Remark 1

l'exercice noté par $(\star)$ ou supplémentaire ne sera pas corrigé dans le sience de **TD**

Exercise correction 1 ☆

We compute the following integrals on the specified domains

1 $\iint_D \frac{dxdy}{(1+x)(1+y)} = \Big(\int_0^1 \frac{dx}{1+x}\Big)\Big(\int_0^1 \frac{dy}{1+y}\Big) = \Big[\ln(1+x)\Big]_0^1\Big[\ln(1+y)\Big]_0^1 = \ln^2(2).$

2

$$\begin{aligned}
\iint_D \frac{y}{x^2+y^2}dxdy &= \frac{1}{2}\int_0^1 \Big(\int_0^1 \frac{2y}{x^2+y^2}dy\Big)dx \\
&= \frac{1}{2}\int_0^1 (\ln(x^2+4)-\ln(x^2+1))dx \\
&= \frac{1}{2}\int_0^1 \ln\Big(\frac{x^2+4}{x^2+1}\Big)dx = \frac{1}{2}\int_0^1 \ln\Big(1+\frac{3}{x^2+1}\Big)dx.
\end{aligned}$$

We calculate this integral by part, so we set

$$\begin{cases} u(x) = \ln\Big(1+\dfrac{3}{x^2+1}\Big) \\ v'(x) = 10 \end{cases} \Rightarrow \begin{cases} u'(x) = -\dfrac{6x}{(x^2+1)(x^2+4)} \\ v(x) = x \end{cases}$$

So we can see that

$$\begin{aligned}
\iint_D \frac{y}{x^2+y^2}dxdy &= \frac{1}{2}\Big[x\ln\Big(1+\frac{3}{x^2+1}\Big)\Big]_0^1 + \frac{1}{2}\int_0^1 \frac{6x^2}{(x^2+1)(x^2+4)}dx \\
&= \frac{1}{2}\ln\frac{5}{2} - \frac{1}{2}\int_0^1 \frac{1}{x^2+1}dx + 4\int_0^1 \frac{1}{x^2+4}dx \\
&= \frac{1}{2}\ln\frac{5}{2} + \Big[-\arctan(x) + 2\arctan(\frac{x}{2})\Big]_0^1 \\
&= \frac{1}{2}\ln\frac{5}{2} - \frac{\pi}{2} + 2\arctan(\frac{1}{2}).
\end{aligned}$$

3 $\iint_D \frac{x\sin y}{1+x^2}dxdy = \Big(\int_0^1 \frac{2x}{1+x^2}dx\Big)\Big(\int_0^\pi \sin y dy\Big) = \Big[\ln(1+x^2)\Big]_0^1\Big[\cos y\Big]_0^\pi = \ln 2.$

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$$\begin{aligned}\iint\limits_D xydxdy &= \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\Big(\int_0^x \frac{2y}{1+y^2}dy\Big)dx\\ &= \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\Big[\arctan y\Big]_0^x dx = \frac{1}{2}\int_0^1 \frac{\arctan x}{1+x^2}dx\\ &= \Big[\arctan^2 x\Big]_0^1 = \frac{\pi^2}{32}.\end{aligned}$$

5 Using integration by parts, we obtain,

$$\begin{aligned}\iint\limits_D x\cos ydxdy &= \int_0^1 x\int_{x^2}^x \Big(\cos ydy\Big)dx\\ &= \frac{1}{2}\ln\frac{5}{2} - \frac{1}{2}\int_0^1 \frac{1}{x^2+1}dx + 4\int_0^1 \frac{1}{x^2+4}dx\\ &= \int_0^1 x\Big[\sin ydy\Big]_{x^2}^x dx = \frac{1}{2} + \sin 1 - \frac{3}{2}\cos 1.\end{aligned}$$

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$$\begin{aligned}\iint\limits_D \frac{dxdy}{(x+y)^3} &= \int_0^3 \Big[\int_1^{4-x} \frac{dy}{(x+y)^3}\Big]dx = \int_1^3 \Big[\frac{1}{(x+y)^2}\Big]_1^{4-x} dx = -\frac{1}{2}\Big[\frac{1}{16}x + \frac{1}{x+1}\Big]_1^3\\ &= \int_1^3 \Big[\frac{1}{16} - \frac{1}{(x+1)^2}\Big]dx = \frac{1}{16}.\end{aligned}$$

