University of M'sila

Faculty of Mathematics and computer science

Complex Analysis

Course designed for second-year (L2) Mathematics students

Dr: Abdelhamid Tallab

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Introduction

Chapter 1

Generalities on Complex Numbers

In this chapter, we assume that $\mathbb{k}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$.

1.1 Algebraic form of complex numbers

Definition 1.1.1 *A complex number z can be written in the form $z = x+iy$ with $x, y \in \mathbb{R}$, and i is called the imaginary unit, by convention $i^2 = -1$ $z = x+iy$ is known as the algebraic form of z.*
The number x is called the real part of z, written $x = Re(z)$. The number y is called the imaginary part of z, written $y = Im(z)$. The set of complex numbers is denoted by $\mathbb{C}$.

Remark 1.1.1 *We have the following set inclusion*

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

Definition 1.1.2 (The sum and the product of two complex numbers) *1) The sum of two complex numbers $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ is defined to be the complex number*

$$\begin{cases} z = z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2) \\ where \\ Re(z_1 + z_2) = x_1 + x_2 \text{ and } Im(z_1 + z_2) = y_1 + y_2. \end{cases}$$

2)The product of two complex numbers, $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ is defined to be the complex number

$$\begin{cases} z = z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2) \\ \text{where} \\ Re(z_1z_2) = x_1x_2 - y_1y_2 \text{ et } Im(z_1z_2) = x_1y_2 + y_1x_2. \end{cases}$$

Remark 1.1.2 $(\mathbb{C}, +, \times)$ *is a commutative field*

Definition 1.1.3 *Let $z \in \mathbb{C}$,*

1) The module or the absolute value of z is defined by

$$|z| = |x + iy| = \sqrt{x^2 + y2}.$$

2) The conjugate of z is defined by

$$\overline{z} = \overline{x + iy} = x - iy.$$

3) We have $\operatorname{Re}(z) = \dfrac{z + \overline{z}}{2}$ *and* $\operatorname{Im}(z) = \dfrac{z + \overline{z}}{2i}$.

Remark 1.1.3 *1) In particular, we have identically*

$$|z|^2 = z\overline{z}.$$

2) For $z_1, z_2 \in \mathbb{C}$, $|z_1 - z_2|$ is the Euclidean distance between z_1 and z_2. This provides a geometric proof of the triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Definition 1.1.4 *The argument $\arg(z)$ of a non-zero complex number z is the angle, defined modulo 2π, between the vector z and the x-axis. We thus have*

$$\arg z = \theta, \text{ then } \cos\theta = \frac{\operatorname{Re}(z)}{|z|} \text{ and } \sin\theta = \frac{\operatorname{Im}(z)}{|z|}.$$

A non-zero complex number is completely determined by its modulus and its argument: if $|z| = r$ and $\arg z = \theta$, then $x = r\cos\theta$, $y = r\sin\theta$, and hence

$$z = r(\cos\theta + i\sin\theta). \tag{1.1.1}$$

It is said that r and θ are the polar coordinates of z. And the form (1.1.1) is called the trigonometric form of z.

Remark 1.1.4 *The principal argument or the principal value of the argument is denoted* $Arg\,(z)$ *with*

$$-\pi \leq Arg\,(z) \leq \pi.$$

Proposition 1.1.1 *Let* $z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)\,, z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)\,,$ *we have*

1) the trigonometric form of $z_1 z_2$ *is given by*

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))\,,$$

then $|z_1 z_2| = |z_1|\,|z_2| = r_1 r_2$ *and* $\arg(z_1 z_2) = \arg z_1 + \arg z_2 = \theta_1 + \theta_2.$

2) the trigonometric form $\dfrac{z_1}{z_2}\ (z_2 \neq 0)$ *is given by*

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))\,,$$

then $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} = \dfrac{r_1}{r_2}$ *and* $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 = \theta_1 - \theta_2.$

3) If $z_1 = z_2 = ... = z_n = r(\cos\theta + i\sin\theta)\,,$ *we have*

$$z^n = r^n (\cos n\theta + i\sin n\theta)\,, n \in \mathbb{N},$$

then $|z^n| = |z|^n$ *and* $\arg(z^n) = n\arg(z) = n\theta.$

1.2 Exponential form of a complex number

In the following definition, we introduce another form of a complex number, called the exponential form. Thus, if we apply Euler's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

Then we can represent a complex number with modulus r and argument θ in the following definition.

Definition 1.2.1 (Exponential form of a complex number) *The exponential form of a complex number*

$$z = r(\cos\theta + i\sin\theta)$$

is

$$z = re^{i\theta} \text{ or } z = |z|\, re^{i\arg z}.$$

Example 1.2.1 *We have* $i = e^{i\frac{\pi}{2}}, -3 = e^{i\pi}.$

1.2.1 Powers, Roots, and Moivre's Formula

Definition 1.2.2 *Let $n \in \mathbb{N}$, we have*

1) $z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}$. If $r = 1$, we find $\left(e^{i\theta}\right)^n = e^{in\theta}$ or $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$ this formula is called Moivre's Formula.

2) A complex number ω is called n^{th} roots of z if

$$\omega = z^{\frac{1}{n}} \text{ or } \omega^n = z$$

if $\omega = r_1(\cos\theta + i\sin\theta)$ and $z = r_2(\cos\beta + i\sin\beta)$, then

$$\omega^n = z \Leftrightarrow \begin{cases} r_1^n = r_2 \\ n\theta = \beta + 2k\pi \end{cases} \Leftrightarrow \begin{array}{c} r_1 = r_2^{\frac{1}{n}} \\ \theta = \frac{\beta+2k\pi}{n} \end{array}, k = 0, 1, ..., n-1. \text{ i.e.,}$$

$$\begin{aligned} \omega &= |z|^{\frac{1}{n}}\left(\cos\left(\frac{Arg(z) + 2k\pi}{n}\right) + i\sin\left(\frac{Arg(z) + 2k\pi}{n}\right)\right), \\ k &= 0, 1, ..., n-1. \end{aligned}$$

Example 1.2.2 *Find the third roots of (-8). Since $(-8) = 8(\cos\pi + i\sin\pi) = 8e^{i\pi}$, then*

$$(-8)^{\frac{1}{3}} = \left(8e^{i\pi}\right)^{\frac{1}{3}} \Leftrightarrow z_k = 2\exp\left(\frac{\pi + 2k\pi}{3}\right), k = 0, 1, 2$$

Hence

$$(-8)^{\frac{1}{3}} = \begin{cases} z_0 = 2e^{i\frac{\pi}{3}} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + i\sqrt{3} \\ z_1 = 2e^{i\pi} = 2(\cos\pi + i\sin\pi) = -2 \\ z_2 = 2e^{-i\frac{\pi}{3}} = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = 1 - i\sqrt{3}. \end{cases}$$

2) We find the fourth roots of (-1). We have $(-1) = 1(\cos\pi + i\sin\pi) = e^{i\pi}$, then

$$(-1)^{\frac{1}{4}} = \left(8e^{i\pi}\right)^{\frac{1}{4}} \Leftrightarrow z_k = \exp\left(\frac{\pi + 2k\pi}{4}\right), k = 0, 1, 2, 3$$

hence

$$(-1)^{\frac{1}{4}} = \begin{cases} z_0 = e^{i\frac{\pi}{4}} \\ z_1 = e^{i3\frac{\pi}{4}} \\ z_3 = e^{i\frac{5\pi}{4}} \\ z_4 = e^{i\frac{7\pi}{4}}. \end{cases}$$

1.3 Topology on $\mathbb{C}$

Definition 1.3.1 (Circle, Open disk, Closed disk) *Let r be a positive real and $z_0 \in \mathbb{C}$, we call*

- The circle with center z_0 and radius r, the set

$$C_r = \{z \in \mathbb{C}, |z - z_0| = r\}.$$

- The open disk with center z_0 and radius r, the set

$$D(z_0, r) = \{z \in \mathbb{C}, |z - z_0| < r\}.$$

- The closed disk with center z_0 and radius r, the set

$$\overline{D}(z_0, r) = \{z \in \mathbb{C}, |z - z_0| \leq r\}.$$

Definition 1.3.2 (Open, neighborhood, connected) *1) A set V of $\mathbb{C}$ is a neighborhood of $z_0 \in \mathbb{C}$ If it contains an open ball centered at z_0 with a strictly positive radius de rayon.*
2) A subset $\Omega \subset \mathbb{C}$ is said to be open if and only if, for every point $z \in \Omega$, there exists an open disk centered at z_0 with radius $r > 0$ entirely contained within Ω. i.e., $D(z_0, r) \subset \Omega$.
3) A subset $U \subset \mathbb{C}$ is said to be arc-connected if any two points in U can be joined by an arc contained within U.

Exercise 1.3.1 (Exercise 1(series1)) *1) Write the following complex numbers in algebraic form: $z_1 = \dfrac{(1+i)^9}{(1-i)^7}$ and $z_2 = \dfrac{1+\lambda i}{2\lambda + \left(\lambda^2 - 1\right) i}, \lambda \in \mathbb{R}$.*
2) Write the following complex numbers in trigonometric form: $z_1 = 1 - i\sqrt{3}$ and $z_2 = \dfrac{1+i\sqrt{3}}{1-i}$.
3) Write the following complex numbers in exponential form: $a = 1 - i, b = -3, c = \sin x + i\cos x, x \in \mathbb{R}$.

***Solution:** 1) We have $z_1 = \dfrac{(1+i)^9}{(1-i)^7} = (1+i)^2 \dfrac{(1+i)^7}{(1-i)^7} = 2i\dfrac{(2i)^7}{2^7} = 2$, then $\mathrm{Re}(z_1) = 2$ and $\mathrm{Im}(z_1) = 0$.*
On a

$$\begin{aligned} z_2 &= \frac{1+\lambda i}{2\lambda + \left(\lambda^2 - 1\right) i} = \frac{i(-i+\lambda)}{i\left(-2\lambda i + \left(\lambda^2 - 1\right)\right)} \\ &= \frac{\lambda - i}{(\lambda - i)^2} = \underbrace{\frac{\lambda}{\lambda^2 + 1}}_{\mathrm{Re}(z_2)} + i\underbrace{\frac{1}{\lambda^2 + 1}}_{\mathrm{Im}(z_2)}. \end{aligned}$$

2) Trigonometric Form

(a) $z_1 = 1 - i\sqrt{3}$. *Modulus:* $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$. *Argument:* $\theta = -\frac{\pi}{3}$.

Therefore

$$z_1 = 2\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3}\right) = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

(b)

$$z_2 = \frac{1 + i\sqrt{3}}{1 - i}.$$

Multiply by the conjugate:

$$z_2 = \frac{(1 + i\sqrt{3})(1 + i)}{2} = \frac{(1 - \sqrt{3}) + i(1 + \sqrt{3})}{2}.$$

Modulus:

$$r = \sqrt{2}.$$

Argument:

$$\theta = \frac{5\pi}{12}.$$

Hence

$$z_2 = \sqrt{2}\left(\cos\frac{5\pi}{12} + \sin\frac{5\pi}{12}\right).$$

3) Exponential Form

Recall $z = re^{i\theta}$.

(a) $a = 1 - i \Rightarrow r = \sqrt{2}, \qquad \theta = -\frac{\pi}{4}$. *Then* $a = \sqrt{2}e^{-i\frac{\pi}{4}}$.

(b) $b = -3 \Rightarrow r = 3, \qquad \theta = \pi$. *Then* $b = 3e^{i\pi}$

(c) $c = \sin x + i\cos x$. *Since*

$$\sin x + i\cos x = \cos\left(\frac{\pi}{2} - x\right) + i\sin\left(\frac{\pi}{2} - x\right),$$

we obtain

$$c = e^{i\left(\frac{\pi}{2} - x\right)} = ie^{-ix}$$

Chapter 2

The functions with complex variable

2.1 Complex functions

Definition 2.1.1 *If to each z (variable) there corresponds one or more values w, we will say that w is a function of z. In general, a complex function of the complex variable z is any mapping f from $\mathbb{C}$ to $\mathbb{C}$ that associates with each complex number z a complex number w such that*

$$w = f(z); z \in \mathbb{C} \xrightarrow{f} w = f(z) \in \mathbb{C}.$$

Or if $w = f(z)$ is a complex number, so it is written in the form

$$w = f(z) = P + iQ$$

Since $z = x + iy$, then

$$P = P(x, y); Q = Q(x, y),$$

P is the real part of w and Q the imagenary part of w. If $z = re^{i\theta}$, then we can write

$$w = f(z) = f(re^{i\theta}) = P(r; \theta) + iQ(r; \theta).$$

Example 2.1.1 1) $f : \mathbb{C} \to \mathbb{C}$*;* $x + iy \mapsto e^x \cos y + ie^x \sin y$; $z \mapsto e^z$ *(complex exponential).*

2) $z \mapsto \cos z$ *(complex cosinus).*

4) *Let be the function $z \mapsto z^2$, we will write the function $z \mapsto z^2$ under the forme $P + iQ$,*

since $z = x + iy$, *then*

$x + iy \mapsto x^2 - y^2 + i2xy$, *hence* $P(x, y) = x^2 - y^2$ *and* $Q(x, y) = 2xy$. *By the polar form of* z, *we find*

$$f(z) = f(re^{i\theta}) = \left(re^{i\theta}\right)^2 = \underbrace{r^2 \cos 2\theta}_{P(r;\theta)} + i\underbrace{r^2 \sin 2\theta}_{Q(r;\theta)}.$$

2.1.1 Uniform functions

Definition 2.1.2 *A complex function f of the complex variable z is called a uniform function if, for each given value of the complex variable z, the function* $f(z)$ *can take a unique value. If not, the function is said to be multi-valued (or multiform).*

Example 2.1.2 *1) The function* $z \mapsto f(z) = z^3$ *is uniform, because for* $z = z_0$, $f(z_0)$ *takes a unique number* z_0^3.
2) The function $z \mapsto f(z) = \sqrt{z}$ *is multiform..*

2.2 Elementary (or standard) functions

2.2.1 Exponential function

Definition 2.2.1 *Refers to the function* f *defined on* $\mathbb{C}$ *by*

$$e^z = \sum_{n \geq 0} \frac{z^n}{n!}.$$

This definition makes sense, as Hadamard's theorem guarantees that the radius of convergence of the preceding series $R = \infty$.
If $z = x + iy$, *then* $e^z = e^x (\cos y + i \sin y)$.

