University of M'sila		Faculty of Mathematics and Computer Science
L 2 Mathematics	Correction of Series $N : 04$ - Extrema-	Module: Analysis 4

Exercise correction 1 ☆

Consider the application f defined by

$$\begin{array}{rcl} f:\mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & x^2+y^3-3y. \end{array}$$

1 The application f is polynomial in the variables x and y therefore differentiable on $\mathbb{R}^2$ (also from class C^1 on $\mathbb{R}^2$).
We calculate, for all $(x,y)\in\mathbb{R}^2$.

$$\frac{\partial f}{\partial x}=2x, \quad \text{and, } \frac{\partial f}{\partial y}=3y^2-3.$$

So it's gradient at a point $(x,y)\in\mathbb{R}^2$ is

$$\nabla f=\left(\begin{array}{cc} \frac{\partial f}{\partial x}, & \frac{\partial f}{\partial y} \end{array}\right)=\left(\begin{array}{cc} 2x, & 3y^2-3 \end{array}\right)$$

2 Let $(x,y)\in\mathbb{R}^2$

$$\nabla f(x,y)=(0,0)\Leftrightarrow 2x=0, \text{ and } 3y^2-3=0\Leftrightarrow 2x=0, \text{ and } (y=-1\vee y=1)$$

So the function f has two critical points, $(0,-1)$ and $(0,1)$.

3 We know that f achieves a local extremum at a point, so this point is a critical point of f, according to the necessary condition for the existence of an extremum. There are therefore two points to study.

(a) Study of the critical point $(0,1)$ of f.
Let $(x,y)\in\mathbb{R}^2$

$$\begin{array}{rcl} f(x,1+y)-f(0,1) & = & x^2+y(1+y)^3-3(1+y)-(-2) \\ & = & x^2+y^2(3+y). \end{array}$$

So for all (x,y) belonging to

$$\mathcal{U}:=\mathbb{R}\times]-3,3[,$$

which is an open neighbourhood of $(0,0)$, and $f(x,1+y)\geq f(0,1)$. The function f therefore achieves a local minimum (equal to -2) at the point $(0,1)$.
This local minimum is not global because

$$f(0,y)=y^3-y, \text{and } \lim_{y\to-\infty}f(0,y)=\lim_{y\to-\infty}y^3=-\infty.$$

(b) Study of the critical point $(0,-1)$ of f.
Let$(x,y)\in\mathbb{R}^2$

$$\begin{array}{rcl} f(x,-1+y)-f(0,-1) & = & x^2+y(y-1)^3-3(y-1)-(-2) \\ & = & x^2-3y^2+y^3=x^2+y^2(y-3). \end{array}$$

On the one hand, for each $x \in \mathbb{R}$, $(y = 0)$

$$f(x,-1) - f(0,-1) = x^2 \geq 0.$$

On the other hand, if $(x = 0)$

$$f(0,-1+y) = -3y^2 + y^3 \underset{y \to 0}{\sim} -3y^2 < 0.$$

So the quantity $f(x,-1+y) - f(0,-1)$ has no constant sign in any neighbourhood of $(0,0)$. So the function f does not achieve a local extremum (and in particular not a global extremum) at the point $(0,-1)$.

Exercise correction 2 ☆

We have f defined on $\mathbb{R}^2$ by

$$f(x,y) = x^2 + 2xy + y - y^3.$$

1 f is of class C^2 (also of class C^∞), can therefore apply the method of first and second derivatives for the study of local extrema. We have, therefore

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y} \right) = \left(\ 2(x+y), \ \ 2x + 1 - 3y^2 \ \right)$$

Solve the system of equations. Let $(x,y) \in \mathbb{R}^2$

$$\nabla f(x,y) = (0,0) \Leftrightarrow \begin{cases} x + y = 0 \\ -3y^2 - 2y + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} y = -x \\ y = -1 \vee y = \dfrac{1}{3} \end{cases}$$

So the function f has two critical points, $A(1,-1)$ and $B(-\frac{1}{3}, \frac{1}{3})$. We study the nature of the critical points with the Hessian, and as,

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2}(x,y) = 2 \\ \dfrac{\partial^2 f}{\partial x \partial y}(x,y) = 2 \\ \dfrac{\partial^2 f}{\partial y^2}(x,y) = -6y \end{cases}$$

So we find,

$$Hess_f(x,y) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(x,y) & \dfrac{\partial^2 f}{\partial x \partial y}(x,y) \\ \dfrac{\partial^2 f}{\partial x \partial y}(x,y) & \dfrac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & -6y \end{pmatrix}$$

So, the determinant of the Hessian matrix at the critical point $A(1,-1)$ is

$$\det\Big(Hess_f(1,-1)\Big) = \begin{vmatrix} 2 & 2 \\ 2 & 6 \end{vmatrix} = 8 > 0, \text{ with, } \frac{\partial^2 f}{\partial x^2}(1,-1) = 2 > 0$$

Then, A corresponds to a local minimum for f.