Exercise correction 2 ☆

Firstly, we give the hierarchical description of the domain K, where

$$K = \{(x,y)\in\mathbb{R}^2 : \ x+y\geq 1, \ x^2+y^2\leq 1\}.$$

Then, we have $x+y\geq 1 \Rightarrow (x+y)^2\geq 1 \Rightarrow x^2+2xy+y^2\geq 1$. So, we find, $2xy\geq 1-x^2-y^2 \Rightarrow xy\geq 0$. As a result $x\geq 0$ and $y\geq 0$ with $y^2\leq 1$. Consequently

$$K = \{(x,y)\in\mathbb{R}^2 : \ 0\leq y\leq 0, \ 1-y\leq x\leq \sqrt{1-y^2}\}.$$

$$\begin{aligned}I = \iint\limits_K xy^2dxdy &= \int_0^1 \Big[\int_{1-y}^{\sqrt{1-y^2}} xdx\Big]y^2dy = \int_0^1 \Big[\frac{x^2}{2}\Big]_{1-y}^{\sqrt{1-y^2}} y^2dy\\ &= \int_0^1 \Big[\frac{1-y^2}{2} - \frac{(1-y)^2}{2}\Big]y^2dy = \int_0^1 \Big[y^3-y^4\Big]dy = \frac{1}{20}.\end{aligned}$$

Exercise correction 3 ☆

Draw the integration domains and calculate the following double integrals, changing the order of integration if necessary

1 $I_1 = \int_{-1}^{2}\int_{x^2-1}^{x+1}(x^2+y)dxdy.$

$D_1 = \{(x,y)\in\mathbb{R}^2 : \ -1 \le x \le 2, \ x^2-1 \le y \le x+1\}$. In other words, x is fixed. Therefore,

$$\begin{aligned} I_1 = \int_{-1}^{2}\int_{x^2-1}^{x+1}(x^2+y)dxdy &= \int_{-1}^{2}\Big[\int_{x^2-1}^{\sqrt{x+1}}(x^2+y)dy\Big]dx = \int_{-1}^{2}\Big[yx^2+\frac{1}{2}y^2\Big]_{x^2-1}^{x+1}dx \\ &= \int_{-1}^{2}\Big[(x+1)x^2+\frac{1}{2}(x+1)^2-(x^2-1)x^2-\frac{1}{2}(x^2-1)^2\Big]dx \\ &= \Big[-\frac{3}{5}x^5+\frac{1}{4}x^4+\frac{7}{6}x^3+\frac{1}{2}x^2\Big]_{-1}^{2} = \frac{117}{20}. \end{aligned}$$

2 $I_2 = \int_{0}^{4}\int_{\frac{y}{2}}^{\sqrt{y}}(xy+y^3)dxdy.$

$D_2 = \{(x,y)\in\mathbb{R}^2 : \ 0 \le y \le 2, \ \frac{y}{2} \le x \le \sqrt{y}\}$. In this case y is fixed, and we have,

$$\begin{aligned} I_2 = \int_{0}^{4}\int_{\frac{y}{2}}^{\sqrt{y}}(xy+y^3)dxdy &= \int_{0}^{4}\Big[\int_{\frac{y}{2}}^{\sqrt{y}}(xy+y^3)dx\Big]dy = \int_{0}^{4}\Big[\frac{yx^2}{2}+xy^3\Big]_{\frac{y}{2}}^{\sqrt{y}}dy \\ &= \int_{0}^{4}\Big[\frac{y^2}{2}+y^{\frac{7}{2}}-\frac{y^3}{8}-\frac{y^4}{2}\Big]dy \\ &= \Big[\frac{y^3}{6}+\frac{2}{9}y^{\frac{9}{2}}-\frac{y^4}{32}-\frac{y^5}{20}\Big]_0^4 = -\frac{364}{15}+\frac{2}{9}\sqrt{262144}. \end{aligned}$$