Proposition 2.2.1 *We have the following properties:*
1) $e^0 = 1$.
2) $e^z \neq 0, \forall z \in \mathbb{C}$.
3) $e^{z_1 + z_2} = e^{z_1} e^{z_2}$.

4) $|e^z| = e^x$.

5) $\overline{\exp(z)} = \exp(\overline{z})$.

6) *The complex exponential function is not bijective on* $\mathbb{C}$, *but The restriction of the exponential function to* $\mathbb{R}$ *is a strictly increasing bijection from* $\mathbb{R}$ *into* $]0, +\infty[$.

7) $e^z = 1$ *if and only if* $z = 2k\pi i; k \in \mathbb{Z}$.

8) $e^{z_1} = e^{z_2}$ *if and only if* $z_1 = z_2 + 2k\pi i; k \in \mathbb{Z}$.

2.2.2 The complex logarithm

Definition 2.2.2 *In the real case, we conceive the logarithmic function as the inverse of the exponential. The formula* $z = |z| e^{i \arg z}$ *shows that a number has an infinite number of logarithms, defined modulo* 2π *by*

$$\log z = \ln |z| + i \arg z.$$

where $\ln |z|$ *is the natural logarithm, to the base* e, *of a real number.*

Remark 2.2.1 *The function* $f(z) = \log z$ *is a multi-valued function, which are called branches.,*

$$f(z) = \log z = \log r + i\left(Arg(z) + 2k\pi\right), k \in \mathbb{Z}.$$

In particular, when $k = 0$, *the quantity* $\ln\, r + i\theta$ *is called the principal value (or principal branch) of* $log(z)$ *and is denoted by*

$$Log(z) = \ln |z| + iArg(z) = \ln r + i\theta$$

with $\theta = Arg(z), i.e., (-\pi \leq \theta \leq \pi)$ *is the principal argument of* z.

Example 2.2.1 1) *As* $Arg(3) = 0$, *then* $\log(3) = \ln 3 + i2k\pi, k \in \mathbb{Z}$ *but* $Log(3) = \ln 3$.

2) *We have* $\left|i + \sqrt{3}\right| = 2$ *and* $Arg\left(i + \sqrt{3}\right) = \dfrac{\pi}{6}$, *then* $\log\left(i + \sqrt{3}\right) = \ln 2 + i\left(\frac{\pi}{6} + 2k\pi\right), k \in \mathbb{Z}$ *and* $Log\left(i + \sqrt{3}\right) = \ln 2 + \dfrac{\pi}{6}$.

Proposition 2.2.2 *The logarithm of a complex variable* z *has the following properties:*

1) $\log(z_1 z_2) = \log(z_1) + \log(z_2)$.

2) $\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2)$.

3) $e^{\log z} = z$.

4) $\log(e^z) = z + 2k\pi i, k \in \mathbb{Z}$.

Example 2.2.2 *If* $z_1 = z_2 = -1$, *we have* $Log\,(z_1 z_2) = Log\,(1) = 0$ *et* $Log\,(-1) + Log\,(-1) = 2\pi i$, *then in general* $Log\,(z_1 z_2) \neq Log\,(z_1) + Log\,(z_2)$.

2.2.3 The trigonometric and hyperbolic functions

Definition 2.2.3 1) *The sine and cosine functions are defined by*

$$\cos z = \sum_{n\geq 0} \frac{(-1)^n z^{2n}}{2n!}, \sin z = \sum_{n\geq 0} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

Moreover, we have

$$\begin{cases} e^{iz} = \cos z + i \sin z \\ e^{-iz} = \cos z - i \sin z \end{cases} \Rightarrow \begin{cases} \cos z = \dfrac{e^{iz} + e^{-iz}}{2} \\ \sin z = \dfrac{e^{iz} - e^{-iz}}{2i} \end{cases}.$$

2) Other trigonometric and hyperbolic functions are defined in the same way as follows:

$$\begin{aligned} \tan z &= -i\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}, \\ \cot z &= i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}, \\ \cosh z &= \frac{e^{z} + e^{-z}}{2}, \\ \sinh z &= \frac{e^{z} - e^{-z}}{2}, \\ \tanh z &= \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}, \\ \coth z &= \frac{e^{z} + e^{-z}}{e^{z} - e^{-z}}. \end{aligned}$$

Remark 2.2.2 1) *We have*

$$\cos z = \cosh\,(iz)\,, \sinh\,(iz) = i \sin z.$$

2) *By changing* z *by* iz, *we find*

$$\cos\,(iz) = \cosh z, \sin\,(iz) = i \sinh z.$$

3) *The sine and cosine functions are not bounded on* $\mathbb{C}$.

2.2.4 Others functions

1) Polynomial functions. The polynomial function, or polynomial, is a function of the form

$$\begin{aligned} P_n \quad &: \quad \mathbb{C} \to \mathbb{C} \\ z \quad &\longmapsto \quad P_n(z) = a_0 + a_1 z + a_2 z^2 ... + a_n z^n. \end{aligned}$$

The equation $P_n(z) = 0$ has exactly n complex roots in $\mathbb{C}$.

1) Rational functions. A rational function is a function of the form $z \longmapsto \dfrac{P_n(z)}{Q_n(z)}$ with P_n, Q_n are polynomials with $Q_n(z) \neq 0$.

In the following, the notation 'function' represents a uniform function

2.3 Countinuity of functions

The concepts of limit and continuity are similar to those in the real case.

2.3.1 Limites

Definition 2.3.1 *We say that the complex number l is a limit of f at the point z_0 if*

$$\forall \varepsilon > 0, \exists \eta > 0; \mid z - z_0 \mid < \eta \Longrightarrow \mid f(z) - l \mid < \varepsilon$$

and we say $l = \lim\limits_{z \to z_0} f(z)$. As $f(z) = P(x, y) + iQ(x, y)$ then $a = \lim\limits_{\substack{x \to x_0 \\ y \to y0}} P(x; y)$ and $b = \lim\limits_{\substack{x \to x_0 \\ y \to y_0}} Q(x; y)$ such that $l = a + ib$.
We have also

- $\lim\limits_{z \to \infty} f(z) = l \Leftrightarrow \forall \varepsilon > 0, \exists A > 0$ such that $|z| > A \Rightarrow |f(z) - l| < \varepsilon$.
- $\lim\limits_{z \to z_0} f(z) = \infty \Leftrightarrow \forall A > 0, \exists \eta > 0$ such that $|z - z_0| < \eta \Rightarrow |f(z)| > A$.
- $\lim\limits_{z \to \infty} f(z) = \infty \Leftrightarrow \forall A > 0, \exists B > 0$ such that $|z| > B \Rightarrow |f(z)| > A$.

Remark 2.3.1 *For infinity in complex analysis, we have*

$$z \rightarrow \infty \Leftrightarrow \left(\frac{1}{z} \rightarrow 0 \text{ and } |z| \rightarrow \infty\right)$$

and

$$\left.\begin{array}{l} z \rightarrow \infty \\ \omega \rightarrow a \end{array}\right\} \Leftrightarrow z + \omega \rightarrow \infty$$

and

$$\left.\begin{array}{l} z \rightarrow \infty \\ \omega \rightarrow a \end{array}\right\} \Leftrightarrow z\omega \rightarrow \infty \ \ (a \neq 0).$$

Proposition 2.3.1 *Let f and g be two complex functions, and l and l' their respective limits at the point z_0, then we have the following properties:*

1. $\lim\limits_{z \to z_0} (f \pm g) = l \pm l'$,

2. $\lim\limits_{z \to z_0} (fg) = ll'$,

3. $\lim\limits_{z \to z_0} (\frac{f}{g}) = \frac{l}{l'}$ *si* $l' \neq 0$,

4. For $\lambda \in \mathbb{R}$, *then* $\lim\limits_{z \to z_0} (\lambda f) = \lambda l$.

2.3.2 Continuity

Definition 2.3.2 *Let f be a complex function (uniform) in the neighborhood of a point z_0. We say that the function f is continuous at z_0 if*

$$\lim_{z \to z_0} f(z) = f(z_0).$$

The following three conditions are simultaneously satisfied:

1) The limit of $f(z)$ at point z_0 does exist, i.e., $\lim\limits_{z \to z_0} f(z) = l$ does exist and finit.

2) $f(z)$ is defined at point z_0 i.e., $f(z_0)$ does exists.

3) $l = f(z_0)$.

Remark 2.3.2 *1. A function $f(z)$ is said to be continuous in the domain of the complex plan ,if it is continuous at every point in this domain.*

2. If the function $f(z)$ is continuous z_0, then

$$\forall \varepsilon > 0; \exists \delta > 0 \text{ such that for } \mid z - z_0 \mid < \delta \Rightarrow \mid f(z) - f(z_0) \mid \leq \varepsilon.$$

Example 2.3.1 *Let $f(z) = \sin(z)$ we have*

$$l = \lim_{z \to 0} f(z) = \lim_{z \to 0} \sin(z) = 0 \text{ and } f(0) = exp(0) = 0,$$

then this function is is continuous at $z_0 = 0$.

The following theorem holds.

Theorem 2.3.1 *A function $f(z) = u(x, y) + iv(x, y)$ is continuous at the point $z_0 = x_0 + iy_0$ if and only if its real and imaginary parts, $u(x, y)$ and $v(x, y)$, are continuous at the point (x_0, y_0).*

2.3.3 Extension by Continuity

Definition 2.3.3 *If the function f admits a limit l at z_0, but is not defined at this point, then we can associate with f a function g such that $\lim\limits_{z \to z_0} g(z) = l = g(z_0)$, where*

$$g(z) = \begin{cases} f(z) & \text{if } z \neq z_0 \\ l \text{ if } z = z_0 \end{cases}$$

That is, the function g is continuous at z_0. In this case, we say that the function f can be extended by continuity at z_0, and its extension is g.

Example 2.3.2 *Let f be a complex function given by:*

$$f(z) = \frac{z^2 - 2iz - 1}{z^4 + 2z^2 + 1}.$$

a) Compute $\lim_{z \to i} f(z)$.

b) Is f continuous at $z = i$?

c) Is f extendable by continuity at $z = i$?

Solution:

We have

$$\lim_{z \to i} f(z) = \lim_{z \to i} \frac{z^2 - 2iz - 1}{z^4 + 2z^2 + 1} = \lim_{z \to i} \frac{(z - i)^2}{(z - i)^2 (z + i)^2} = -\frac{1}{4}.$$

Since f is not defined at $z = i$, f is extendable by continuity at $z = i$. Its extension by continuity is:

$$g(z) = \begin{cases} \dfrac{z^2 - 2iz - 1}{z^4 + 2z^2 + 1} \text{ if } z \neq i \\ -\frac{1}{4} \text{ if } z = i. \end{cases}$$

2.4 Holomorphic Functions

2.4.1 Complex Differentiation

We give two equivalent definitions of the differentiability of a function of a complex variable.

Definition 2.4.1 *Let f be a complex function defined in a domain (open connected set) $\Omega \subseteq \mathbb{C}$ containing z_0. If the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and is finite, then it is called the derivative of the function f at the point z_0 and is denoted by $f'(z_0)$. In this case, we write

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

and we say that f is differentiable at z_0. That is,

$$f \text{ is differentiable at } z_0 \iff \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = l,$$

where l is a finite complex constant, and we write $f'(z_0) = l$.

Definition 2.4.2 *Let f be a function defined and continuous in a simply connected domain D. The function f is differentiable at z_0 if there exists a function $L : D \to \mathbb{C}$ that is continuous at z_0 such that*

$$f(z) = f(z_0) + (z - z_0)L(z) \quad \text{for all } z \in D.$$

If L exists, it is determined by f:

$$L(z) = \frac{f(z) - f(z_0)}{z - z_0} \quad \text{for all } z \in D \setminus \{z_0\}.$$

Letting $h = z - z_0$, the continuity of L implies that

$$L(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

The constant $L(z_0) \in \mathbb{C}$ is called the derivative of f at z_0, and we write

$$L(z_0) = f'(z_0) = \frac{df}{dz}(z_0).$$

Definition 2.4.3 *If the derivative of f exists at every point z in a domain D, then f is said to be holomorphic in D. A function f is said to be holomorphic at a point z_0 if it is differentiable in an open disk centered at z_0 (denoted $D(z_0, r)$). That is,*

$$f \text{ is holomorphic at } z_0 \iff f \text{ is differentiable at } z_0.$$

Example 2.4.1 *1) For the function $f(z) = z^3$ at $z_0 \in \mathbb{C}$, we have*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^3 - z_0^3}{z - z_0} = 3z_0^2.$$

Thus, f is holomorphic on $\mathbb{C}$.