At the point $B(-\frac{1}{3}, \frac{1}{3})$, we have

$$\det\Big(Hess_f(-\frac{1}{3}, \frac{1}{3})\Big) = \begin{vmatrix} 2 & 2 \\ 2 & -2 \end{vmatrix} = -8 < 0.$$

so B is a saddle point (neither a local minimum nor a local maximum).

2 The value of f at point A is $f(1,-1) = -1$, or we have for example $f(0,2) = -6 < f(1,-1)$, so A is not a global minimum.

Remark 1

Note that, without even looking for local minima, it was clear from the beginning that f could not admit a global minimum on $\mathbb{R}^2$, since

$$\lim_{y\to\pm\infty} f(0,y) = \lim_{y\to\pm\infty}(y-y^3) = \lim_{y\to\pm\infty}(-y^3) = -\infty.$$

The absence of a global minimum of f results from a compactness of $\mathbb{R}^2$ despite the continuity of f.

Exercise correction 3 ☆

We have $f : (x,y) \longmapsto \left(x(\ln^2 x) + y^2\right)$.

1 $\mathcal{D}_f = \left\{(x,y) \in \mathbb{R}^2 : x > 0\right\} = \mathbb{R}_+^* \times \mathbb{R}$.

2 (a) f is of class C^{infty} on its domain and has the following partial derivatives

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \ \frac{\partial f}{\partial y}(x,y) \right) = \left(\ln^2 x + 2\ln x + y^2, \ \ 2xy \right)$$

$$\nabla f(x,y) = (\ 0, \ \ 0\ ) \Rightarrow \left(\begin{array}{c} y = 0, \ (x>0) \\ \ln^2 x + 2\ln x + y^2 = 0 \end{array} \right)$$

Consequently, $\ln^2 x + 2\ln x + y^2 = 0 \Rightarrow (x = 1 \ \vee \ x = e^{-2})$. We therefore have two critical points, $A(1,0)$ and $B(e^{-2},0)$.

(b) We use Monge's notation, so let's calculate the determinant of the Hessian. Then we have,

$$Hess_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 2\left(\frac{1+\ln x}{x}\right) & 2y \\ 2y & 2x \end{pmatrix}$$

So, the determinant of the Hessian matrix at the critical point $A(1,0)$ is

$$\det\left(Hess_f(1,0)\right) = pr - q^2 = 4, \text{ and } p = 2 > 0,$$

where $p = \frac{\partial^2 f}{\partial x^2}(1,0)$, $r = \frac{\partial^2 f}{\partial y^2}(1,0)$ and $q = \frac{\partial^2 f}{\partial x \partial y}(1,0)$. Then, A is a local minimum for f.

The determinant of the Hessian matrix at the critical point $B(e^{-2},0)$ is

$$\det\left(Hess_f(e^{-2},0)\right) = pr - q^2 = -4.$$

So B is a col point for f.

3 (a) We have $f(1,0) = 0$, but f is always positive, because,

$$\forall (x,y) \in \mathcal{D}_f : \ f(x,y) \geq 0.$$

We deduce that A is the global minimum of f.

(b) We note that $\mathcal{D}_f$ is open and since f does not admit a local maximum. Then, f does not admit a global maximum. We can also see that f is not bounded above, because,

$$\lim_{x\to+\infty} f(x,0) = \lim_{x\to+\infty} x\ln^2 x = +\infty.$$

Exercise correction 4 ☆

For each of the following functions, we study the nature of the stationary points,

1 $f(x,y) = 4x^2 + 2xy + y^2$. f is of class C^∞ on it's domain. Therefore, we have

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \ \ \frac{\partial f}{\partial y}(x,y) \right) = \left(\ 8x+2y, \ \ 2x+2y \ \right)$$

$$\nabla f(x,y) = \left(\ 0, \ \ 0 \ \right) \Rightarrow \left(\begin{array}{c} x=0 \\ y=0 \end{array} \right)$$

Then only the point $A(0,0)$ can be an extremum for f, and the nature of this extremum depends on the sign of $\Delta(Hess_f)$ at $A(0,0)$.

$$Hess_f(x,y) = \left(\begin{array}{cc} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x\partial y}(x,y) \\ \frac{\partial^2 f}{\partial x\partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{array} \right) = \left(\begin{array}{cc} 8 & 2 \\ 2 & 2 \end{array} \right)$$

Comme,

$$\det\left(Hess_f(0,0)\right) = \left| \begin{array}{cc} 8 & 2 \\ 2 & 2 \end{array} \right| = 12 > 0, \ \text{ and } \ \frac{\partial^2 f}{\partial x^2}(0,0) = 8 > 0.$$

is a local minimum at $(0,0)$ and this minimum is $f(0,0) = 0$.