3 We calculate $I_3 = \int_{1}^{10}\int_{0}^{\frac{1}{y}} ye^{xy}dxdy$. So we find

$$\begin{aligned} I_3 = \int_{1}^{10}\int_{0}^{\frac{1}{y}} ye^{xy}dxdy &= \int_{0}^{10}\Big[\int_{0}^{\frac{1}{y}} ye^{xy}dx\Big]dy = \int_{0}^{10}\Big[e^{xy}\Big]_0^{\frac{1}{y}}dy \\ &= \int_{0}^{10}(e-1)dy = 9(e-1). \end{aligned}$$

4 For the integral, $I_4 = \int_{0}^{2\sqrt{\ln 3}}\int_{\frac{y}{2}}^{\sqrt{\ln 3}} e^{x^2}dxdy$. on a

$$\begin{aligned} I_4 = \int_{0}^{2\sqrt{\ln 3}}\int_{\frac{y}{2}}^{\sqrt{\ln 3}} e^{x^2}dxdy &= \int_{0}^{\sqrt{\ln 3}}\Big[\int_{0}^{x^2} e^{x^2}dy\Big]dx = \int_{0}^{\sqrt{\ln 3}} e^{x^2}\Big[y\Big]_0^{x^2}dx \\ &= \int_{0}^{\sqrt{\ln 3}} 2xe^{x^2}dx = \Big[e^{x^2}\Big]_0^{\sqrt{\ln 3}} = 2. \end{aligned}$$

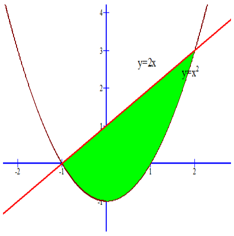

Figure 1: D_1.

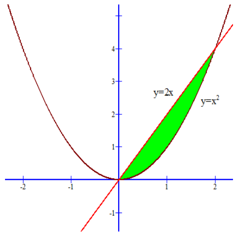

Figure 2: D_2.

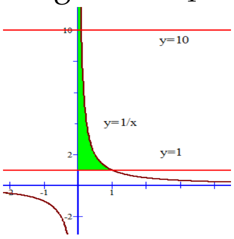

Figure 3: D_3.

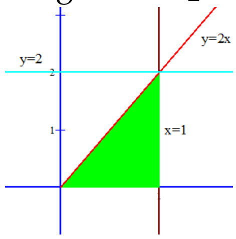

Figure 4: D_4.

Exercise correction 4 ☆☆

We calculate the integral of the function f defined by, $f(x,y,z) = xyz$ in the volume V bounded by the planes $x=0,\ y=0,\ z=0$ and $x+y+z-1=0$. So,

$$\begin{aligned}
\iiint\limits_V xyz dxdydz &= \int_0^1 x\Big[\int_0^{1-x} y\Big(\int_0^{1-x-y} zdz\Big)dy\Big]dx = \int_0^1 x\Big[\int_0^{1-x} y\Big(\frac{1}{2}z^2\Big)_0^{1-x-y}dy\Big]dx \\
&= \int_0^1 x\Big[\int_0^{1-x}\frac{1}{2}y(1-x-y)^2dy\Big]dx \\
&= \int_0^1 x\Big[\frac{1}{4}y^2(1-x)^2+\frac{1}{8}y^4-\frac{1}{8}y^4(1-x)\Big]_0^{1-x}dx \\
&= \frac{1}{4}\int_0^1 x(1-x)^4dx = \frac{1}{720}
\end{aligned}$$

Exercise correction 5 ☆

$I = \iint\limits_D (x+y)^2 e^{x^2-y^2}dxdy,\ D = \{(x,y)\in\mathbb{R}^2 :\ x\geq 0,\ y\geq 0 \text{ and } x+y\leq 1\}$. First, we express x and y as functions of u and v. Thus, we have $\begin{cases} u = x+y \\ v = x-y \end{cases} \Rightarrow \begin{cases} x = \dfrac{u+v}{2} \\ y = \dfrac{u-v}{2} \end{cases}$ Since, $x\geq 0$ then, we will have, $u+v\geq 0 \Rightarrow u\geq -v$.
and we also get $y\geq 0\Rightarrow u-v\geq 0\Rightarrow v\leq u$. Consequently we find $-u\leq v\leq u$.
Taking into account $x+y\leq 1$, we get $0\leq u\leq 1$.
This gives the new integration domain,

$$Im(D) = \{(u,v)\in\mathbb{R}^2 :\ 0\leq u\leq 1 \text{ and } -u\leq v\leq u\}.$$