2) For $f(z) = x^2 + y^2$ at $z_0 = 0$, we have

$$f'(0) = \lim_{z \to 0} \frac{x^2 + y^2 - 0}{x + iy} = \lim_{r \to 0} r(\cos\theta - i\sin\theta) = 0.$$

Remark 2.4.1 *The properties of differentiation remain valid as in $\mathbb{R}$ (sum, scalar multiplication, product, quotient, composition, etc...).*

2.5 Cauchy-Riemann Conditions

In Cartesian Coordinates

Definition 2.5.1 *Let $f(z) = P(x, y) + iQ(x, y)$ be a function defined and continuous in a domain $\Omega \subseteq \mathbb{C}$. We say that the functions P and Q satisfy the Cauchy-Riemann conditions (C.R.C.) if*

$$\begin{cases} \dfrac{\partial P}{\partial x}(x, y) = \dfrac{\partial Q}{\partial y}(x, y), \\ \dfrac{\partial P}{\partial y}(x, y) = -\dfrac{\partial Q}{\partial x}(x, y). \end{cases}$$

Remark 2.5.1 *The expression for $f'(z)$ is given in terms of P and Q as follows:*

$$f'(z) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} + i\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial x} - i\frac{\partial P}{\partial y}.$$

If the functions P and Q are differentiable at a point (x_0, y_0) and their partial derivatives are continuous and satisfy the Cauchy-Riemann conditions, then the function

$$z \mapsto f(z) = P(x, y) + iQ(x, y)$$

is differentiable at $z_0 = x_0 + iy_0$.

2.5.1 In polar cordinates

Proposition 2.5.1 *Let $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$. The Cauchy-Riemann conditions in polar coordinates are given by*

$$\frac{\partial u}{\partial r}(r, \theta) = \frac{1}{r}\frac{\partial v}{\partial \theta}(r, \theta), \quad \frac{\partial v}{\partial r}(r, \theta) = -\frac{1}{r}\frac{\partial u}{\partial \theta}(r, \theta).$$

Additionally,

$$f'(re^{i\theta}) = e^{-i\theta}\frac{\partial}{\partial r}\left(u(r, \theta) + iv(r, \theta)\right).$$

Preuve. Let $z = re^{i\theta}$, where $x = r\cos\theta$ and $y = r\sin\theta$. Thus, $\theta = \arg z$ and $|z| = r$. On the other hand, it is clear that for $f(z) = u(x, y) + iv(x, y)$:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta; \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta; \\
\frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta; \\
\frac{\partial v}{\partial y} &= \frac{\partial v}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta.
\end{aligned}$$

Now, using the Cauchy-Riemann conditions (CCR), we obtain:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\cos\theta + \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta = 0$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \quad \left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin\theta + \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta = 0$$

The solution to this system gives exactly equation. ■

Remark 2.5.2 *1) Note that we have,*

$$\begin{aligned}\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y} &= \frac{\partial(P+iQ)}{\partial x}+i\frac{\partial(P+iQ)}{\partial y}\\ &= (\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y})+i(\frac{\partial Q}{\partial x}+\frac{\partial P}{\partial y})\\ &= 0\end{aligned}$$

A condensed form of the Cauchy-Riemann conditions is

$$\forall(x,y)\in\Omega\subset\mathbb{R}^2, \frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}=0. \tag{2.5.1}$$

2) We also have $df(z)=\frac{\partial f}{\partial z}dz+\frac{\partial f}{\partial \overline{z}}d\overline{z}$. *Since*

$$\frac{\partial f}{\partial \overline{z}}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \overline{z}}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \overline{z}} \tag{2.5.2}$$

Proposition 2.5.2 *If f is holomorphic, then* $\frac{\partial f}{\partial \overline{z}}=0$.

Preuve. We have

$$\begin{cases} x=\dfrac{z+\overline{z}}{2} \\ y=\dfrac{z-\overline{z}}{2i}\end{cases} \Rightarrow \begin{cases}\dfrac{\partial x}{\partial \overline{z}}=\dfrac{1}{2}\\ \dfrac{\partial y}{\partial \overline{z}}=-\dfrac{1}{2i}\end{cases}.$$

Substituting these relations into 2.5.2 and by using 2.5.1, *we have*

$$\begin{aligned}\frac{\partial f}{\partial \overline{z}} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \overline{z}}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \overline{z}}\\ &= \frac{1}{2}\frac{\partial f}{\partial x}-\frac{1}{2i}\frac{\partial f}{\partial y}\\ &= \frac{1}{2}\frac{\partial f}{\partial x}-\frac{1}{2i}\frac{\partial f}{\partial y}\\ & \frac{1}{2}(\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y})-\frac{1}{2}(\frac{\partial Q}{\partial x}+\frac{\partial P}{\partial y})=0.\end{aligned}$$

■

Corollary 2.5.1 *1) Finally, if f is differentiable, then*

$$df(z)=\frac{\partial f}{\partial z}dz=f'(z)dz.$$

2) Therefore, if f is differentiable, $f(z)$ must not contain terms in $\overline{z}$. That is, if $\frac{\partial f}{\partial \overline{z}} \neq 0$, then f is not holomorphic.

3) We must think of holomorphic functions as functions independent of $\overline{z}$.

Example 2.5.1 *Consider the function $f : \mathbb{C}^* \to \mathbb{C}$ defined by $f(z) = \frac{1}{z} + zRe(z)$. We have*

$$f(z) = \frac{1}{z} + z\left(\frac{z + \overline{z}}{2}\right),$$

so

$$\frac{\partial f}{\partial \overline{z}} = \frac{z}{2} \neq 0.$$

Thus, f is not differentiable on $\mathbb{C}^*$.

Remark 2.5.3 *The Cauchy-Riemann conditions are necessary but not sufficient. This can be seen in the following example.*

Example 2.5.2 *Let f be a function defined by:*

$$f(z) = f(x + iy) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} + i\dfrac{x^3 + y^3}{x^2 + y^2} & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0, \end{cases}$$

It is clear that f is continuous on $\mathbb{C}^*$, and we have

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \left(\frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2}\right) = 0 = f(0),$$

so f is continuous on $\mathbb{C}$. On the other hand, we have

$$\frac{\partial P}{\partial x}(0,0) = \frac{\partial Q}{\partial y}(0,0) = 1, \quad \frac{\partial Q}{\partial x}(0,0) = -\frac{\partial P}{\partial y}(0,0) = -1,$$

so f satisfies the Cauchy-Riemann conditions at $z = 0$. However,

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{1}{x + iy} \left(\frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2}\right).$$

For $y = 0$, we obtain

$$f'(0) = \lim_{x \to 0} \frac{(1 + i)x^3}{x^3} = 1 + i.$$

For $y = x$, we have

$$f'(0) = \lim_{x \to 0} \frac{ix}{(1+i)x} = \frac{1}{2}(1+i).$$

Thus, f is not differentiable at 0 (since the limit is not unique), despite satisfying the Cauchy-Riemann conditions.

Example 2.5.3 *Show that the Cauchy–Riemann conditions are satisfied for*

$$f(z) = z^2 + z$$

at every point of the complex plane $\mathbb{C}$.

Solution.

Let $z = x + iy$, where $x, y \in \mathbb{R}$. Then

$$f(z) = z^2 + z = (x+iy)^2 + (x+iy).$$

Compute:

$$(x+iy)^2 = x^2 - y^2 + 2ixy.$$

Hence

$$f(z) = (x^2 - y^2 + x) + i(2xy + y).$$

We write

$$f(z) = P(x,y) + iQ(x,y),$$

where

$$P(x,y) = x^2 - y^2 + x, \qquad Q(x,y) = 2xy + y.$$

Now compute the partial derivatives:

$$P_x = 2x + 1, \qquad P_y = -2y,$$

$$Q_x = 2y, \qquad Q_y = 2x + 1.$$

The Cauchy–Riemann equations are

$$P_x = Q_y, \qquad P_y = -Q_x.$$

We verify:

$$P_x = 2x + 1 = Q_y,$$

$$P_y = -2y = -Q_x.$$

Thus, the Cauchy–Riemann equations are satisfied at every point $(x, y) \in \mathbb{R}^2$. Since the Cauchy–Riemann conditions are satisfied everywhere and the first-order partial derivatives are continuous on $\mathbb{R}^2$, it follows that

$$f(z) = z^2 + z \text{ is holomorphic on } \mathbb{C}.$$

2.6 Harmonic Functions

Definition 2.6.1 *Let ψ be a function from $\Omega \subset \mathbb{R}^2$ to $\mathbb{R}$. The function ϕ is said to be of class C^2 on Ω (denoted $\psi \in C^2(\Omega)$) if, for all $(x, y) \in \Omega \subset \mathbb{R}^2$, the partial derivatives $\frac{\partial \psi}{\partial x}$, $\frac{\partial \psi}{\partial y}$, $\frac{\partial^2 \psi}{\partial x^2}$, $\frac{\partial^2 \psi}{\partial x \partial y}$, and $\frac{\partial^2 \psi}{\partial y^2}$ exist and are continuous.*

Remark 2.6.1 *For functions of class C^2, Schwarz's theorem ensures the following equality:*

$$\forall (x, y) \in \Omega \subset \mathbb{R}^2, \quad \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}.$$

Definition 2.6.2 *Let ψ be a function from $\Omega \subset \mathbb{R}^2$ to $\mathbb{R}$ of class C^2. We say that ψ is harmonic if*

$$\forall (x, y) \in \Omega \subset \mathbb{R}^2, \quad \Delta \psi(x, y) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0,$$

where $\Delta \psi = \dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2}$ is called the Laplace operator or Laplacian of ψ.

Example 2.6.1 *The function $(x, y) \mapsto f(x, y) = x^2 - y^2$ is harmonic, because*

$$\Delta f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Proposition 2.6.1 *If $f(z) = P(x, y) + iQ(x, y)$ is holomorphic and $P, Q \in \mathcal{C}^2$, then its real and imaginary parts are harmonic.*

Preuve. Let $f(z) = P(x,y) + iQ(x,y)$ be a holomorphic function with $P, Q \in C^2$. Since P and Q satisfy the Cauchy-Riemann conditions, we have

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \implies \frac{\partial^2 P}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial P}{\partial y}\right) = -\frac{\partial^2 P}{\partial y^2},$$

which implies $\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0$. Thus, P is harmonic. The same method can be used to show that Q is harmonic. ■

Proposition 2.6.2 (Laplacien in polar coordinates) *In polar coordinates, the Laplace equation becomes*

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 P}{\partial \theta^2} = 0.$$

2.6.1 Harmonic Conjuguate

Let $f(z) = P(x,y) + iQ(x,y)$ be a holomorphic function in a domain D. Then P and Q are harmonic in D. Now, suppose $P(x,y)$ is a real-valued harmonic function in D. If we can find a function $Q(x,y)$ such that $P(x,y) + iQ(x,y)$ is holomorphic in D, then $Q(x,y)$ is called the harmonic conjugate of $P(x,y)$.

Example 2.6.2 *Let $P(x,y) = x^2 - y^2 + x$. Find the harmonic conjugate of P. We have $\Delta P = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 2 - 2 = 0$, so P is harmonic. To find a function Q such that $f = P + iQ$ is holomorphic, we use the Cauchy-Riemann conditions:*

$$\begin{cases} \dfrac{\partial Q}{\partial y} = \dfrac{\partial P}{\partial x} = 2x + 1 \\ \dfrac{\partial Q}{\partial x} = -\dfrac{\partial P}{\partial y} = 2y \end{cases}$$

Integrating the first equation with respect to y, we obtain

$$Q(x,y) = \int (2x+1)\, dy = 2xy + y + h(x),$$

where $h(x)$ is a real-valued function of x. From the second equation, we have

$$\frac{\partial Q}{\partial x} = 2y + h'(x) = 2y,$$

so $h(x) = c$, where $c \in \mathbb{R}$. Thus, the harmonic conjugate is $Q(x,y) = 2xy + y + c$, and the holomorphic function is

$$f(z) = P(x,y) + iQ(x,y) = x^2 - y^2 + x + i(2xy + y + c).$$

2.7 Analytic functions

Definition 2.7.1 *A power(Integer) series in the complex variable z is a series of functions $\sum f_n$, where*

$$f_n : \mathbb{C} \to \mathbb{C}, \quad z \mapsto a_n z^n,$$

with $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}$. We call a_n the coefficient of index n of the series, and a_0 is the constant term. The series is denoted by $\sum a_n z^n$.

2.7.1 Radius of Convergence

Theorem 2.7.1 (Abel's lemma) *Let $\sum a_n z^n$ be a power series and $z_0 \in \mathbb{C}$. Suppose the sequence $(a_n z_0^n)_n$ is bounded. Then, the series $\sum a_n z^n$ is uniformly convergent in every disk $D(0, r)$, where $r < |z_0|$.*

Preuve. We may assume $z_0 \neq 0$. Let M be an upper bound for the sequence $(|a_n z_0^n|)_n$, and let r be a positive real number such that $r < |z_0|$. If $|z| \leq r$, we have

$$|a_n z^n| = |a_n z_0^n \left(\frac{z}{z_0}\right)^n| \leq M \left|\frac{z}{z_0}\right|^n \leq M \left|\frac{r}{z_0}\right|^n,$$

and the series $\sum \left|\frac{r}{z_0}\right|^n$ is a convergent geometric series because $r < |z_0|$. Therefore, $\sum a_n z^n$ is normally convergent in the disk $D(0, r)$. ■

Corollary 2.7.1 *If the entire series $\sum a_n z^n$ converges for $z_0 \in \mathbb{C}$, it converges normally in the disk $D(0, r)$, where $r < |z_0|$.*

Definition 2.7.2 *Given an entire series $\sum a_n z^n$, the number R is called the radius of convergence of this series, such that*

$$R = \sup\{r \in \mathbb{R}_+ : (a_n r^n) \textit{is bounded}\} \in [0, +\infty).$$

Furthermore:

i) For every $z \in \mathbb{C}$ such that $|z| < R$, the series $\sum a_n z^n$ is absolutely convergent.

ii) For every $z \in \mathbb{C}$ such that $|z| > R$, the series $\sum a_n z^n$ is divergent.

We say that $D(0, r)$ is the disk of convergence, and $C(0, r)$ is the circle of convergence.

Proposition 2.7.1 *Let $\sum a_n z^n$ be an entire series whose coefficients are nonzero starting from a certain index.*
1)If the sequence $\left|\frac{a_{n+1}}{a_n}\right|$ tends to l as n approaches infinity, then the radius of convergence of the entire series $\sum a_n z^n$ is $R = \frac{1}{l}$ (with the convention $R = +\infty$ if $l = 0$ and $R = 0$ if $l = \infty$).
2)Similarly, if the sequence $\sqrt[n]{|a_n|}$ tends to l as n approaches infinity, then $R = \frac{1}{l}$ with the same convention.