2 $f(x,y) = -x^2 + x - xy + y - y^2$. We have $f \in C^\infty(\mathbb{R}^2)$. Therefore, the necessary extremum conditions are realized at the solution points of the following system,

$$\nabla f(x,y) = \left(\ 0, \ \ 0 \ \right) \Rightarrow \left(\frac{\partial f}{\partial x}(x,y), \ \ \frac{\partial f}{\partial y}(x,y) \right) = \left(\ 0, \ \ 0 \ \right)$$

$$\left\{ \begin{array}{l} -2x+1-y=0 \\ -x+1-2y=0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} x = \frac{1}{3} \\ y = \frac{1}{3}. \end{array} \right.$$

So the point $A\left(\frac{1}{3}, \frac{1}{3}\right)$ can be an extremum for f. Now let's study the nature of this extremum. We can see that,

$$Hess_f\left(\frac{1}{3}, \frac{1}{3}\right) = \left(\begin{array}{cc} -2 & -1 \\ -1 & -2 \end{array} \right)$$

When it was,

$$\det\left(Hess_f\left(\frac{1}{3}, \frac{1}{3}\right)\right) = \left| \begin{array}{cc} -2 & -1 \\ -1 & -2 \end{array} \right| = 3 > 0, \ \text{ and } \ \frac{\partial^2 f}{\partial x^2}\left(\frac{1}{3}, \frac{1}{3}\right) = -2 < 0.$$

So, $f(\frac{1}{3}, \frac{1}{3}) = \frac{1}{3}$ is a local maximum of f at point, $(\frac{1}{3}, \frac{1}{3}) = \frac{1}{3}$.

3 $f(x,y,z) = x^2 + 2x + y^2 - 3y + 2z^2$. We note that $f \in C^\infty(\mathbb{R}^3)$. Therefore, we pose,

$$\nabla f(x,y,z) = \begin{pmatrix} 0, & 0, & 0 \end{pmatrix}$$

As a result, we have

$$\left(\frac{\partial f}{\partial x}(x,y,z), \quad \frac{\partial f}{\partial y}(x,y,z) \quad \frac{\partial f}{\partial z}(x,y,z) \right) = \begin{pmatrix} 0, & 0, & 0 \end{pmatrix}$$

Hence, we find,

$$\begin{cases} 2x + 2 = 0 \\ 2y - 3 = 0 \\ 2z = 0 \end{cases} \Rightarrow \begin{cases} x = -1 \\ y = \dfrac{3}{2} \\ z = 0 \end{cases}$$

The stationary point is $A(-1, \frac{3}{2}, 0)$, so the nature of this extremum depends on the sign of $\Delta(Hess f)$.

$$Hess_f(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y,z) & \frac{\partial^2 f}{\partial x \partial y}(x,y,z) & \frac{\partial^2 f}{\partial x \partial z}(x,y,z) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y,z) & \frac{\partial^2 f}{\partial y^2}(x,y,z) & \frac{\partial^2 f}{\partial y \partial z}(x,y,z) \\ \frac{\partial^2 f}{\partial z \partial x}(x,y,z) & \frac{\partial^2 f}{\partial z \partial y}(x,y,z) & \frac{\partial^2 f}{\partial z^2}(x,y,z) \end{pmatrix}$$

Consequently,

$$\det\left(Hess_f(-1, \frac{3}{2}, 0)\right) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 16 > 0.$$

Then $f(-1, \frac{3}{2}, 0) = -\dfrac{13}{4}$ is a minimum for the function f.

Exercise correction 5 ☆

We study the extrema of f under the constraint $g(x,y) = 0$, such that

1 $f(x,y) = 2x^2 + 2xy + y^2 - 6x - 2y - 3$ et $g(x,y) = x + y - 4 = 0$.
The constraint implicitly expresses y as a function of x. So we find

$$y = 4 - x.$$

We then study the variations of the function F defined by

$$F(x) = f(x, y(x)) = 2x^2 + 2x(4-x) + (4-x)^2 - 6x - 2(4-x) - 3 = x^2 - 4x + 5.$$

F is a polynomial function of the second degree, with dominant coefficient $a = 1 > 0$ and $\Delta = 0$. Then, F reaches its minimum at $x = -\dfrac{b}{2a} = 2$.
Conclusion: f has a relative minimum under the constraint $x + y - 4 = 0$ at $(2,2)$.. This minimum is equal to $f(2,2) = 1$.

