.$$

Hence

$$z_k = e^{i\frac{-\pi/2+2k\pi}{6}}, \qquad k = 0, 1, \dots, 5.(1pt)$$

III)

Let

$$f(z) = -4z^5, \qquad g(z) = z^8 + z^2 - 1.$$

On $|z| = 1$:

$$|f(z)| = 4,$$

and

$$|g(z)| \leq 1 + 1 + 1 = 3.$$

Hence

$$|g(z)| < |f(z)|.$$

By Rouch's theorem,

$$z^8 - 4z^5 + z^2 - 1$$

has the same number of zeros as $-4z^5$ inside $|z| < 1$.

Therefore the equation has exactly

$$\boxed{5}$$

roots inside the unit disk. (1,5pt)

IV)
We write

$$\frac{1}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2}.$$

For $|z| < 3$:

$$\frac{1}{z-3} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n.$$

For $|z| > 2$:

$$\frac{1}{z-2} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n.$$

Hence for $2 < |z| < 3$:

$$f(z) = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n - \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}.$$

Now

$$I = \int_{|z|=1} f(z)\, dz.(2pt)$$

The poles 2 and 3 are outside $|z| = 1$, hence

$$\boxed{I = 0}.$$

Exercise 3
I)
Parametrize:

$$z(t) = it, \qquad 0 \le t \le 1.$$

Then

$$dz = i\, dt, \qquad \bar{z} = -it.$$

Thus

$$\bar{z} + z = 0.$$

Hence

$$\boxed{\int_\gamma (\bar{z} + z)\, dz = 0}.(1.5pt)$$

II)

$$I_1 = \int_{|z|=2} \frac{z^2}{(z+4)(z+i)^2}\, dz.$$

Only the pole $z = -i$ lies inside $|z| = 2$.
Let

$$h(z) = \frac{z^2}{z+4}.$$

Using Cauchy's formula for derivatives:

$$I_1 = 2\pi i\, h'(-i).$$

Compute

$$h'(z) = \frac{2z(z+4) - z^2}{(z+4)^2} = \frac{z^2 + 8z}{(z+4)^2}.$$

Hence

$$h'(-i) = \frac{-1-8i}{(4-i)^2}.$$

Therefore

$$\boxed{I_1 = 2\pi i \frac{-1-8i}{(4-i)^2}}.(1.5pt)$$

Next:

$$I_2 = \int_{|z|=1} \frac{e^z}{z^{2026}}\, dz.$$

By generalized Cauchy formula:

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(a).$$

Here

$$f(z) = e^z, \qquad a = 0, \qquad n = 2025.$$

Since

$$f^{(2025)}(0) = 1,$$

we get

$$\boxed{I_2 = \frac{2\pi i}{2025!}}.(1.5pt)$$

III)

$$f(z) = \frac{2}{z^2 + 4iz - 1}.$$

Solve

$$z^2 + 4iz - 1 = 0.$$

The poles are

$$z = -2i \pm i\sqrt{3}.$$

Only

$$z_1 = -2i + i\sqrt{3}$$

belongs to $|z| < 1$.
The residue is

$$\mathrm{Res}(f, z_1) = \frac{2}{z_1 - z_2} = \frac{2}{2i\sqrt{3}} = \frac{1}{i\sqrt{3}}.$$

Therefore

$$I = 2\pi i \,\mathrm{Res}(f, z_1) = 2\pi i \cdot \frac{1}{i\sqrt{3}}.(1.5pt)$$

Hence

$$\boxed{I = \frac{2\pi}{\sqrt{3}}(0.5pt)}.$$

ii. Using the substitution

$$z = e^{ix},$$

we obtain

$$\boxed{J = \frac{2\pi}{\sqrt{3}}}.(1.5pt)$$

Bonus Exercise

$$I = \int_{-\infty}^{+\infty} \frac{dx}{1+x^6}.$$

Using residues in the upper half-plane:

$$\boxed{I = \frac{2\pi}{3}}.(2pt)$$