Example 2.7.1 *For $a_n = \frac{1}{n}$ and $z \neq 0$, we have*

$$\left(\frac{|a_{n+1}|}{|a_n|}\right) = \frac{n}{n+1} \to 1 = l,$$

so the radius of convergence of $\sum \frac{z^n}{n}$ is $R = \frac{1}{l} = 1$. This implies that:
1) If $|z| < 1$, the series converges absolutely.
2) If $|z| > 1$, we have $\lim |a_n z^n| = +\infty$ and the series $\sum \frac{z^n}{n}$ diverges.
3) If $|z| = 1$, the series $\sum \frac{z^n}{n}$ diverges.
The domain of convergence of $\sum \frac{z^n}{n}$ is

$$D(0,1) - \{1\} = \{z \in \mathbb{C} : |z| \leq 1\} - \{1\}.$$

Definition 2.7.3 *Let U be an open set in $\mathbb{C}$. We say that a function $f : U \to \mathbb{C}$ is analytic at $z_0 \in U$ if there exists a real number $R > 0$ and an entire series $\sum a_n z^n$ with radius of convergence greater than or equal to R such that:*
1) $D(z_0, R) \subset U$,
2) For all $z \in D(z_0, R)$, $f(z) = \sum a_n (z - z_0)^n$.

The function is said to be analytic on U if it is analytic at every point of U.

Example 2.7.2 *1) $f(z) = \exp(z)$ is analytic on $\mathbb{C}$ because*

$$\begin{aligned} \forall z_0 &\in \mathbb{C} : \exp(z) = \exp(z - z_0 + z_0) \\ &= \exp(z_0) \sum \frac{(z - z_0)^n}{n!}, \quad \text{where } a_n = \frac{\exp(z_0)}{n!}. \end{aligned}$$

2) $f(z) = \frac{1}{z}$ is analytic on $\mathbb{C}^*$, because

$$\begin{aligned} \forall z_0 \;\in\; \mathbb{C} : \frac{1}{z} &= \frac{1}{z_0\left(1 + \frac{z - z_0}{z_0}\right)} \\ &= \frac{1}{z_0}\sum \frac{(-1)^n}{z_0^n}(z - z_0)^n, \quad \textit{where } |z - z_0| \le |z_0|. \end{aligned}$$

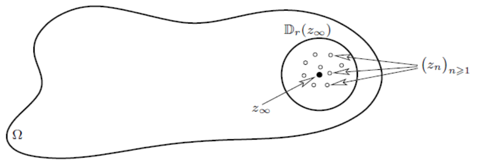
Corollary 2.7.2 *If f is analytic on U, then f' is also analytic on U.*

Remark 2.7.1 *1) Like holomorphic functions, elementary operations remain valid for analytic functions.*
2)Holomorphy and analyticity are equivalent concepts. Many mathematicians use them as synonyms.
3) It is clear that on a domain Ω, f analytic $\iff$ f holomorphic.
4)The region of analyticity only contains interior points, so analytic functions are defined only in domains (open and connected).
5)Thus, analyticity $\Rightarrow$ differentiability, but the converse is false.

2.7.2 Principle of Analytic Continuation(Extension)

Theorem 2.7.2 *Let Ω be a connected open set of $\mathbb{C}$, $a \in \Omega$, and $f : \Omega \to \mathbb{C}$ an analytic function at a. The following conditions are equivalent:*
1) $f = 0$ identically in Ω.
2) There exists $R > 0$ such that $\forall z \in D(a, R), f(z) = 0$.
3) $\forall n \in \mathbb{N}, f^{(n)}(a) = 0$.

Corollary 2.7.3 *Let Ω be a connected open set in $\mathbb{C}$, and suppose f, g are analytic. If there exists $a \in \Omega$ and $R > 0$ such that $\forall z \in D(a, R), f(z) = g(z)$, then $f \equiv g$ for all $z \in \Omega$.*



2.7.3 Principle of Isolated Zeros

Definition 2.7.4 *Let $\Omega \subseteq \mathbb{C}$. A point $a \in \Omega$ is said to be isolated in Ω if there exists R such that*

$$D(a, R) \cap \Omega = \{a\}.$$

Example 2.7.3 *The points in $\mathbb{N}$ are isolated in $\mathbb{N}$, because for any $n \in \mathbb{N}$, we have*

$$D(n, \frac{1}{2}) \cap \mathbb{N} = \{n\}.$$

Theorem 2.7.3 *Let Ω be a connected open set (domain) and f an analytic function on Ω that is not identically zero. Then the zeros of f in Ω are isolated.*

2.7.4 Order of a Zero of an Analytic Function

Theorem 2.7.4 *Let Ω be a connected open set (domain) and f an analytic function on Ω, with $z_0 \in \Omega$. The following two assertions are equivalent:*
1)There exists $k \in \mathbb{N}$ such that $f(z_0) = f'(z_0) = \cdots = f^{(k-1)}(z_0) = 0$ and $f^{(k)}(z_0) \neq 0$.
2)There exists an analytic function g on Ω such that $g(z_0) \neq 0$ and for all $z \in \Omega$,

$$f(z) = (z - z_0)^k g(z).$$

Definition 2.7.5 *We say that $z_0 \in \Omega$ is a zero of order k of the function f if one of the assertions of Theorem 2.7.4 is true.*

Example 2.7.4 *The point $z_0 = 0$ is a zero of order 2 of the function $f(z) = e^{z^2} - 1$, because $f(0) = f'(0) = 0$ and $f''(0) \neq 0$.*

Chapter 3

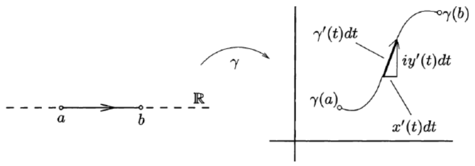
Complexe integration

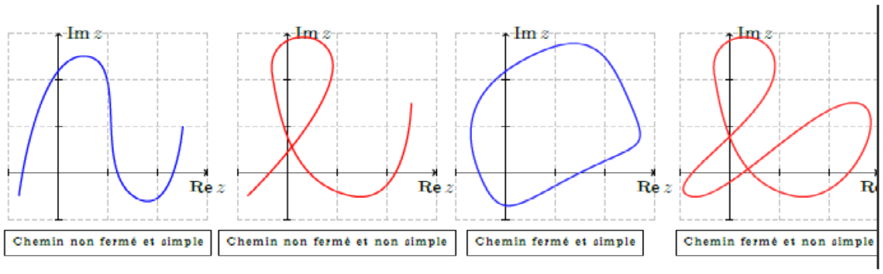
In this chapter, we are interested in the integral calculation of a complex function along a path or loop using specific theorems.

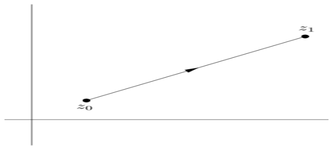
3.1 Curvilinear Integrals

3.1.1 Paths(Curves)

Definition 3.1.1 *Let $\gamma : I = [a, b] \to \mathbb{C}$ be an application defined by $\gamma(t) = x(t) + iy(t)$*
-The image $\gamma(I)$ is called a path, and γ is a parametrization of the path.
-$\gamma(a)$ and $\gamma(b)$ are the ***beginning point****(or* ***initial point****) and the* ***end point****(or* ***terminal point****) of the path, respectively.*
A path is said to be closed (or a loop) if $\gamma(a) = \gamma(b)$, where I is a closed and bounded interval of $\mathbb{R}$ (not reduced to a point) in $\mathbb{C}$.
- γ is said to be differentiable (smooth) if $x(t), y(t)$ are continuously differentiable on $[a, b]$.
- A path is said to be piecewise smooth (differentiable) if there exists a subdivision of $I = [a, b]$ $(a = a_0 < a_1 < \cdots < a_n = b)$ such that γ is differentiable on each interval $[a_i, a_{i+1}]$ for all







$0 \le i \le n-1$.

- *A path is said to be simple if it does not intersect itself, i.e., if $\gamma(t_1) \neq \gamma(t_2)$ when $t_1 \neq t_2$.*

- *Any simple closed curve is called a Jordan curve.*

Example 3.1.1 *1) Let $z_1, z_2 \in \mathbb{C}$, we define:*

$$\begin{aligned} \gamma: \quad & [0,1] \to \mathbb{C} \\ & t \mapsto \gamma(t) = z_1(1-t) + z_2 t \end{aligned}$$

We have $\gamma(0) = z_1$ and $\gamma(1) = z_2$. The image of γ is a segment joining z_1 and z_2.
2)2) Let $a \in \mathbb{C}$ and $r > 0$, we define:

$$\begin{aligned} \gamma: \quad & [0,2\pi] \to \mathbb{C} \\ & t \mapsto \gamma(t) = a + re^{it} \end{aligned}$$

The path is a circle with center a and radius r, and we have $\gamma(0) = \gamma(2\pi)$, so the path is closed.
3) The ellipse is parametrized by the path:

$$\begin{aligned} \gamma: \quad & [0,2\pi] \to \mathbb{C} \\ & t \mapsto \gamma(t) = a\cos t + ib\sin t \text{ with } a, b \in \mathbb{C}. \end{aligned}$$

Definition 3.1.2 (Opposit Path) *Let $\gamma : [a, b] \to \mathbb{C}$ be a path, the opposite path is defined by*

$$-\gamma : [a, b] \to \mathbb{C}, \quad t \mapsto -\gamma(t) = \gamma(a + b - t)$$

with origin $\gamma(b)$ and end point $\gamma(a)$.

Definition 3.1.3 (Length of a Path) *Let $\gamma : [a, b] \to \mathbb{C}$ be a (smooth) differentiable path defined by $\gamma(t) = x(t) + iy(t)$. Then the length $L(\gamma)$ of the path γ is defined by*

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{(x'^2 + (y'^2}\, dt = \int_a^b \left|\frac{dz}{dt}\right| dt.$$

3.1.2 Line integral or contour integral of a function along the path

Definition 3.1.4 *Let $\gamma : [a, b] \to D$ be a piecewise $\mathcal{C}^1$ path in a domain D, and let $f : D \to \mathbb{C}$ be a complex function defined in D. The curvilinear integral of f along the path γ is the complex number*

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

Proposition 3.1.1 *Let $\gamma : [a, b] \to \Omega$ be a piecewise $\mathcal{C}^1$ path in Ω. Then for any continuous function f on Ω, we have*

$$\left|\int_\gamma f(z)\, dz\right| \leq \sup_{z\in\gamma} |f(z)| L(\gamma).$$

Preuve.

$$\left|\int_\gamma f(z)\, dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\, dt\right| \leq \int_a^b |f(\gamma(t))||\gamma'(t)|\, dt \leq \sup_{z\in\gamma} |f(z)| L(\gamma).$$

■

Remark 3.1.1 *If $f(z) = P(x, y) + iQ(x, y)$, then*

$$\int_\gamma f(z)\, dz = \int_\gamma (P(x, y)+iQ(x, y))(dx+idy) = \int_\gamma (P(x, y)\, dx-Q(x, y)\, dy)+i\int_\gamma (Q(x, y)\, dx+P(x, y)\, dy).$$

Thus, the complex integral is a combination of two real curvilinear integrals.

Proposition 3.1.2 *Suppose that $\gamma(t), a \leq t \leq b$ is a differentiable path included in a domain D and that f and g are continuous complex functions in D. The following properties hold:*

1)

$$\int_\gamma (\alpha f(z) + \beta g(z))\, dz = \alpha \int_\gamma f(z)\, dz + \beta \int_\gamma g(z)\, dz, \quad \alpha, \beta \in \mathbb{C}.$$

2) If $\gamma = \gamma_1 \cup \gamma_2$ (with $\gamma_1 \cap \gamma_2$ being a finite set), then

$$\int_{\gamma=\gamma_1\cup\gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz.$$

3)

$$\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz.$$

Definition 3.1.5 *Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth (differentiable) path defined by $\gamma(t) = x(t) + iy(t)$. Then the length $L(\gamma)$ of the path γ is defined by*

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{(x'^2 + (y'^2}\, dt = \int_a^b \left|\frac{dz}{dt}\right| dt.$$

Example 3.1.2 *Let $\gamma_1(t) = e^{it}, 0 \leq t \leq 2\pi$ and $\gamma_1(t) = e^{2i\pi t}, 0 \leq t \leq 1$. We find the path of γ_1 and γ_2. Indeed*

$$\begin{aligned} L(\gamma) = \quad & \int_0^{2\pi} |\gamma'(t)| dt \quad = \int_0^{2\pi} |ie^{it}| dt = \int_0^{2\pi} dt = 2\pi \\ & \textit{and} \\ L(\gamma) = \quad & \int_0^1 |\gamma'(t)| dt \quad = \int_0^1 |2i\pi e^{it}| dt = \int_0^1 2\pi dt = 2\pi. \end{aligned}$$

Example 3.1.3 *[8, Example 6] Let $f(z) = z^2$, $g(z) = z$, $\gamma_1(t) = t + it$, and $\gamma_2(t) = t + it^2$, where $0 \leq t \leq 1$. Note that the paths are different, but they have the same origin $z = 0$ and the same endpoint $z = 1 + i$.*

(a)

$$\int_{\gamma_1} f(z)\, dz = \int_0^1 t^2(1+i)^2(1+i)\, dt = \frac{2}{3}(1+i)$$

Proposition 3.1.3 *(b)*

$$\int_{\gamma_2} f(z)\,dz = \int_0^1 (t+it^2)^2(1+2it)\,dt = \frac{2}{3}(1+i)$$

(c)

$$\int_{\gamma_1} g(z)\,dz = \int_0^1 (t-it)(1+i)\,dt = 1$$

(d)

$$\int_{\gamma_2} g(z)\,dz = \int_0^1 (t-it^2)(1+2it)\,dt = \frac{1+i}{3}$$

Theorem 3.1.1 (Green-Stokes Theorem) *Let $P, Q : U \subset \mathbb{R}^2 \to \mathbb{R}$ be functions whose partial derivatives are continuous on U, and let γ be a positively oriented closed path in U with an interior R entirely contained in U. Then*

$$\int_\gamma P(x,y)\,dx + Q(x,y)\,dy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy.$$

3.1.3 Index of a point relative to a closed path (Loop)

Definition 3.1.6 *Let $z_0 \in \mathbb{C}$ and $\gamma \subset \mathbb{C}$ be a closed (loop) that does not pass through z_0. The index of z_0 relative to γ is defined by*

$$Ind_\gamma(z_0) = \frac{1}{2i\pi}\oint_\gamma \frac{dz}{z-z_0}.$$

Proposition 3.1.4 *1) We have*

$$Ind_{-\gamma}(z_0) = -Ind_\gamma(z_0).$$

2) If γ_1 and γ_2 are two loops with the same origin that do not pass through z_0, then

$$Ind_{\gamma_1+\gamma_2}(z_0) = Ind_{\gamma_1}(z_0) + Ind_{\gamma_2}(z_0$$

Example 3.1.4 *Let $\gamma : [0, 2\pi] \to \Omega$, avec $\gamma(t) = a + re^{it}$. We have the following cases:*

- If $|z_0 - a| = r$ (i.e., $\gamma(t) = a + re^{it}$) $\Rightarrow \gamma$ does not pass by z_0 and $Ind_\gamma(z_0)$.