2 $f(x,y) = 10\sqrt[4]{x} + 2\sqrt[3]{y}$ and $g(x,y) = \dfrac{1}{4}\ln x + \dfrac{1}{3}\ln y - 3$. Keep it as an exercise.

Exercise correction 6 ☆☆

We search for the maximum of the function $f(x,y,z) = x + y + z$ under the constraint $x^2 + y^2 + z^2 = 1$.

1 The surface
$$S=\{(x,y,z)\in\mathbb{R}^3_+ : x^2+y^2+z^2=1\}.$$
Clearly $\mathcal{S}$ is a compact, since it is closed and bounded. The function f is continuous on its domain, so it is bounded and reaches its limits, in particular its maximum.

2 The Lagrangian associated with the problem is the function with 4 variables,
$$L(x,y,z,\lambda)=x+y+z-\lambda(x^2+y^2+z^2-1).$$

3 We search for the critical points of the Lagrangian, which gives, $\nabla f(x,y,z)-\lambda\nabla g(x,y,z)=0 \Rightarrow \begin{cases} 1-2\lambda x=0\\ 1-2\lambda y=0\\ 1-2\lambda z=0\\ x^2+y^2+z^2=1\end{cases} \Rightarrow \begin{cases} x=y=z=\dfrac{1}{2\lambda}\\ 3\dfrac{1}{4\lambda^2}=1\end{cases}$ It follows that the unique critical point is $A(\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2})$. It's a maximum or a minimum. It can't be a minimum, since we have for example $f(1,0,0)=1<f(\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2})=\sqrt{3}$, so that's the maximum.

Exercise correction 7 ☆☆

We define the function f on $\mathbb{R}^3$ by, $f(x,y,z)=x^5+xyz+y^3+3xz^4-2$

1 Then, the function f is of class $C^\infty(\mathbb{R}^3)$ for all $(x,y,z)\in\mathbb{R}^3$, and we have
$$\frac{\partial f}{\partial z}(x,y,z)=xy+12xz^3.$$
Furthermore, $f(1,-1,1)=0$ and $\frac{\partial f}{\partial z}(1,-1,1)=11\neq 0$. Then, by the implicit function theorem there exists a neighborhood $\mathcal{V}(1,-1)\subseteq\mathbb{R}^2$ and a unique function $g:\mathcal{V}(1,-1)\longrightarrow\mathbb{R}$ of class $C^1(\mathcal{V}(1,-1))$ such that, $g(1,-1)=1.$ and $f(x,y,g(x,y))=0$.

2 The equation of the tangent plane to the surface $z=g(x,y)$ at $(1,-1)$. We have, therefore,
$$\nabla f(x,y,z)=\begin{pmatrix} 5x^4+yz+3z^4, & xz+3y^2, & xy+12xz^3\end{pmatrix}$$
Donc, on obtient
$$\nabla f(1,-1,1)=\begin{pmatrix} 7, & 4, & 11\end{pmatrix}$$
So, the equation is given by,
$$\langle\nabla f(1,-1,1),\begin{pmatrix} x-1\\ y+1\\ z-1\end{pmatrix}\rangle=0\Leftrightarrow 7x+4y+11z-14=0.$$

Exercise correction 8 ☆

Let the function f be defined on $\mathbb{R}^2$ by, $f(x,y)=\frac{x^2}{9}+\frac{y^2}{4}$, with the constraint $g(x,y)=x^2+y=1$.

1 The Lagrangian of the problem is thus written,
$$L(x,y,\lambda)=\frac{x^2}{9}+\frac{y^2}{4}-\lambda(x^2+y-1).$$

$$\nabla f(x,y)-\lambda\nabla g(x,y)=0 \Rightarrow \begin{cases} \frac{2}{9}x-2\lambda x=0 \\ \frac{1}{2}y-\lambda=0 \\ x^2+y=1 \end{cases} \Rightarrow \begin{cases} y=2\lambda \\ x(\frac{2}{9}x-y)=0 \\ x^2+y=1 \end{cases} \Rightarrow \begin{cases} y=2\lambda \\ x=0 \vee y=\frac{2}{9} \\ x^2+y=1 \end{cases}$$

So, by substituting $x=0$ by in the third equation, we obtain the first critical value $A(0,1)$. Similarly, by substituting $y=\frac{2}{9}$ into the third equation, we obtain $x=\pm\frac{\sqrt{7}}{3}$, so we find the other two two other critical values $B(\frac{\sqrt{7}}{3},\frac{2}{9})$ and $C(-\frac{\sqrt{7}}{3},\frac{2}{9})$

