University of M'sila	**Correction of Series N : 03** **- Differentiability-**	Faculty of Mathematics and Computer Science
L 2 Mathematics		Module: Analysis 4

Exercise correction 1 ☆

Let's calculate the first partial derivatives of the following functions.

1 If $f(x,y)=x^2+3xy^2-4y^5$. So, we have

$$\frac{\partial f}{\partial x}(x,y)=2x+3y^2, \text{ and } \frac{\partial f}{\partial y}(x,y)=6xy-20y^4.$$

2 For $f(x,y)=y\cos(e^{xy+3y})$. So we find,

$$\frac{\partial f}{\partial x}(x,y)=-y^2\sin(e^{xy+3y}), \text{ and } \frac{\partial f}{\partial y}(x,y)=\cos(e^{xy+3y})-y(x+3)\sin(e^{xy+3y}).$$

3 The first partial derivatives of $f(x,y,z)=x\sin(yz)-\ln(3-e^{x+y})$ are,
$\frac{\partial f}{\partial x}(x,y)=\sin(yz)-\frac{-e^{x+y}}{3-e^{x+y}}$, $\frac{\partial f}{\partial y}(x,y)=xz\cos(yz)-\frac{-e^{x+y}}{3-e^{x+y}}$ and $\frac{\partial f}{\partial z}(x,y)=xy\cos(yz)$.

Exercise correction 2 ☆

f is a function defined on $\mathbb{R}^2$ by

$$f(x,y)=\begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) & : (x,y)\neq(0,0)\\ 0 & : (x,y)=(0,0).\end{cases}$$

1 We study the continuity of f on $\mathbb{R}^2$.
f is continuous on $\mathbb{R}^2-\{(0,0)\}$. Because the functions $(x,y)\longmapsto xy$, $(x,y)\longmapsto x^2+y^2$ and $(x,y)\longmapsto x^2-y^2$ (are polynomials). So, it remains to prove the continuity in point $(0,0)$.
So, using polar coordinates, let's put,

$$\begin{cases} x=r\cos\theta,\\ x=r\sin\theta, \quad \text{where } r>0,\ \theta\in[0,2\pi[.\end{cases}$$

So, we find,

$$f(r\cos\theta,r\sin\theta)=r^2\sin\theta\cos\theta\cos 2\theta.$$

As a result, $\lim\limits_{(x,y)\to(0,0)}\left|\frac{xy(x^2-y^2)}{x^2+y^2}\right|=\lim\limits_{r\to 0}|(r^2\sin\theta\cos\theta)\cos 2\theta|\leq\lim\limits_{r\to 0}r^2=0$. Therefore,

$$\lim_{(x,y)\to(0,0)}f(x,y)=0=f(0,0).$$

Hence the continuity of f on $\mathbb{R}^2$.

2 On calcule $\nabla f(x,y)$.

(a) For $(x,y) \neq (0,0)$, we get

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} \frac{yx^4 - y^5 + 4x^2y^3}{(x^2+y^2)^2} \\ \frac{xy^4 - x^5 + 4x^3y^2}{(x^2+y^2)^2} \end{pmatrix}$$

.

(b) For $(x,y) = (0,0)$, we have,

$$\nabla f(0,0) = \begin{pmatrix} \frac{\partial f}{\partial x}(0,0) \\ \frac{\partial f}{\partial y}(0,0) \end{pmatrix} = \begin{pmatrix} \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

3 From the above we have,

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \begin{cases} \frac{4xy^3(x^2-3y^2)}{(x^2+y^2)^3} & : (x,y) \neq (0,0) \\ \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(h,0) - \frac{\partial f}{\partial x}(0,0)}{h} = 0 & : (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \begin{cases} \frac{4x^3y(3x^2-y^2)}{(x^2+y^2)^3} & : (x,y) \neq (0,0) \\ \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(0,h) - \frac{\partial f}{\partial y}(0,0)}{h} = 0 & : (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \begin{cases} \frac{x^6 + 9x^4y^2 - 9x^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} & : (x,y) \neq (0,0) \\ \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = 1 & : (x,y) = (0,0) \end{cases}$$

et

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \begin{cases} \frac{x^6 + 9x^4y^2 - 9x^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} & : (x,y) \neq (0,0) \\ \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(0,h) - \frac{\partial f}{\partial y}(0,0)}{h} = -1 & : (x,y) = (0,0) \end{cases}$$

4 Since $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$ the Schwarz theorem allows us to conclude that the crossed second derivatives $\frac{\partial^2 f}{\partial x \partial y}(x,y)$ and $\frac{\partial^2 f}{\partial y \partial x}(x,y)$ are not continuous at $(0,0)$.

Exercise correction 3 ☆

The function f defined on $\mathbb{R}^2$ by $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$

1 The function f is continuous on $\mathbb{R}^2 - \{(0,0)\}$. Because it is a quotient of polynomial functions whose denominator does not cancel (vanish). It is also continuous at $(0,0)$, in fact, using polar coordinates, we find

$$\lim_{(x,y)\to(0,0)} |f(x,y)| = \lim_{r\to 0} |r^2 \sin^2\theta \cos^2\theta| \leq \lim_{r\to 0} r^2 = 0 = f(0,0).$$

So f is continuous on $\mathbb{R}^2$.