-For $r > 0$, we have for

$$\gamma(z) = \begin{cases} z \in \mathbb{C} : |z_0 - a| > r \\ z \in \mathbb{C} : |z_0 - a| < r \end{cases},$$

then

$$Ind_\gamma(z_0) = \begin{cases} 0, & \text{if } \gamma(z) = \{z : |z_0 - a| > r\} \\ 1, & \text{if } \gamma(z) = \{z : |z_0 - a| < r\} \end{cases}.$$

3.2 Cauchy Formula, the Generalization of the Cauchy Theorem

In 1825, the French mathematician Louis-Augustin Cauchy proved one of the most crucial theorems in complex analysis.

Proposition 3.2.1 *Let Ω be a simply connected open set, and γ be a closed curve (path) in Ω. For any holomorphic function $f : \Omega \to \mathbb{C}$, we have*

$$\int_\gamma f(z)\,dz = 0.$$

Preuve. *Let $f(z) = P(x,y) + iQ(x,y)$ and $dz = dx + idy$. According to Green's theorem,we have*

$$\int_\gamma f(z)\,dz = \int_\gamma P\,dx - Q\,dy + i\int_\gamma Q\,dx + P\,dy$$

By Green's theorem

$$\begin{aligned} & \int_\gamma f(z)\,dz \\ = \; & \iint_R \left(-\tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}\right) dx\,dy + i\iint_R \left(\tfrac{\partial P}{\partial x} - \tfrac{\partial Q}{\partial y}\right) dx\,dy \\ = \; & 0 \end{aligned}$$

The Cauchy-Riemann conditions are satisfied, since f is holomorphic, leading to the result.

■

Remark 3.2.1 *The Cauchy theorem is important because it frees us from the necessity of knowing the existence of a primitive of f over the set Ω. If f had a primitive in Ω, we have the following theorem.*

Theorem 3.2.1 *Let Ω be an open subset of $\mathbb{C}$ and f a continuous function on Ω. The following conditions are equivalent:*

1) f has a primitive in Ω,

2)For every closed path γ in Ω, we have $\int_\gamma f(z)\,dz = 0$.

Theorem 3.2.2 (Cauchy Formula) *Let Ω be a simply connected open set, and $\gamma : [a,b] \to \mathbb{C}$ be a closed curve in Ω. For any holomorphic (or analytic) function $f : \Omega \to \mathbb{C}$, and for any $z_0 \notin \gamma([a,b])$, we have*

$$f(z_0) \cdot Ind_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\,dz.$$

Preuve. *We have*

$$\frac{f(z)}{z-z_0} = \frac{f(z) - f(z_0)}{z - z_0} + \frac{f(z_0)}{z-z_0}.$$

Let

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0, \\ f'(z_0) & \text{if } z = z_0. \end{cases}$$

Since f is holomorphic in Ω, g is continuous and holomorphic in Ω, which is simply connected. Thus,

$$\int_\gamma g(z)\,dz = 0 \Rightarrow \int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\,dz = 0.$$

This gives us by Cauchy's Theorem

$$\begin{aligned} \int_\gamma \frac{f(z)}{z - z_0}\,dz &= f(z_0) \int_\gamma \frac{dz}{z - z_0} \\ &= 2\pi i \cdot f(z_0) \cdot Ind_\gamma(z_0). \end{aligned}$$

■

Theorem 3.2.3 (Cauchy's integral formula for derivatives) *Sous les mêmes hypothèses que le théorème précédent, on a*

$$f^{(n)}(z_0) \cdot Ind_\gamma(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

ou bien

$$\frac{2\pi i}{n!} f^{(n)}(z_0) \cdot Ind_\gamma(z_0) = \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

Corollary 3.2.1 *En particulier, si $\gamma = z_0 + C_r$ est un cercle inclus dans Ω ainsi que le disque, $z_0 + D_r$ qu'il délimite, la formule de Cauchy se simplifie en*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{z_0+C_r} \frac{f(z)}{(z-z_0)^{n+1}}dz.$$

Example 3.2.1 *1) Compute $\int_{|z|=1} \frac{e^z}{z-z_0}\,dz,\quad$ for all $z_0 \in \mathbb{C}$. Indeed,*

-If $|z_0| > 1$, the function $\frac{e^z}{z-z_0}$ is holomorphic inside the disk $|z| = 1$, and hence by Cauchy's Theorem, we have

$$\int_{|z|=1} \frac{e^z}{z-z_0}\,dz = 0.$$

- -If $|z_0| < 1$, define $f(z) = e^z$, which is holomorphic inside the disk $|z| = 1$, so by Cauchy's Integral Formula, we find

$$\int_{|z|=1} \frac{e^z}{z-z_0}\,dz = 2\pi i\, e^{z_0}.$$

2) Compute

$$\int_{|z|=2} \frac{z^2}{4z+1}\cdot\frac{1}{z+i}\,dz.$$

Let $f(z) = \dfrac{z^2}{4z+1}$, and $z_0 = i$. Note that f is analytic in $|z| \leq 2$ and z_0 is inside the disk $|z| \leq 2$. By Cauchy's Integral Formula, we obtain

$$\int_{|z|=2} \frac{z^2}{4z+1}\cdot\frac{1}{z+i}\,dz = 2\pi i\, f(i) = 2\pi i \cdot 4i = 8\pi i.$$

3) Compute

$$\int_{|z-2i|=4} \frac{z}{z^2+9}\,dz.$$

Note that $z^2 + 9 = (z-3i)(z+3i)$, and that only $z_0 = 3i$ is inside the contour $|z - 2i| = 4$. Let $f(z) = \frac{z}{z+3i}$. Using Cauchy's Integral Formula, we get:

$$\int_{|z-2i|=4} \frac{z}{z^2+9}\,dz = \int_{|z-2i|=4} \frac{f(z)}{z-3i}\,dz = 2\pi i\, f(3i) = 2\pi i \cdot \frac{3i}{6i} = \pi i.$$

4) Let C be the circle $|z| = 4$ traversed once in the counterclockwise direction. Let's evaluate the integral

$$\int_C \frac{\cos z}{z^2-6z+5}\,dz.$$

We simply write the integrand as

$$\frac{\cos z}{z^2-6z+5}=\frac{\cos z}{(z-5)(z-1)}=\frac{f(z)}{z-1},$$

where

$$f(z)=\frac{\cos z}{z-5}.$$

Observe that $f(z)$ is analytic on and inside C, and so, by Cauchy's Integral Formula:

$$\int_C \frac{\cos z}{z^2-6z+5}\,dz=\int_C \frac{f(z)}{z-1}\,dz=2\pi i\, f(1).$$

Now, we evaluate $f(1)$:

$$f(1)=\frac{\cos 1}{1-5}=\frac{\cos 1}{-4}.$$

Therefore, the integral becomes:

$$2\pi i\cdot\frac{\cos 1}{-4}=-\frac{\pi i}{2}\cos 1.$$

5) Let $I=\int_{|z|=1}\frac{e^z}{z^{2023}}\,dz$. We observe that the singularities are $z_0=0$ and $z_0=0$ inside C. By setting $f(z)=e^z$, it is holomorphic inside C. Using Cauchy's Integral Formula, we have

$$I=\int_{|z|=1}\frac{e^z}{z^{2025}}\,dz=\frac{2\pi i}{2024!}f^{(2024)}(0)=\frac{2\pi i}{2024!}.$$

Morera's Theorem

Theorem 3.2.4 *Suppose that f is continuous on a simply connected domain D and*

$$\int_\gamma f(z)\,dz=0$$

for every closed curve γ in D. Then, f is analytic on D.

Preuve. Since f is continuous, it admits a primitive

$$F(z)=\int_{z_0}^{z} f(s)\,ds \quad \text{for all} \quad z\in D,$$

where z_0 is a fixed point inside D. Therefore, F is analytic in D, and consequently, its derivative f is also analytic. ■

3.2.1 Cauchy Inequality

Theorem 3.2.5 *Let f be a holomorphic function on the disk $D(z_0; R)$, where $R > 0$. Then $f(z)$ can be developed in a power series on this disk. Moreover, we have the inequality:*

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}, \quad \text{for all } 0 < r < R,$$

where $M = \sup_{|z|=r} |f(z)|$.

Preuve. Let $\gamma : [0, 2\pi] \to \mathbb{C}$, where $\gamma(\theta) = z_0 + re^{i\theta}$. We have

$$a_n = \frac{1}{2\pi i} \int \frac{f(z)}{(z - z_0)^{n+1}}\, dz = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{r^n e^{in\theta}}\, d\theta.$$

Therefore,

$$|r^n a_n| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta \leq M \cdot \frac{1}{2\pi} \int_0^{2\pi} d\theta = M.$$

Thus,

$$|a_n| \leq \frac{M}{r^n}.$$

Since $f^{(n)}(z_0) = n! a_n$, we obtain

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}, \quad \text{for all } 0 < r < R.$$

■

3.2.2 Liouville's Theorem

Theorem 3.2.6 *Every bounded analytic (entire) function is constant. Equivalently: A nonconstant analytic (entire) function cannot be bounded.(i.e., $\exists M \in \mathbb{R}^* : |f(z)| \leq M$, then f is constant).*

Preuve. Let $f(z)$ be an entire and bounded function, i.e., $|f(z)| \leq M$ for all $z \in \mathbb{C}$. From the Cauchy inequality, for any $z_0 \in \mathbb{C}$, we have

$$|f'(z_0)| \leq \frac{M}{r} \to 0 \quad \text{as} \quad r \to \infty.$$

Thus, $|f'(z_0)| = 0$ for all $z_0 \in \mathbb{C}$, which shows that f is constant. ■

Example 3.2.2 *Let f be a holomorphic function on the complex plane $\mathbb{C}$, and suppose that*

$$\frac{f(z)}{z} \leq M \quad on \quad \mathbb{C}, \quad with \quad M > 0.$$

Show that $f(z)$ is a polynomial of degree ≤ 1 on $\mathbb{C}$. Define $g(z) = \frac{f(z)}{z}$ for $z \neq 0$ and $g(0) = 0$. This function satisfies the conditions of Liouville's Theorem, so it is constant on $\mathbb{C}$. This gives

$$\frac{f(z)}{z} = C, \quad so \quad f(z) = Cz,$$

which shows that $f(z)$ is a polynomial of degree ≤ 1 on $\mathbb{C}$.

Corollary 3.2.2 *If an entire function satisfies $|f(z)| \leq M|z|^n$ as $|z| \to \infty$, then the function f is a polynomial of degree at most n.*

3.3 Maximun principle

Theorem 3.3.1 (The Maximum Modulus Principle) *Let $D \subset \mathbb{C}$ be a domain (open and connected), and let $f : D \to \mathbb{C}$ be analytic. If there exists $a \in D$ such that*

$$|f(a)| \geq |f(z)| \text{ for all } z \in D,$$

then f is constant. In other words, $|f(z)|$ cannot have a maximum in D unless f is constant.

Corollary 3.3.1 (The Maximum Modulus Principle) *Let $D \subset \mathbb{C}$ be a bounded domain (open and connected), and let $f : D \to \mathbb{C}$ be continuous and analytic on D. Then,*

$$\max_{z \in D} |f(z)| = \max_{z \in \partial D} |f(z)|.$$

In other words, $|f|$ attains its maximum on the boundary ∂D and nowhere else.