2 We have $y=1-x^2$ so $y\in]-\infty,1]$. By substitution in f, we obtain the new single-variable function

$$F(x)=f(x,y(x))=\frac{x^2}{9}+\frac{(1-x^2)^2}{4}=\frac{1}{4}x^4-\frac{7}{18}x^2=0\Rightarrow x=0\vee x=\mp\frac{\sqrt{7}}{3}.$$

Hence the three critical points of the Lagrangian $A(0,1)$, $B(\frac{\sqrt{7}}{3},\frac{2}{9})$ et $C(-\frac{\sqrt{7}}{3},\frac{2}{9})$.
It is clear from the study of F that the minimum of this function is reached at the points $x=\mp\frac{\sqrt{7}}{3}$ and the maximum at $x=0$. We can easily see that A is a local maximum for f, B and C are global minima.

3 Then,

$$F'(x)=0\Leftrightarrow x^3-\frac{7}{9}x+\frac{1}{4}.$$

4 If f has a global maximum, it is necessarily achieved (reached) at its unique local maximum A, but we note that,

$$\lim_{x\to+\infty} f(x,1-x^2)=\lim_{x\to+\infty} F(x)=+\infty.$$

Therefore, f does not admit a global maximum under the constraint g.

Exercise correction 9 ☆☆

We use the Hessian to study the nature of the critical points for each of the following applications

1 (a) The first-degree extremum conditions $f(x,y,z)=x^2+2x-y^2+y+z^2$, are satisfied at the system's solution points (stationary points) $\begin{cases} \partial_x f(x,y,z)=0 \\ \partial_y f(x,y,z)=0 \\ \partial_z f(x,y,z)=0 \end{cases} \Rightarrow \begin{cases} 2x+2=0 \\ -2y+1=0 \\ 2z=0 \end{cases} \Rightarrow$
$\begin{cases} x=-1 \\ y=\frac{1}{2} \\ z=0 \end{cases}$ Therefore, only the point $(-1,\frac{1}{2},0)$ can be an extremum for f.

(b) Nature of the extremum. We have

$$\begin{cases} \partial^2_{x^2} f(x,y,z)=2 \\ \partial^2_{y^2} f(x,y,z)=-2 \\ \partial^2_{z^2} f(x,y,z)=2 \\ \partial^2_{xy} f(x,y,z)=\partial^2_{xz} f(x,y,z)=\partial^2_{yz} f(x,y,z)=0 \end{cases}$$

As the determinant of the Hessian matrix at the point $(-1,\frac{1}{2}0)$ is

$$\det\left(Hess_f(-1,\frac{1}{2},0)\right)=\begin{vmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{vmatrix}=-8<0.$$

So, the point $(-1,\frac{1}{2},0)$ is a saddle point of f.

2 (a) For the function f where $f(x,y,z)=x^2+4x+xy+2y^2-xz+z^2$. We find,

$$\begin{cases}\partial_x f(x,y,z)=0\\ \partial_y f(x,y,z)=0\\ \partial_z f(x,y,z)=0\end{cases} \Rightarrow \begin{cases}2x+y+4=0\\ 4y+x=0\\ -x+2z=0\end{cases} \Rightarrow \begin{cases}x=-\frac{16}{7}\\ y=\frac{4}{7}\\ z=-\frac{8}{7}\end{cases}$$

So, only the point $(-\frac{16}{7},\frac{4}{7},-\frac{8}{7})$ can be an extremum for f.

(b) Nature of the extremum. We have

$$\begin{cases}\partial^2_{x^2}f(-\frac{16}{7},\frac{4}{7},-\frac{8}{7})=2\\ \partial^2_{y^2}f(-\frac{16}{7},\frac{4}{7},-\frac{8}{7})=4\\ \partial^2_{z^2}f(-\frac{16}{7},\frac{4}{7},-\frac{8}{7})=2\\ \partial^2_{xy}f(-\frac{16}{7},\frac{4}{7},-\frac{8}{7})=1,\ \partial^2_{xz}f(-\frac{16}{7},\frac{4}{7},-\frac{8}{7})=-1 \text{ et } \partial^2_{yz}f(-\frac{16}{7},\frac{4}{7},-\frac{8}{7})=0\end{cases}$$

As the determinant of the Hessian matrix at point $(-\frac{16}{7},\frac{4}{7},-\frac{8}{7})$ is

$$\det\left(Hess_f(-\frac{16}{7},\frac{4}{7},-\frac{8}{7})\right)=\begin{vmatrix}2&1&-1\\1&4&0\\-1&0&2\end{vmatrix}=-8<0.$$

Then, the point $(-\frac{16}{7},\frac{4}{7},-\frac{8}{7})$ is a saddle point of f.