We also need to be Jacobian,

$$\det(J(u,v)) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{vmatrix} = -\frac{1}{2}\neq 0.$$

The new integral is expressed as follows

$$\begin{aligned} I &= \iint\limits_{D}(x+y)^2 e^{x^2-y^2}dxdy = \iint\limits_{D}|\det(J(u,v))|f(u,v)dudv \\ &= \frac{1}{2}\int_0^1 u^2\Big[\int_{-u}^{u} e^{uv}dv\Big]du = \frac{1}{2}\int_0^1 u^2\Big[\frac{e^{uv}}{2u}\Big]_{-u}^{u}du \\ &= \frac{1}{2}\int_0^1 u(e^{u^2}-e^{u^2})du = \frac{1}{4}\Big[e^{u^2}+e^{u^2}\Big]_0^1 \\ &= \frac{1}{4}\Big[e+e^{-1}-2\Big] = \frac{ch1-1}{2}. \end{aligned}$$

Exercise correction 6 ☆☆

1 Put, $x = r\cos\theta,\ y = r\sin\theta$, with $0 \le r \le 1$ and $\theta \in [0, 2\pi[$. Then, we see that

$$I_1 = \iint\limits_{D_1}\frac{dxdy}{1+x^2+y^2} = \int_0^1\int_0^{2\pi}\frac{rdrd\theta}{1+r^2} = 2\pi\Big[\frac{1}{2}\ln(1+r^2)\Big]_0^1 = \pi\ln(2).$$

2 Put, $x = r\cos\theta,\ y = r\sin\theta$, with $0 \le r \le 1$ and $\theta \in [0, \frac{\pi}{2}[$. Then we find, $I_2 = \iint\limits_{D_2} e^{-(x^2+y^2)}dxdy =$
$\int_0^1\int_0^{2\pi} e^{-r^2}rdrd = -\frac{\pi}{4}\Big[e^{-r^2}\Big]_0^1 = \frac{\pi(e-1)}{4e}$

3 Remains as an exercise.

Exercise correction 7 ☆

Let's calculate the double integral $I = \iint\limits_{D}\frac{\cos(x^2+y^2)}{2+\sin(x^2+y^2)}dxdy$, where,

$$D = \{(x,y) \in \mathbb{R}^2 :\ 1 \le x^2+y^2 \le 4\}.$$

Making the change, let's say $x = r\cos\theta,\ y = r\sin\theta$, such that $r \in [1,2]$ and $\theta \in [0, 2\pi]$. we see,

$$\begin{aligned} I &= \iint\limits_{D}\frac{\cos(x^2+y^2)}{2+\sin(x^2+y^2)}dxdy = \int_1^{2\pi}\Big[\int_1^2\frac{r\cos r^2}{2+\sin r^2}d\theta\Big]dr \\ &= \pi\Big[\ln(2+\sin r^2)\Big]_1^2 = \pi\ln\Big(\frac{\ln(2+\sin 4)}{\ln(2+\sin 1)}\Big). \end{aligned}$$

Supplementary exercise correction 1 ☆

Draw the integration domains and calculate the following double integrals

1 We compute I_1 on the domain D_1 where,
$I_1 = \iint\limits_{D_1}(x\sin y)dxdy$, and $D_1 = \{(x,y) \in \mathbb{R}^2 :\ 0 \le x \le \pi,\ 0 \le y \le x\}$. Then, we obtain

$$\begin{aligned} I_1 = \iint\limits_{D_1}(x\sin y)dxdy &= \int_0^{\pi}\int_0^{x} x\sin ydxdy = \int_0^{\pi}\Big[\int_0^{x} x\sin ydy\Big]dx \\ &= \int_0^{\pi}\Big[-x\cos y\Big]_0^{x}dx = \int_0^{\pi}\Big[x - x\cos x\Big]dx \\ &= \Big[\frac{x^2}{2} - \cos x - x\sin x\Big]_0^{\pi} = \frac{\pi^2}{2}+2 \end{aligned}$$