Preuve. Indeed, since D is closed and bounded in $\mathbb{C}$, it is compact. The continuous function $|f(z)|$ therefore attains its maximum at a point z_0, which cannot lie inside D by the maximum modulus principle. Thus, the maximum of $|f(z)|$ is attained on the boundary ∂D. ■

Example 3.3.1 *1) Find the maximum of $f(z) = 2z + 5i$ on $|z| \leq 2$. We have*

$$|2z + 5i|^2 = (2z + 5i)(2z - 5i) = 4|z|^2 + 20Im(z) + 25.$$

By the maximum modulus principle, we have

$$\begin{aligned}\max_{|z|\leq 2} |2z + 5i| &= \max_{|z|=2} |2z + 5i| \\ &= \max_{|z|=2} \sqrt{4|z|^2 + 20Im(z) + 25} \\ &= \sqrt{4 \cdot 2^2 + 20 \cdot 2 + 25} = 9.\end{aligned}$$

2) Find the maximum and minimum of $f(z) = z^2 - 2$ on $|z| \leq 1$. By the maximum modulus principle, we have:

$$\max_{|z|\leq 1} |z^2 - 1| = \max_{|z|=1} |z^2 - 1| = \max_{t\in[0,2\pi]} |e^{2it} - 1| = |1 - 2| = 3$$

and

$$\min_{|z|\leq 1} |z^2 - 1| = \min_{|z|=1} |z^2 - 1| = \min_{t\in[0,2\pi]} |e^{2it} - 1| = |1 - 1| = 1.$$

Proposition 3.3.1 *Let U be an open subset of $\mathbb{C}$, and let f be a continuous function on U. The following conditions are equivalent:*

(i) For every disk $D \subseteq (a, r)$ contained in U, we have

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it})\, dt.$$

(ii) For every disk $D \subseteq (a, r)$ contained in U, we have

$$f(a) = \frac{1}{\pi r^2} \int \int_{D'(a,r)} f(x + iy)\, dx\, dy.$$

If these conditions are satisfied, we say that f possesses the mean property in U.

Preuve. Using polar coordinates, we get

$$\frac{1}{\pi r^2} \int\limits_{D(a,r)} f(x + iy)\, dx\, dy = \frac{1}{\pi r^2} \int_0^r \left(\int_0^{2\pi} f(a + \rho e^{it})\, dt \right) \rho\, d\rho.$$

The implication $(i) \Rightarrow (ii)$ is immediate. Now, assume (ii) holds. Let $R > 0$ be such that $D(a, R) \subset U$. For $0 \leq r < R$, we have:

$$f(a) = \frac{1}{2\pi} \int_0^r \left(\int_0^{2\pi} f(a + \rho e^{it})\, dt \right) \rho\, d\rho.$$

Taking the derivative of both sides with respect to r, we obtain (i). ■

Proposition 3.3.2 *Let U be an open subset of $\mathbb{C}$ and let f be a holomorphic function on U. Then, f possesses the mean property in U.*

Preuve. If $D(a,r) \subset U$, there exists $R > r$ such that $D(a,R) \subset U$. Let C be the circle $C(a,r)$ traversed in the counterclockwise direction. By applying Cauchy's formula to $D(a,R)$, we get:

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt.$$

This proves the assertion. ■

Remark 3.3.1 *Let U be an open subset of $\mathbb{C}$ and let f be a function on U. We say that f has a local maximum at $a \in U$ if there exists a neighborhood V of a in U such that $|f(z)| \leq |f(a)|$ for all $z \in V$. We say that f satisfies the maximum principle in U if it is constant in any neighborhood of every point $a \in U$ where it has a local maximum.*

Assume that f satisfies the maximum principle in U, and that U is connected. It is immediate that the set of points in U where f has a local maximum is both open and closed in U. Therefore, if f has a local maximum at a point of U, then f must be constant.

Theorem 3.3.2 *Let U be an open subset of $\mathbb{C}$ and let f be a continuous function on U that satisfies the mean property in U. Then, f satisfies the maximum principle in U.*

3.4 Rouché's Theorem

Theorem 3.4.1 *Let f and g be two holomorphic functions on a domain Ω, and let the closed disk $D(a,r)$ be contained in Ω, with $|g(z)| < |f(z)|$ on the circle $C(a,r)$. Then f and $f+g$ have the same number of zeros in the open disk $D(a,r)$, counting multiplicities.*

Example 3.4.1 *1. Calculate the number of zeros of $z^8 - 4z^5 + z^2 - 1 = 0$ in $D = \{z \in \mathbb{C} : |z| < 1\}$. Let $f(z) = 4z^5$ and $g(z) = z^8 + z^2 - 1$ on the circle $C(0,1)$, i.e., $|z| = 1$. Then,*

$$|f(z)| = 4 \quad \textit{and} \quad |g(z)| = |z^8 + z^2 - 1| \leq z^8 + z^2 + |1| = 3.$$

Thus, $|g(z)| < |f(z)|$ on $C(0,1)$. Therefore, $f(z) = 4z^5$ and $f(z) + g(z) = z^8 - 4z^5 + z^2 - 1$ have the same number of zeros in $D(0,1)$. Since $z_0 = 0$ is a root of $f(z) = 4z^5$ of order 5, *it follows that* $z^8 - 4z^5 + z^2 - 1$ *has 5 roots in* $D(0,1)$.

2. Calculate the number of zeros of $z^4 - 5z + 1 = 0$ *in* $D = \{z \in \mathbb{C} : |z| < \frac{1}{4}\}$. *Let* $f(z) = 5z$ *and* $g(z) = z^4 + 1$ *on the circle* $C(0, \frac{1}{4})$, *i.e.,* $|z| = \frac{1}{4}$. *Then,*

$$|f(z)| = \frac{5}{4} \quad \textit{and} \quad |g(z)| = |z^4 + 1| \leq z^4 + |1| = \frac{1}{4^4} + 1 = \frac{1}{256} + 1 \approx 1.0039.$$

Thus, $|g(z)| < |f(z)|$ *on* $C(0, \frac{1}{4})$. *Therefore,* $f(z) = 5z$ *and* $f(z) + g(z) = z^4 - 5z + 1$ *have the same number of zeros in* $D(0, \frac{1}{4})$. *Since* $z_0 = 0$ *is a simple root of* $f(z) = 5z$, *it follows that* $z^4 - 5z + 1$ *has one root in* $D(0, \frac{1}{4})$.

3. Calculate the number of zeros of $3z^n e^z = 0$ *in* $D = \{z \in \mathbb{C} : |z| < 1\}$. *Let* $f(z) = 3z^n$ *and* $g(z) = e^z$ *on the circle* $C(0,1)$, *i.e.,* $|z| = 1$. *Then,*

$$|f(z)| = 3 \quad \textit{and} \quad |g(z)| = e^{Re(z)} \leq e = 3.$$

Thus, $|g(z)| < |f(z)|$ *on* $C(0,1)$. *Therefore,* $f(z) = 3z^n$ *and* $f(z) + g(z) = 3z^n e^z$ *have the same number of zeros in* $D(0,1)$. *Since* $z_0 = 0$ *is a root of* $f(z) = 3z^n$ *of order* n, *it follows that* $3z^n e^z = 0$ *has* n *roots in* $D(0,1)$.

Chapter 4

Residue Theorem

Calculation of the Residue

Laurent Series

Definition 4.0.1 (annulus) *Let $a \in \mathbb{C}$ and $R_1, R_2 \in [0, +\infty[$ such that $R_1 < R_2$. We denote*

$$C(z_0, R_1, R_2) = \{z \in \mathbb{C}; R_1 < |z - z_0| < R_2\}.$$

We say that $C(z_0, R_1, R_2)$ (respectively $C(z_0, R_1, R_2)$) is the open (respectively closed) annulus associated with z_0, R_1, R_2.

Definition 4.0.2 *A series of functions of the form*

$$\sum \lambda_n (z - z_0)^n$$

is called a Laurent series.

Theorem 4.0.2 (Laurent's theorem) *Let $\Omega \subset \mathbb{C}$ be a domain containing the annulus $C(z_0, R_1, R_2)$ and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then for $R_1 < |z - z_0| < R_2$,*

$$f(z) = \sum \lambda_n (z - z_0)^n = \sum \frac{b_n}{(z - z_0)^n} + \sum a_n (z - z_0)^n,$$

where

$$\lambda_n = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi,$$

and C_ρ is the circle centered at z_0 with radius ρ, where $R_1 < \rho < R_2$, traversed in the positive direction.

Definition 4.0.3 *1) The part $\sum a_n(z-z_0)^n$ is called the analytic or regular part of the Laurent series.*
2) The part $\sum \frac{b_n}{(z-z_0)^n}$ is called the principal or singular part of the Laurent series.

Practical Methods of Obtaining Laurent Series

In practice, one tries to avoid computing the coefficients λ_n of the Laurent series by means of integration. Other practical methods are used instead. The background for these methods is the geometric series:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1. \tag{4.4.15}$$

Replacing z with $\frac{1}{z}$ in 4.4.15, we obtain

$$\frac{1}{1-\left(\frac{1}{z}\right)} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n, \quad |z| > 1,$$

so that

$$\frac{1}{1-z} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}, \quad |z| > 1.$$

Differentiating 4.4.15 and 4, we obtain

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1}, \quad |z| < 1,$$

and

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^{n+2}, \quad |z| > 1.\textbf{??}$$

Similarly, one can get Laurent series expansions for $\log(1-\frac{1}{z})$, $e^{1/z}$, $\sin(1/z)$, and $\cos(1/z)$ by replacing z with $\frac{1}{z}$ in 4.4.15 and **??**.

Example 4.0.2 *Find the Laurent series of the function*

$$f(z) = \frac{1}{(z-2)(z-3)}$$

(a) in the annulus $2 < |z| < 3$,
(b) in the region $3 < |z| < \infty$.

Solution:

In case (a), we first expand $f(z)$ into partial fractions:

$$f(z) = \frac{1}{z-3} - \frac{1}{z-2}. \tag{4.4.19}$$

Hence

$$\frac{1}{z-3} = -\frac{1}{3}\frac{1}{1-(z/3)} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n, \quad |z| < 3.$$

This is what we need. Hence

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-(2/z)} = \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}, \quad |z| > 2.$$

Then, we obtain the desired formula:

$$\begin{aligned} f(z) &= -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^{n+1} \\ &= -\sum_{n=0}^{\infty}\left(\frac{2^n}{z^{n+1}} + \frac{z^n}{3^{n+1}}\right), \quad 2 < |z| < 3. \end{aligned}$$

Example 4.0.3 *Expanding* $f(z) = \frac{1}{z(z-1)}$ *into a Laurent series valid in the annulus* $1 < |z-2| < 2$.

Solution: We can write the function as a sum of partial fractions:

$$f(z) = \frac{1}{z} + \frac{1}{z-1} = f_1(z) + f_2(z)$$

In the domain $|z-2| < 2$, we have:

$$f_1(z) = \frac{1}{z} = \frac{1}{2}\frac{1}{1+\frac{z-2}{2}} = \frac{1}{2}\left(\frac{1}{1+\frac{z-2}{2}}\right)$$

Expanding the geometric series, we get:

$$f_1(z) = \frac{1}{2}\left(1 + \frac{z-2}{2} + \frac{1}{4}(z-2)^2 + \frac{1}{8}(z-2)^3 + \frac{1}{16}(z-2)^4 + \dots\right)$$

So:

$$f_1(z) = \frac{1}{2} + \frac{1}{4}(z-2) + \frac{1}{8}(z-2)^2 + \frac{1}{16}(z-2)^3 + \dots$$

Now, in the domain $1 < |z-2|$, we have:

$$f_2(z) = \frac{1}{z-1} = \frac{1}{z-2}\frac{1}{1+\frac{1}{z-2}} = \frac{1}{z-2}\left(\frac{1}{1+\frac{1}{z-2}}\right)$$

Expanding the geometric series for this term, we get:

$$f_2(z) = \frac{1}{z-2}\left(\frac{1}{z-2} + \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} + \dots\right)$$

Thus, for $1 < |z-2| < 2$, we have the following Laurent series:

$$f(z) = \frac{1}{(z-2)^3} + \frac{1}{(z-2)^2} + \frac{1}{z-2} + \frac{1}{2} + \frac{1}{4}(z-2) + \frac{1}{8}(z-2)^2 + \dots$$

4.0.1 Classification of Singular Points

Definition 4.0.4 *A point z_0 is called a singularpoint of $f(z)$ if f is not defined at z_0. A singular point z_0 of $f(z)$ is called an isolated singular point if there exists $\delta > 0$ such that $f(z)$ is analytic in the punctured disk $0 < |z - z_0| < \delta$.*

Definition 4.0.5 *Let $z = z_0$ be an isolated singularity of f. We say that the singularity is:*
1) ***Removable singularity*** *(also called artificial or false) if the principal part of the Laurent series of f is zero, i.e., all the coefficients $b_n = 0$ for $n = 1, 2, \dots,$ or if a uniform function f is not defined at $z = z_0$, but $\lim_{z\to z_0} f(z)$ exists.*
2) ***Pole of order*** *m if the coefficients b_n are not all zero, i.e., if there exists a $b_n \neq 0$ such that $b_n = 0$ for all $n > m$. Then z_0 is a pole of order m, and for all z in the punctured disk $0 < |z - z_0| < \delta$:*

$$f(z) = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \dots$$

If $m = 1$, we say that z_0 is a simple pole.
*3)****Essential singularity*** *if the principal part contains an infinite number of non-zero terms b_n.*

Examples

1-Let $f(z) = \frac{\sin z}{z}$, with $z_0 = 0$ being a singularity of f. Expanding $\frac{\sin z}{z}$ as a series:

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots$$

The coefficients b_n are all zero, so $z_0 = 0$ is a removable singularity.2-Let $f(z) = \exp\left(\frac{1}{z}\right)$, with $z_0 = 0$ being a singularity of f. Expanding $\exp\left(\frac{1}{z}\right)$ as a series:

$$\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n}.$$

Hence, there are an infinite number of non-zero b_n, so $z_0 = 0$ is an essential singularity.

2-Let $f(z) = \frac{\sin z}{z^2}$, with $z_0 = 0$ being a singularity of f. Expanding $\frac{\sin z}{z^2}$ as a series:

$$f(z) = \frac{\sin z}{z^2} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} + \dots$$

Thus, $z_0 = 0$ is a simple pole.

Residues

Definition 4.0.6 *Let $f : D \to \mathbb{C}$ be an analytic function at the point z_0, where z_0 is an isolated singularity of f, and $0 < |z - z_0| < r$ is a punctured disk that does not contain singular points of f. We have*

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The coefficient b_1 of a Laurent series in a neighborhood of an isolated singular point z_0 is called the residue of the analytic functione f and noted by $Res(f, z_0)$.