Exercise correction 10 ☆

Recalling the formula for the distance of a point (x,y) from a straight line of equation $ax+by+c=0$, we need to minimize the function

$$f(x,y)=\frac{ax+by+c}{\sqrt{a^2+b^2}}=\frac{\sqrt{3}x+y-4}{2},$$

in $\mathcal{D}(x,y)=\{(x,y)\in\mathbb{R}:\ x^2+y^2\leq 1\}$. Since $\sqrt{3}x+y-4\leq 0$ for any point of $\mathcal{D}$, it is then a question of minimizing the function

$$d(x,y)=\frac{4-\sqrt{3}x-y}{2}$$

The extrema are found either in $\mathcal{D}^\circ$ or on $\partial\mathcal{D}$. Since $\nabla d(x,y)=(\sqrt{3},-1)\neq(0,0)$, there are no extrema in $\mathcal{D}^\circ$. It is then necessary to look for them on $\partial\mathcal{D}$. To do this, we express $\partial\mathcal{D}$ in parametric form $x=\cos\alpha$ and $y=\sin\alpha$ for $\alpha\in[0,2\pi]$. Restricting d to this curve gives the function of a single variable

$$g(\alpha)=\frac{4-\sqrt{3}\cos\alpha-\sin\alpha}{2}$$

Since $g'(\alpha)=\sin(\alpha-\frac{\pi}{6})$ we have that $g'(\alpha)=0$ if and only if $\alpha=\frac{\pi}{6}$ or $\alpha=\frac{7\pi}{6}$. We must compare the value of g at these points with the value at 0 and at 2π. as,

$$g(0)=g(2\pi)=\frac{4-\sqrt{3}}{2},\ g(\frac{\pi}{6})=1,\ \text{and } g(\frac{7\pi}{6})=3.$$

We conclude that the point that minimizes the distance corresponds to , that is to say the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let us note that, once established that the extrema are on the curve of equation $x^2 + y^2 = 1$, this amounts to calculating the minimum of d under the constraint $h(x, y) = 0$ with $h(x, y) = x^2 + y^2 - 1$. We can then use the method of Lagrange multipliers. So, we introduce the Lagrangian $L(x, y, \lambda) = \dfrac{4 - \sqrt{3}x - y}{2} - \lambda(x^2 + y^2 - 1)$, we calculate its gradient

$$\nabla L(x, y, \lambda) = \begin{cases} -\dfrac{\sqrt{3}}{2} - 2\lambda x \\ -\dfrac{1}{2} - 2\lambda y \\ 1 - x^2 - y \end{cases}$$

As a result, we have $\nabla L(x, y, \lambda) = (0, 0, 0)$ if and only if $(x, y) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ or $(x, y) = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, and as $d(\frac{1}{2}, \frac{\sqrt{3}}{2}) < d(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, we conclude that $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ is the minimum.

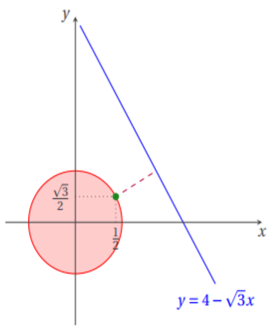

Figure 1:

Exercise correction 11 ☆

Let $g(x, y, z) = x + y - z$ and $h(x, y, z) = x^2 + 2z^2 - 6$. In this case, the Lagrangian is given by by $L(x, y, z, \lambda, \mu) = x + y - z + \lambda(x + yz) + \mu(x^2 + 2z^2 - 6)$. We are searching the critical points of Lagrangian L.

$$\nabla L(x, y, z, \lambda, \mu) = 0_{\mathbb{R}^5} \Rightarrow \begin{cases} 3 + \lambda + 2\mu x = 0 \\ 1 - \lambda = 0 \\ -3 - \lambda + 4\mu z = 0 \\ x + y - z = 0 \\ x^2 + 2z^2 - 6 = 0 \end{cases}$$

After solving the system, we obtain,

$$(x, y, z, \lambda, \mu) \in \{(-2, 3, 1, 1, 1), (2, -3, -1, 1, -1)\}$$

The associated Hessian matrix is,

$$\left(Hess_f(-1, \frac{3}{2}, 0)\right) = \begin{pmatrix} 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2x & 0 & 4z \\ 1 & 2x & 2\mu & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 4z & 0 & 0 & 4\mu \end{pmatrix}.$$

Consequently,

$$\det\left(Hess_f(-2,3,1,1,1)\right)=\begin{vmatrix} 0 & 0 & 1 & 1 & -1\\ 0 & 0 & -4 & 0 & 4\\ 1 & -4 & 2 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ -1 & 4 & 0 & 0 & 4 \end{vmatrix}=96>0.$$

So f admits a bound local minimum at $(-2,3,1)$. For the second critical point, we find,

$$\det\left(Hess_f(-2,3,1,1,1)\right)=\begin{vmatrix} 0 & 0 & 1 & 1 & -1\\ 0 & 0 & 4 & 0 & -4\\ 1 & 4 & -2 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ -1 & -4 & 0 & 0 & -4 \end{vmatrix}=-96<0.$$

Then, f admits a bound local maximum at point $(2,-3,-1,1,-1)$.