2 $I_2 = \iint\limits_{D_1} 3y^3 e^{xy} dxdy, \;\; D_2 = \{(x,y) \in \mathbb{R}^2 : \; 0 \le x \le 1, \; \sqrt{x} \le y \le 1\}.$

$$\begin{aligned} I_2 &= \iint\limits_{D_2} 3y^3 e^{xy} dxdy = \int_0^{\pi}\int_0^{x} x \sin y dxdy \\ &= \int_0^1 \Big[\int_{\sqrt{x}}^{1} 3y^3 e^{xy} dy\Big] dx = \int_0^1 \Big[\int_0^{y^2} 3y^3 e^{xy} dx\Big] dy \\ &= \int_0^1 \Big[3y^2 e^{xy}\Big]_0^{y^2} dy = \Big[e^{y^3} - y^3\Big]_0^1 = e - 2. \end{aligned}$$

3 For the integral $I_3 = \iint\limits_{D_3} \dfrac{xe^{2y}}{4-y} dxdy$, on the domain D_3, where, $D_3 = \{(x,y) \in \mathbb{R}^2 : \; 0 \le x \le 2, \; 0 \le y \le 4 - x^2\}$. We have,

$$\begin{aligned} I_3 &= \iint\limits_{D_3} \frac{xe^{2y}}{4-y} dxdy = \int_0^4 \Big[\int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} dx\Big] dy = \int_0^4 \Big[\frac{x^2 e^{2y}}{2(4-y)}\Big]_0^{\sqrt{4-y}} dy \\ &= \frac{1}{2}\int_0^4 e^{2y} dy = \frac{1}{4}\Big[e^{2y}\Big]_0^4 = \frac{e^8 - 1}{4}. \end{aligned}$$

4 $I_4 = \iint\limits_{D_4} (x^2 + y^2) dxdy, \;\; D_4 = \{(x,y) \in \mathbb{R}^2 : \; x \ge 2, \; y \ge 2, \text{ and } x + y \le 1\}.$

$$\begin{aligned} I_4 &= \iint\limits_{D_4} (x^2 + y^2) dxdy = \int_0^1 \Big[\int_0^{1-x} (x^2 + y^2) dy\Big] dx = \int_0^1 \Big[(x^2 y + \frac{y^3}{3})\Big]_0^{1-x} dx \\ &= \int_0^1 \Big[(x^2(1-x) + \frac{(1-x)^3}{3})\Big] dx = \int_0^1 \Big[-\frac{4}{3}x^3 + 2x^2 - x + \frac{1}{3}\Big] dx \\ &= \Big[-\frac{1}{3}x^4 + \frac{2}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x\Big]_0^1 = \frac{1}{6}. \end{aligned}$$

Supplementary exercise correction 2

We start by writing the expressions of the variables $x, \; y, \; z$ as a function of the new variables $u, \; v$ and w. We therefore have,

$$\begin{cases} u = x + y + z \\ v = y + z \\ w = z \end{cases} \Rightarrow \begin{cases} x = u - v \\ y = v - w \\ z = w \end{cases}$$

Then the Jacobian of the variable change is given by following formula

$$\det(J(u,v,w)) = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Since we know that $x+y+z \leq 1$ we find $0 \leq u \leq 1$, and as $x \geq 0,\ y \geq 0,\ z \geq 0$, we have $0 \leq w \leq v \leq u \leq 1$. Consequently, we express the integral I as follows,

$$\begin{aligned} I &= \iiint\limits_D \frac{z^3 dxdydz}{(y+z)(x+y+z)} = \int_0^1 \Big[\int_0^u \Big[\int_0^v \frac{w^3}{uv} dw\Big] dv\Big] du \\ &= \int_0^1 \Big[\int_0^u \frac{v^3}{4u} dv\Big] du = \int_0^1 \frac{u^3}{16} du = \frac{1}{64}. \end{aligned}$$

Supplementary exercise correction 3

1 The domain $A = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}$, it can be characterized as

$$A = \left\{(x,y) \in \mathbb{R}^2 : -a \leq x \leq a, \text{ and } g_1(x) = -b\sqrt{1-\frac{x^2}{a^2}} \leq y \leq b\sqrt{1-\frac{x^2}{a^2}} = g_2(x)\right\},$$

Using Fubini's theorem we find,

$$Area(A) = \iint\limits_A 1dxdy = \int_{-a}^{a} \Big[\int_{g_1(x)}^{g_2(x)} dy\Big] dx = 2b \int_{-a}^{a} \sqrt{1-\frac{x^2}{a^2}} dx.$$

By changing the variable $x = a\sin t$ we see that the area of the ellipse is $Area(A) = ab\pi$.