Proposition 4.0.1 *We have*

$$Res\,(f(z), z_0) = b_1 = \frac{1}{2\pi i} \int_C f(\zeta)\, d\zeta.$$

and C is any closed path that contains the only singular point z_0 inside and is taken in the positive direction

Remark 4.0.1 *If z_0 is a removable singularity of $f(z)$, then the Laurent series does not contain negative powers of $z - z_0$, and therefore $b_1 = 0$.*

Proposition 4.0.2 ***(Residue Calculation).*** *Let $z = z_0$ be an isolated singularity of f, then*

(1) If $z = z_0$ is a simple pole of f, then

$$Res(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z). \tag{1}$$

(2) If $f(z)=\frac{g(z)}{h(z)}$ where $g(z_0)\neq 0$ and $h(z_0)=0$, but $h'(z_0)\neq 0$, then

$$Res(f;z_0)=\frac{g(z_0)}{h'(z_0)}. \tag{2}$$

(3) If $z=z_0$ is a pole of order n of f, then

$$Res(f;z_0)=\frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right]. \tag{3}$$

Example 4.0.4 *Let $f(z)=\frac{1}{(z-1)^2(z-3)}$ and we want to calculate $Res(f;3)$ and $Res(f;1)$.*

Solution.

Since $z=3$ is a simple pole, then

$$\operatorname{Res}(f;3)=\lim_{z\to 3}(z-3)f(z)=\lim_{z\to 3}\frac{1}{(z-1)^2}=\frac{1}{4}.$$

We have

$$f(z)=\frac{1}{(z-1)^2(z-3)}=\frac{g(z)}{h(z)},$$

and

$$\operatorname{Res}(f;3)=\frac{g(3)}{h'(3)}=\frac{1}{4}\cdot 1=\frac{1}{4}.$$

Since $z=1$ is a pole of order 2, then

$$\begin{aligned}\operatorname{Res}(f;1) &= \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2 f(z)\right]\\ &= \lim_{z\to 1}\frac{d}{dz}\left(\frac{1}{z-3}\right)=\lim_{z\to 1}\frac{1}{(z-3)^2}=\frac{1}{4}.\end{aligned}$$

Computing the residue at an essential singularity

There are no general formulae for computing residues at essential singularities. One may compute the integral 4.0.1 or use some special ways of getting the Laurent expansion and determining the coefficient b_1. For example, if $f(z)$ is an even function with respect to $z-z_0$, then the Laurent series contains only even powers of $z-z_0$, and hence

$$\operatorname{Res}(f(z),z_0)=0.$$

Example 4.0.5 *Find the residue of the function*

$$f(z) = \frac{e^{-\frac{1}{z^4}}}{z^2+1}$$

at $z = 0$.

Solution: In this case, $z = 0$ is an essential singularity. Since $f(z)$ is an even function, we conclude that

$$\operatorname{Res}\left(\frac{e^{-\frac{1}{z^4}}}{z^2+1}, z = 0\right) = 0.$$

Example 4.0.6 *Find the residue of*

$$f(z) = z^2 e^{\frac{1}{z}}$$

at $z = 0$.

Solution: The point $z = 0$ is an essential singularity. We expand $f(z)$ in a Laurent series:

$$f(z) = z^2\left(1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right).$$

It follows from this expansion that the coefficient c_{-1} is

$$b_1 = c_{-1} = \operatorname{Res}\left(z^2 e^{\frac{1}{z}}, z = 0\right) = \frac{1}{3!} = \frac{1}{6}.$$

Example 4.0.7 *Find the residue of*

$$f(z) = \frac{1}{1-z} e^{\frac{1}{z}}$$

at $z = 0$.

Solution: The point $z = 0$ is an essential singularity. We expand $\frac{1}{1-z}$ and $e^{\frac{1}{z}}$ in Taylor and Laurent series, respectively:

$$\frac{1}{1-z} e^{\frac{1}{z}} = \left(1 + z + z^2 + \cdots\right)\left(1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right).$$

To determine the coefficient at z^{-1}, one multiplies terms such as 1 by $\frac{1}{z}$, z by $\frac{1}{2!z^2}$, z^2 by $\frac{1}{3!z^3}$, and so on, and adds the results. Thus,

$$c_{-1} = \text{Res}\left(\frac{1}{1-z}e^{\frac{1}{z}}, z=0\right) = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = e - 1.$$

Theorem 4.0.3 (Residue Theorem) *Suppose the function $f(z)$ is analytic in a simply connected closed region D bounded by the path C taken in the positive direction, except for a finite number of isolated singularities z_k, $k = 1, 2, \ldots, N$, located inside D. Then*

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{N} Res\,(f, z_k)\,.$$

Preuve. We surround each singular point z_k, $k = 1, 2, \ldots, N$, by a sufficiently small closed path γ_k, containing only the singular point z_k (see Fig. 5.2). Then $f(z)$ is analytic in the region D_0 bounded by the paths $C, \gamma_1, \gamma_2, \ldots, \gamma_N$. Therefore, by Cauchy's Theorem 3.3.4 for multiply connected domains, we have

$$\int_C f(z)\,dz = \sum_{k=1}^{N} \int_{\gamma_k} f(z)\,dz.$$

where the paths C and γ_k are taken counterclockwise. By the definition of the residue at z_k, we have

$$\int_{\gamma_k} f(z)\,dz = 2\pi i\, Res\,(f, z_k)\,.$$

Substituting equation **??** into equation **??** yields

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{N} Res\,(f, z_k)\,,$$

which is the desired result, as stated in equation **??**. ■

Theorem 4.0.4 (Argument Principle) *Let f be a holomorphic function inside a closed contour C, except for P poles. Suppose it has Z zeros (counted with multiplicity). Then:*

$$Z - P = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\,dz,$$

where the contour C is traversed in the positive (counterclockwise) direction.

Example 4.0.8 *Compute*

$$\int_{|z|=8} \frac{e^z}{e^z-1}\,dz.$$

We note that $e^z - 1$ has 3 zeros and 0 poles inside the circle $|z| = 8$. Therefore,

$$\int_{|z|=8} \frac{e^z}{e^z-1}\,dz = 2\pi i(Z-P) = 2\pi i(3-0) = 6\pi i.$$

4.0.2 Residue of an Analytic Function at Infinity

Definition 4.0.7 *The function $f(z)$ is said to be analytic at $z = \infty$ if the function*

$$f^*(z) = f\left(\frac{1}{z}\right)$$

is analytic at $z = 0$. For example, $f(z) = \sin\left(\frac{1}{z}\right)$ is analytic at $z = \infty$ since

$$f^*(z) = \sin z$$

is analytic at $z = 0$.

Definition 4.0.8 *We call the residue of f at infinity, and denote it by $\operatorname{Res}(f;\infty)$, the residue at $\omega = 0$ of the function*

$$-\frac{1}{\omega^2} f\left(\frac{1}{\omega}\right).$$

If

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{z^n} + \sum_{n=0}^{\infty} a_n z^n$$

is the Laurent expansion of f in the domain $\{z \in \mathbb{C} : |z| > r\}$, then

$$\operatorname{Res}(f;\infty) = -b_1,$$

where b_1 is the coefficient of $\frac{1}{z}$.

Therefore,

$$\int_\Gamma f(z)\,dz = -2\pi i \operatorname{Res}(f;\infty),$$

where f is analytic in the domain $\{z \in \mathbb{C} : |z| > r\}$, and Γ is a positively oriented closed circle centered at the origin with radius $\rho > r$.

Theorem 4.0.5 *If an analytic function $f(z)$ has a finite number of singularities z_k, for $k = 1, 2, \ldots, N$, in the complex plane, then the sum of all the residues of $f(z)$, including the residue at $z = \infty$, is equal to zero:*

$$Rés(f, \infty) + \sum_{k=0}^{n} (f, z_k) = 0.$$

Let $f(z) = \dfrac{z}{z+2}$. The point $z = -2$ is a singularity of f.

Method 1

$$\lim_{z \to -2} (z+2) f(z) = -2,$$

so $z_0 = -2$ is a simple pole, and

$$\operatorname{Res}(f; -2) = -2 \quad \Rightarrow \quad \operatorname{Res}(f; \infty) = -\operatorname{Res}(f; -2) = 2.$$

Method 2

$$\operatorname{Res}(f; \infty) = \operatorname{Res}\left(-\frac{1}{\omega^2} f\left(\frac{1}{\omega}\right), 0 \right) = \operatorname{Res}\left(\frac{-1}{\omega^2 + 2\omega^3}, 0 \right).$$

Let $g(\omega) = \frac{-1}{\omega^2 + 2\omega^3}$. Then

$$\lim_{\omega \to 0} \omega^2 g(\omega) = -1,$$

so $\omega = 0$ is a double pole and

$$\operatorname{Res}(g; 0) = \frac{1}{1!} \lim_{\omega \to 0} \left[\frac{d}{d\omega} \left(\omega^2 g(\omega) \right) \right] = 2.$$

Chapter 5

Application of the Residue Theorem to Integral Calculus

The evaluation of definite integrals can often be performed using the residue theorem applied to a suitable function and contour, the choice of which may require considerable ingenuity. The following types of integrals are frequently encountered in practice.

1. Let f be a continuous function on $\mathbb{C}$ such that

$$\lim_{|z|\to+\infty} zf(z) = 0,$$

   then we have

$$\lim_{r\to+\infty} \int_{\gamma_r} f(z)\, dz = 0,$$

   where $\gamma_r(t) = re^{it}$ is defined on the interval $[\theta_1, \theta_2]$ with $\theta_i \in [0, 2\pi]$ and $r \in \mathbb{R}_+^*$.

2. Let f be a continuous function on $\mathbb{C}$ such that

$$\lim_{z\to 0} zf(z) = 0,$$

   then we have

$$\lim_{r\to 0} \int_{\gamma_r} f(z)\, dz = 0,$$

   where $\gamma_r(t) = re^{it}$ is defined on the interval $[\theta_1, \theta_2]$ with $\theta_i \in [0, 2\pi]$ and $r \in \mathbb{R}_+^*$.

Integrals of Real Rational Functions

Let us consider ff a real rational function with no real poles, whose integral we wish to compute.

$$I = \int_{-\infty}^{+\infty} f(x)\,dx$$

where

(1) $f(x) = \dfrac{P(x)}{Q(x)}$, with P and Q polynomials.

(2) $\lim_{|x|\to\infty} xf(x) = 0$, i.e., $\deg P \le \deg Q + 2$.

(3) $Q(x) \neq 0$ on $\mathbb{R}$.

We consider the contour integral over $C = [-r, r] \cup \gamma_r$, where γ_r is the upper semicircle $\{z \in \mathbb{C} : |z| = r, \mathrm{Im}(z) \ge 0\}$.

Then:

$$\int_C f(z)\,dz = \int_{-r}^{r} f(z)\,dz + \int_{\gamma_r} f(z)\,dz = 2\pi i \sum_{\mathrm{Im}(z_k)>0} \mathrm{Res}(f(z), z_k).$$

As $r \to \infty$, $\int_{\gamma_r} f(z)\,dz \to 0$, so:

$$\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \sum_{\mathrm{Im}(z_k)>0} \mathrm{Res}(f(z), z_k).$$

- If $f(x)$ is even, then:

$$\int_0^{+\infty} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} f(x)\,dx.$$

(2) If we choose the lower half-disk, the formula becomes:

$$\int_{-\infty}^{+\infty} f(x)\,dx = -2\pi i \sum_{\mathrm{Im}(z_k)<0} \mathrm{Res}(f(z), z_k).$$

Example 5.0.9 *1) Evaluate:*

$$J = \int_0^{+\infty} \frac{dx}{1+x^2}.$$

The function $f(x) = \frac{1}{1+x^2}$ satisfies the previous conditions. The singularities are $z_1 = i$, $z_2 = -i$. Only z_1 lies in the upper half-plane:

$$\mathrm{Res}\left(\frac{1}{1+z^2}, i\right) = \frac{1}{2i}.$$

Hence,

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = 2\pi i \cdot \frac{1}{2i} = \pi, \quad J = \frac{\pi}{2}.$$

2) Compute the integral:

$$I = \int_{-\infty}^{+\infty} \frac{dx}{1+x^4}$$

Consider the function $f(z) = \frac{1}{1+z^4}$, *which is holomorphic on* $\mathbb{C} \setminus \{z_k\}$, *where the poles are given by:*

$$z_1 = e^{i\pi/4}, \quad z_2 = e^{3i\pi/4}, \quad z_3 = e^{5i\pi/4}, \quad z_4 = e^{7i\pi/4}$$

The poles z_1 *and* z_2 *lie in the upper half-plane. These are simple poles.*

We compute the residues:

$$\operatorname{Res}(f, z_k) = \lim_{z \to z_k} (z - z_k) f(z) = \frac{1}{4z_k^3}$$

By the residue theorem:

$$I = \int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \sum_{\operatorname{Im}(z_k)>0} \operatorname{Res}(f, z_k) = 2\pi i \left(\frac{1}{4z_1^3} + \frac{1}{4z_2^3} \right) = \frac{\pi}{\sqrt{2}}$$

3) Using the residue theorem, compute:

$$I = \int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2}$$

Let $f(z) = \frac{1}{(z^2+1)^2}$. The singular points are $z_0 = i$ and $z_0 = -i$. Both are double poles since:

$$\lim_{z \to i} (z-i)^2 f(z) = -\frac{1}{4}, \quad \lim_{z \to -i} (z+i)^2 f(z) = -\frac{1}{4}$$

Using the formula for the residue at a double pole:

$$\operatorname{Res}(f, i) = \frac{1}{1!} \lim_{z \to i} \frac{d}{dz} \left((z-i)^2 f(z) \right) = \frac{1}{4i}$$

$$\operatorname{Res}(f, -i) = \frac{1}{1!} \lim_{z \to -i} \frac{d}{dz} \left((z+i)^2 f(z) \right) = \frac{i}{4}$$

Applying the residue theorem:

$$I = \int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2} = 2\pi i \cdot \operatorname{Res}(f, i) = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2}$$

Trigonometric Rational Fraction Integrals

Consider the integral of the form:

$$I = \int_0^{2\pi} F(\cos t, \sin t)\, dt$$

where $F(\cos t, \sin t)$ is a rational function of $\cos t$ and $\sin t$; that is, $F(x, y) = \frac{P(x,y)}{Q(x,y)}$, with $Q(x, y) \neq 0$ on $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Set $z = e^{it}$, then:

$$\sin t = \frac{z - z^{-1}}{2i}, \quad \cos t = \frac{z + z^{-1}}{2}, \quad dt = \frac{dz}{iz}.$$

The integral becomes:

$$I = \int_{|z|=1} F\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) \frac{dz}{iz}.$$

By applying the residue theorem:

$$I = 2\pi i \sum_{|z_k|<1} \operatorname{Res}\left(\frac{1}{iz} F\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right), z_k\right).$$

Example 5.0.10 *1) Evaluate:*

$$I = \int_0^{2\pi} \frac{dt}{(5 - 3\sin t)^2}.$$

Using the substitution $z = e^{it}$, we find:

$$I = \int_{|z|=1} \frac{4iz}{(3z^2 - 10iz - 3)^2}\, dz = 2\pi i \sum_{|z_k|<1} \operatorname{Res}\left(\frac{4iz}{(3z^2 - 10iz - 3)^2}, z_k\right).$$

The singularities are $z_1 = 3i$, $z_2 = \frac{i}{3}$, with $|z_1| > 1$, $|z_2| < 1$. Since z_2 is a pole of order two:

$$\operatorname{Res}\left(\frac{4iz}{(3z^2 - 10iz - 3)^2}, z_2\right) = -\frac{i}{6},$$

which gives $I = \frac{5\pi}{3}$.