Supplementary exercise correction 1

Let the function f be of class C^1 on $\mathbb{R}^2$. We calculate the partial derivatives of the following functions.

1 (a)

$$\begin{array}{llll} F:\mathbb{R}^2 & \xrightarrow{g} & \mathbb{R}^2 & \xrightarrow{f} \mathbb{R}\\ (x,y) & \longmapsto & (y,x) & \longmapsto f(y,x). \end{array}$$

The Jacobian matrix of the first application g is as follows,

$$J_g=\begin{pmatrix} 0 & 1\\ 1 & 0 \end{pmatrix}$$

The Jacobian matrix associated with f is as follows,

$$J_f=\begin{pmatrix} \frac{\partial f}{\partial x}, & \frac{\partial f}{\partial y} \end{pmatrix}$$

Ce qui signifie que

$$\begin{aligned} J_F=\begin{pmatrix} \frac{\partial F}{\partial x}, & \frac{\partial F}{\partial y} \end{pmatrix} &= \begin{pmatrix} \frac{\partial f}{\partial x}, & \frac{\partial f}{\partial y} \end{pmatrix}\times\begin{pmatrix} 0 & 1\\ 1 & 0 \end{pmatrix}\\ &= \begin{pmatrix} \frac{\partial f}{\partial x}, & \frac{\partial f}{\partial y} \end{pmatrix}\begin{pmatrix} y, & x \end{pmatrix} \end{aligned}$$

Finally we find,

$$\frac{\partial F}{\partial x}=\frac{\partial f}{\partial y}(y,x)\quad\text{and,}\quad \frac{\partial F}{\partial y}=\frac{\partial f}{\partial x}(y,x).$$

(b) Application:

$$\frac{\partial F}{\partial x}=\frac{\partial f}{\partial y}(y,x)=2xy\quad\text{and,}\quad \frac{\partial F}{\partial y}=\frac{\partial f}{\partial x}(y,x)=3y^2+x^2.$$

2 (a)

$$\begin{array}{llll} F:\mathbb{R} & \xrightarrow{h} & \mathbb{R}^2 & \xrightarrow{f} \mathbb{R}\\ x & \longmapsto & (x,x) & \longmapsto f(x,x). \end{array}$$

The Jacobian matrix of the first application h is as follows,

$$J_h = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The Jacobian matrix associated with f is as follows,

$$J_f = \begin{pmatrix} \frac{\partial f}{\partial x}, & \frac{\partial f}{\partial y} \end{pmatrix}$$

This gives us,

$$F'(x) = J_F = \begin{pmatrix} \frac{\partial F}{\partial x}, & \frac{\partial F}{\partial y} \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial F}{\partial x}, & \frac{\partial F}{\partial y} \end{pmatrix} \begin{pmatrix} x, & x \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\partial f}{\partial x} + & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} x, & x \end{pmatrix}$$

(b) Application:

$$f(x,y) = x^3 + xy^2 \Rightarrow F(x) = f(x,x) = 2x^3 \Rightarrow F'(x) = (3x^2 + y^2 + 2xy)_{(x,x)} = 6x^2.$$

Exercise correction 12 ☆

Let's put $f(x,y) = xe^y + e^x \sin(2y)$.

1 Note that $(0,0)$ is a solution of the equation $f(x,y) = 0$, so we have

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = e^y + e^x \sin(2y) \\ \frac{\partial f}{\partial y}(x,y) = xe^y + 2e^x \cos(2y) \end{cases} \Rightarrow \begin{cases} \frac{\partial f}{\partial x}(0,0) = 1 \\ \frac{\partial f}{\partial y}(0,0) = 2 \end{cases}$$

Since $\frac{\partial f}{\partial y}(0,0) \neq 0$ there is one and only one function $y = \varphi(x)$ defined in the neighbourhood of 0 such that $f(x, \varphi(x)) = 0$.

2 When it was,

$$\varphi'(0) = -\frac{\frac{\partial f}{\partial x}(0,0)}{\frac{\partial f}{\partial y}(0,0)} = -\frac{1}{2}.$$

Then the equation of the tangent line at $x = 0$ is $y = -\frac{1}{2}x$.