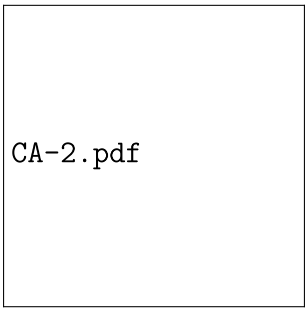

Figure 5: $Area(A)$

2 If we change to polar coordinates, with $x = r\cos\theta$ and $y = r\sin\theta$, we find,

$$\begin{aligned} I = \iint\limits_D x^2 dxdy &= \iint\limits_{]1,\sqrt{3}[\times]-\frac{\pi}{2},\frac{\pi}{2}[} x^2 dxdy = \int_1^{\sqrt{3}} r^3 dr \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta \\ &= \Big[\frac{r^4}{4} d\Big]_1^{\sqrt{3}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} d\theta = \frac{27\pi}{16}. \end{aligned}$$

AE-5.pdf

Figure 6: $Aire(\mathcal{E})$

3 We switch to spherical coordinates, by posing

$$\begin{cases} x = r\sin\theta\sin\alpha \\ y = r\cos\theta\sin\alpha \\ z = r\cos\alpha, \qquad \text{where } 0 \le r \le 1,\ \alpha \in [-\pi, \pi] \text{ and } \theta \in [0, \pi] \end{cases}$$

Par suit, on obtient,

$$\begin{aligned} I = \iiint\limits_B \frac{z^3 dxdydz}{\sqrt{x^2+y^2+(z-a)^2}} &= 2\pi \int_0^1 \int_{-\pi}^{\pi} \int_0^{\pi} \frac{r^2\sin\alpha}{\sqrt{r^2+a^2-2ar\cos\theta}} d\theta d\alpha dr \\ &= 2\pi \int_0^1 r^2 \int_0^{\pi} \frac{r^2\sin\alpha}{\sqrt{r^2+a^2-2ar\cos\theta}} dr. \end{aligned}$$

We calculate the integral below by executing the change of variables $t = r^2 + a^2 - 2ar\cos\theta,\ dt = 2ar\sin\alpha d\alpha$. We find, therefore,

$$\begin{aligned} \iiint\limits_B \frac{z^3 dxdydz}{\sqrt{x^2+y^2+(z-a)^2}} &= 2\pi \int_0^1 \int_{(a-r)^2}^{(a+r)^2} \frac{dt}{2ar\sqrt{t}} dr \\ &= \frac{4\pi}{a} \int_0^1 r^2 dr = \frac{4\pi}{3a}. \end{aligned}$$

Supplementary exercise correction 4

We can do the calculation in the case $a = 1$, to obtain the general case, simply multiply by a^3. we find, therefore, $\begin{cases} x^2+y^2+z^2 \le 1 \\ x^2+y^2+z^2 \le x \\ z \ge 0 \end{cases}$ Switching to cylindrical coordinates, these constraints can be written as follows

$$\begin{cases} r^2+z^2 \le 1 \\ r^2 \le r\cos\theta \\ z \ge 0 \end{cases} \Rightarrow \begin{cases} r^2+z^2 \le 1 \\ r \le \cos\theta, \qquad r \in [0,1] \\ z \ge 0, \end{cases}$$

when r is finite θ between $-\arccos r$ and $\arccos r$. Then, the volume of the domain to be expressed integrally

$$\begin{aligned}
\int_0^1 \int_{-\arccos r}^{\arccos r} \int_0^{\sqrt{1-r^2}} r dz d\theta dr &= \int_0^1 2r\sqrt{1-r^2}\arccos r dr \\
&= \left[\frac{2}{3} - (1-r^2)^{\frac{3}{2}} \arccos r\right]_0^1 - \frac{2}{3}\int_0^1 \frac{(1-r^2)^{\frac{3}{2}}}{\sqrt{1-r^2}} dr \\
&= \frac{\pi}{3} - \frac{2}{3}\int_0^1 (1-r^2) dr = \frac{\pi}{3} - \frac{4}{9}.
\end{aligned}$$