2) Evaluate:

$$I = \int_0^{2\pi} \frac{dt}{2 + \cos t}.$$

Let $z = e^{it}$, then:

$$I = \int_{|z|=1} \frac{dz}{i(z^2 + 4z + 1)}.$$

The singularities of $F(z) = \frac{1}{i(z^2+4z+1)}$ are $z_1 = -2 - \sqrt{3}$, $z_2 = -2 + \sqrt{3}$. Since both lie outside the unit circle, $I = 0$.

Example 5.0.11 *Evaluate*

$$I = \int_0^{2\pi} \frac{dx}{2 + \sin x}$$

Note: $2+\sin x$ *does not vanish on* $\mathbb{R}$. *We use the substitution* $z = e^{ix}$, *which implies* $dx = \frac{dz}{iz}$ *and* $\sin x = \frac{1}{2i}(z - \frac{1}{z})$. *Using this, the integral becomes:*

$$I = \int_{C(0,1)} \frac{2\,dz}{z^2 + 4iz - 1}.$$

We solve

$$z^2 + 4iz - 1 = 0 \Rightarrow z = z_1 = (\sqrt{3} - 2)i, \quad z = z_2 = -(\sqrt{3} + 2)i$$

Then

$$|z_1| < 1, \quad |z_2| > 1$$

So only z_1 *lies inside the unit circle. Therefore, by the Cauchy residue theorem:*

$$I = \int_0^{2\pi} \frac{dx}{2 + \sin x} = 2\pi i \cdot Res(f, z_1), \quad \text{where } f(z) = \frac{2}{z^2 + 4iz - 1}$$

We compute:

$$Res(f, z_1) = \lim_{z \to z_1} (z - z_1) f(z) = \frac{2}{2z_1 + 4i} = \frac{1}{\sqrt{3}i}$$

Hence:

$$I = 2\pi i \cdot \frac{1}{\sqrt{3}i} = \frac{2\pi}{\sqrt{3}}$$

2) Evaluate

$$I = \int_{-\infty}^{+\infty} \frac{dx}{5 + \cos x}$$

We use the substitution $z = e^{ix}$, *then:*

$$\cos x = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad dx = \frac{dz}{iz}$$

Thus:

$$I = \int_{-\infty}^{+\infty} \frac{dx}{5 + \cos x} = \frac{2}{i} \int_{|z|=1} \frac{dz}{z^2 + 10z + 1}$$

Let:

$$g(z) = \frac{1}{z^2 + 10z + 1}$$

Solve the quadratic:

$$z^2 + 10z + 1 = 0 \Rightarrow z_1 = -5 + 2\sqrt{6}, \quad z_2 = -5 - 2\sqrt{6}$$

Then:

$$|z_1| < 1, \quad |z_2| > 1$$

So z_1 lies inside the unit circle. Since the poles are simple, we compute:

$$Res(g, z_1) = \lim_{z \to z_1} (z - z_1) g(z) = \frac{1}{4\sqrt{6}}$$

Then:

$$I = \frac{2}{i} \cdot 2\pi i \cdot Res(g, z_1) = \frac{4\pi}{\sqrt{6}}$$

Integrals of Functions with $e^{i\beta x}$ as Factor

Consider:

$$I = \int_{-\infty}^{+\infty} f(x) \cos(\beta x)\, dx \quad \text{or} \quad \int_{-\infty}^{+\infty} f(x) \sin(\beta x)\, dx.$$

Let $f(z)$ be a rational function analytic in $\{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0\}$ except for a finite number of poles, and $\lim_{|z| \to \infty} f(z) = 0$. Then over $C = [-r, r] \cup \gamma_r$, we have:

$$\int_C f(z) e^{i\beta z}\, dz = \int_{-r}^{r} f(z) e^{i\beta z}\, dz + \int_{\gamma_r} f(z) e^{i\beta z}\, dz = 2\pi i \sum_{\operatorname{Im}(z_k) > 0} \operatorname{Res}(f(z) e^{i\beta z}, z_k).$$

As $r \to \infty$, $\int_{\gamma_r} f(z) e^{i\beta z}\, dz \to 0$, thus:

$$\int_{-\infty}^{+\infty} f(x) e^{i\beta x}\, dx = 2\pi i \sum_{\operatorname{Im}(z_k) > 0} \operatorname{Res}(f(z) e^{i\beta z}, z_k).$$

Remark 5.0.2 *Since $e^{i\beta x} = \cos(\beta x) + i \sin(\beta x)$, we get:*

$$\int_{-\infty}^{+\infty} f(x) e^{i\beta x}\, dx = \int_{-\infty}^{+\infty} f(x) \cos(\beta x)\, dx + i \int_{-\infty}^{+\infty} f(x) \sin(\beta x)\, dx.$$

Example 5.0.12 *Evaluate:*

$$I = \int_0^{+\infty} \frac{\cos x}{(1 + x^2)^2} dx.$$

We use:

$$I = \frac{1}{2} \left(\int_{-\infty}^{+\infty} \frac{e^{ix}}{(1 + x^2)^2} dx \right).$$

Let $f(z) = \frac{1}{(1+z^2)^2}$. *The singularities are* $z_1 = i$, $z_2 = -i$. *Only* z_1 *lies in the upper half-plane and is a double pole. Then:*

$$\operatorname{Res}(f(z)e^{iz}, z_1) = \frac{1}{2ie}, \quad \Rightarrow \quad I = \frac{\pi}{e}.$$

5.0.3 Integral. With the definition of a root

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We consider an integral of the form:

$$I = \int_0^{+\infty} \frac{f(x)}{x^\beta}\, dx, \quad 0 < \beta < 1$$

where f is a rational function with no poles on the positive real axis. Note that such an integral is always convergent at 0, but convergence at infinity imposes the condition:

$$\lim_{z\to\infty} f(z) = 0$$

which we assume to be true.

We define the function $g(z) = \frac{f(z)}{z^\beta}$, which is holomorphic on the domain $\mathbb{C} \setminus \{z \in \mathbb{C} : \Re(z) \geq 0\}$.

Let us choose the argument of z between 0 and 2π, and integrate the function g along the path $C_{\theta,R}$ (as shown in the figure). The paths L_1^- and L_2^+ are contained along the lines of polar angles 0, thus:

$$\int_{C_{\theta,R}} g(z)\, dz = \int_{C_R} g(z)\, dz + \int_{C_\theta} g(z)\, dz + \int_{L_1^-} g(z)\, dz + \int_{L_2^+} g(z)\, dz$$

When z traces the path L_1^-, the chosen argument implies that $z^\beta = |z|^\beta e^{2\pi i\beta}$, and taking the orientation into account:

$$\int_{C_{\theta,R}} g(z)\, dz = \int_{C_R} g(z)\, dz + \int_{C_\theta} g(z)\, dz + (1 - e^{-2\pi i\beta}) \int_{-\infty}^{\infty} g(x)\, dx$$

We know that:

$$zg(z) \to 0 \quad \text{as} \quad z \to 0, \quad zg(z) \to 0 \quad \text{as} \quad |z| \to +\infty$$

Thus, by Jordan's lemma:

$$\lim_{r\to\infty} \int_{C_R} g(z)\, dz = 0, \quad \lim_{\theta\to 0} \int_{C_\theta} g(z)\, dz = 0$$

The integral I is computed by taking the limit and applying the residue theorem:

$$I = \int_0^{+\infty} \frac{f(z)}{z^\beta} dz = 2\pi i(1 - e^{-2\pi i\beta}) \sum_{z_k} \text{Res}\left(\frac{f(z)}{z^\beta}, z_k\right)$$

where the sum is taken over all finite poles z_k of the rational function f that lie in the complex plane.

Example 5.0.13

$$I = \int_0^{+\infty} \frac{dx}{x^\beta(1+x)}, \quad 0 < \beta < 1$$

We have:

$$I = \frac{e^{i\beta}}{\sin \pi\beta} \sum_{z_k} Res\left(\frac{1}{z^\beta(1+z)}, -1\right)$$

The residue at $z = -1$ is:

$$Res\left(\frac{1}{z^\beta(1+z)}, -1\right) = (-1)^\beta = e^{-i\pi\beta}$$

Thus:

$$I = \frac{\pi}{\sin \pi\beta}$$

Example 5.0.14

$$J = \int_0^{+\infty} \frac{dx}{\sqrt{x}(x^2+1)^2} = 2\pi i \frac{1 - e^{-i\pi}}{\sin \pi} \sum_{z_k} Res(g, z_k)$$

where $g(z) = \frac{1}{\sqrt{z}(z^2+1)^2}$. The poles are at $z_0 = i$ and $z_1 = -i$, which are double poles.

The residues are:

$$\text{Res}(g, i) = \frac{1}{8}e^{-3i\pi/4} + \frac{1}{4i}e^{-i\pi/4}$$
$$\text{Res}(g, -i) = \frac{1}{8}e^{-i\pi/4} - \frac{1}{4i}e^{-3i\pi/4}$$

Then:

$$J = 2\pi i \left[\text{Res}(g, i) + \text{Res}(g, -i)\right] = 3\pi^2/8$$

5.0.4 Integral. Application to Fourier Transforms.

Definition 5.0.9 *Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable continuous function. Its Fourier transform is the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by:*

$$\hat{f}(\xi) = F(f)(\xi) = \int_{-\infty}^{\infty} f(y)e^{-i\xi y}\, dy$$

Case 1: f is a rational function with no real poles. Let f be a rational function integrable over $\mathbb{R}$ with poles z_k. Then:

$$F(f)(\xi) = \begin{cases} 2\pi i \sum_{\mathrm{Im}(z_k)>0} \mathrm{Res}(R(z)e^{-iz\xi}, z_k), & \text{if } \xi \leq 0 \\ -2\pi i \sum_{\mathrm{Im}(z_k)<0} \mathrm{Res}(R(z)e^{-iz\xi}, z_k), & \text{if } \xi \geq 0 \end{cases}$$

Example 5.0.15

$$F\left(\frac{1}{1+x^2}\right)(\xi) = \int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{1+x^2}\, dx$$

The poles are $z_0 = i$ and $z_1 = -i$. The residues are:

$$Res\left(\frac{e^{-iz\xi}}{1+z^2}, i\right) = \frac{-e^{i\xi}}{2i}, \quad Res\left(\frac{e^{-iz\xi}}{1+z^2}, -i\right) = \frac{e^{-i\xi}}{2i}$$

Thus:

$$F(f)(\xi) = \begin{cases} 2\pi i \cdot Res\left(\frac{e^{-iz\xi}}{1+z^2}, i\right), & \text{if } \xi \leq 0 \\ -2\pi i \cdot Res\left(\frac{e^{-iz\xi}}{1+z^2}, -i\right), & \text{if } \xi \geq 0 \end{cases} = \pi e^{-|\xi|}$$

Case 2: f is a rational function with real poles.

Example 5.0.16

$$F\left(\frac{1}{x}\right)(1) = \int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\, dx$$

The pole is at $z_0 = 0$, which is a simple pole:

$$Res\left(\frac{e^{-iz}}{z}, 0\right) = 1$$

$$\begin{aligned} \lim_{R\to\infty} \int_{C_{R,r}} f(z)dz &= 0 \\ \lim_{r\to 0} \int_{C_r} f(z)dz &= \int_{-\pi}^{0} \lim_{r\to 0} \frac{-e^{-ire^{it}}}{re^{it}} ire^{it}dt = i\pi \end{aligned}$$

$$\lim_{r\to 0}\left(\int_{-R}^{-r}+\int_{r}^{+R}\right)\frac{e^{-iz}}{z}dz+\lim_{R\to\infty}\int_{C_{R,r}} f(z)dz+\lim_{r\to 0}\int_{C_r} f(z)dz=0$$

Thus,the Fourier transform evaluates to:

$$\int_{-\infty}^{\infty}\frac{e^{-ix}}{x}\,dx=i\pi$$

and:

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx=\pi,\quad \int_{-\infty}^{\infty}\frac{\cos x}{x}\,dx=0$$

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