3 We have,

$$\lim_{(x,y)\to(0,0)} = \lim_{x\to 0} \frac{\varphi(x)}{x} = \lim_{x\to 0} \frac{\varphi(x) - \varphi(0)}{x - 0} = \varphi'(0) = -\frac{1}{2}.$$

Supplementary exercise correction 2 ☆

We put $f(x,y) = 2x^3y + 2x^2 + y^2$.

1 Note that $(1,1)$ is a solution of the equation $f(x,y) = 0$. Then we have,

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 6x^2y + 4x \\ \frac{\partial f}{\partial y}(x,y) = 2x^3 + 2y \end{cases} \Rightarrow \begin{cases} \frac{\partial f}{\partial x}(1,1) = 10 \\ \frac{\partial f}{\partial y}(1,1) = 4 \end{cases}$$

Since $\frac{\partial f}{\partial y}(1,1) \neq 0$ there is one and only one function $y = \varphi(x)$ defined in the neighbourhood of 1 such that $f(x, \varphi(x)) = 1$.

2 When it was,

$$\varphi'(1) = -\frac{\frac{\partial f}{\partial x}(1,1)}{\frac{\partial f}{\partial y}(1,1)} = -\frac{5}{2}.$$

So, the equation of the tangent line to φ at $x=1$ is,

$$y = -\frac{5}{2}(x-1)+1 = -\frac{5}{2}x+\frac{7}{2}.$$

3 The Taylor expansion of φ to order 1 at point 1 is,

$$\varphi(x) = \varphi(1)+\varphi'(1)(x-1)+o(x) = \frac{7}{2}-\frac{5}{2}x+o(x).$$

Supplementary exercise correction 3 ★★

1 Soit f la fonction définie sur $\mathbb{R}^3$ par $f(x,y,z) = x^3+4xy+z^2-3yz^2-3$. Alors, f possède des dérivées partielles continues,

$$\frac{\partial f}{\partial x}(x,y,z) = 3x^2+4y \;\; \frac{\partial f}{\partial y}(x,y,z) = 4x-3z^2 \text{ et } \frac{\partial f}{\partial z}(x,y,z) = 2z-6yz,$$

et au point $(1,1,1)$ on a,

$$\frac{\partial f}{\partial x}(1,1,1) = 7 \;\; \frac{\partial f}{\partial y}(1,1,1) = 1 \text{ et } \frac{\partial f}{\partial z}(1,1,1) = -4,$$

Puisque $f(1,1,1)=0$ et $\frac{\partial f}{\partial z}(x,y,z) \neq 0$, le théorème des fonctions implicites nous assure qu'on peut exprimer z en fonction de (x,y) dans un voisinage de $(1,1,1)$.

2 D'après le théorème des fonctions implicites, on obtient,

$$\frac{\partial z}{\partial x}(1,1) = -\frac{\frac{\partial f}{\partial x}(1,1,1)}{\frac{\partial f}{\partial z}(x,y,z)} = \frac{7}{4} \text{ et } \frac{\partial z}{\partial y}(1,1) = -\frac{\frac{\partial f}{\partial y}(1,1,1)}{\frac{\partial f}{\partial z}(x,y,z)} = \frac{1}{4}.$$

Supplementary exercise correction 4 ★

1 Let f be the function defined on $\mathbb{R}^3$ by $f(x,y,z) = x^3+4xy+z^2-3yz^2-3$. Then, f has continuous partial derivatives,

$$\frac{\partial f}{\partial x}(x,y,z) = 3x^2+4y \;\; \frac{\partial f}{\partial y}(x,y,z) = 4x-3z^2 \text{ and } \frac{\partial f}{\partial z}(x,y,z) = 2z-6yz,$$

and at the point $(1,1,1)$ we have,

$$\frac{\partial f}{\partial x}(1,1,1) = 7 \;\; \frac{\partial f}{\partial y}(1,1,1) = 1 \text{ and } \frac{\partial f}{\partial z}(1,1,1) = -4,$$

Since $f(1,1,1)=0$ and $\frac{\partial f}{\partial z}(x,y,z) \neq 0$, the implicit function theorem assures us that we can express z as a function of (x,y) in a neighbourhood $(1,1,1)$.

2 According to the implicit function theorem, we obtain,

$$\frac{\partial z}{\partial x}(1,1) = -\frac{\frac{\partial f}{\partial x}(1,1,1)}{\frac{\partial f}{\partial z}(x,y,z)} = \frac{7}{4} \text{ and } \frac{\partial z}{\partial y}(1,1) = -\frac{\frac{\partial f}{\partial y}(1,1,1)}{\frac{\partial f}{\partial z}(x,y,z)} = \frac{1}{4}.$$