2 On calcule $\nabla f(x,y)$.

$$\frac{\partial f}{\partial x}(x,y)=\begin{cases}\dfrac{2xy^4}{(x^2+y^2)^2} & : \ (x,y)\neq(0,0)\\ \lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x}=0 & : \ (x,y)=(0,0)\end{cases}$$

$$\frac{\partial f}{\partial y}(x,y)=\begin{cases}\dfrac{2x^4y}{(x^2+y^2)^2} & : \ (x,y)\neq(0,0)\\ \lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y}=0 & : \ (x,y)=(0,0)\end{cases}$$

3 We calculate $\nabla f(x,y)$.

(a) For $(x,y)\neq(0,0)$, we have

$$\nabla f(x,y)=\begin{pmatrix}\dfrac{\partial f}{\partial x}(x,y)\\ \dfrac{\partial f}{\partial y}(x,y)\end{pmatrix}=\begin{pmatrix}\dfrac{2xy^4}{(x^2+y^2)^2}\\ \dfrac{2x^4y}{(x^2+y^2)^2}\end{pmatrix}$$

.

(b) For $(x,y)=(0,0)$, we get

$$\nabla f(0,0)=\begin{pmatrix}\dfrac{\partial f}{\partial x}(0,0)\\ \dfrac{\partial f}{\partial y}(0,0)\end{pmatrix}=\begin{pmatrix}\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x}\\ \lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

. In other words,

$$\nabla f(x,y)=\begin{cases}\dfrac{\partial f}{\partial x}(x,y)\vec{i}+\dfrac{\partial f}{\partial y}(x,y)\vec{j}, & : \ (x,y)\neq(0,0)\\ \dfrac{\partial f}{\partial x}(0,0)\vec{i}+\dfrac{\partial f}{\partial y}(0,0)\vec{j}, & : \ (x,y)=(0,0)\end{cases}$$

$$=\begin{cases}\dfrac{2xy^4}{(x^2+y^2)^2}(y^3\vec{i}+x^3\vec{j}), & : \ (x,y)\neq(0,0)\\ 0\vec{i}+0\vec{i}=\vec{0}, & : \ (x,y)=(0,0)\end{cases}$$

4 $f\in C^1(\mathbb{R}^2)$?
Using the inequality

$$2|xy|\leq x^2+y^2,$$

we can make the following increases

$$\left|\frac{\partial f}{\partial x}(x,y)\right|\leq\frac{|y^3|}{(x^2+y^2)^2}=\frac{|y||y^2|}{(x^2+y^2)^2}\leq|y|.$$

By passing to the limit we obtain,

$$\lim_{(x,y)\to(0,0)}\frac{\partial f}{\partial x}(x,y)=0=\frac{\partial f}{\partial x}(0,0).$$

So the partial derivative $\dfrac{\partial f}{\partial x}(x,y)$ is continuous at $(0,0)$.

On $\mathbb{R}^2-\{(0,0)\}$ the function $\dfrac{\partial f}{\partial x}(x,y)$ is continuous as a quotient of polynomial functions

whose denominator does not cancel. That is to say $\frac{\partial f}{\partial x}(x,y)$ is continuous on $\mathbb{R}^2$.
Similarly, we show that $\frac{\partial f}{\partial y}(x,y)$ is continuous on $\mathbb{R}^2$ and therefore $f \in C^1(\mathbb{R}^2)$.

5 $f \in C^1(\mathbb{R}^2)$, therefore, it is differentiable on $\mathbb{R}^2$.

Exercise correction 4 ☆

Let's use the definition to calculate the differentiable of f at the indicated point.
An application $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(x_0, y_0) \in \mathbb{R}^2$ if and only if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) - \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0.$$

1 For $f(x,y) = xy - 3x^2$, en $(1,2)$, we have, $f(1,2) = -1$, $\frac{\partial f}{\partial x}(x,y) = y - 6x$, $\frac{\partial f}{\partial x}(1,2) = -4$ $\frac{\partial f}{\partial y}(x,y) = x$ and $\frac{\partial f}{\partial y}(1,2) = 1$. So,

$$\frac{f(x,y) - f(2,1) - \frac{\partial f}{\partial x}(1,2)(x-1) - \frac{\partial f}{\partial y}(1,2)(y-2)}{\sqrt{(x-1)^2+(y-2)^2}} = \frac{xy - 3x^2 + 1 + 4(x-1) - (y-2)}{\sqrt{(x-1)^2+(y-2)^2}}.$$

With the change of variables $x = 1 + r\cos\theta$, $y = 2 + r\sin\theta$, this ratio is rewritten as

$$\frac{xy - 3x^2 + 1 + 4(x-1) - (y-2)}{\sqrt{(x-1)^2+(y-2)^2}} = r\cos\theta(\sin\theta - 3\cos\theta).$$

Then we have,

$$\left|\frac{xy - 3x^2 + 1 + 4(x-1) - (y-2)}{\sqrt{(x-1)^2+(y-2)^2}}\right| \le 4r \underset{r\longrightarrow 0}{\longrightarrow} 0\,.$$

Hence,

$$\lim_{(x,y)\to(1,2)} \left|\frac{xy - 3x^2 + 1 + 4(x-1) - (y-2)}{\sqrt{(x-1)^2+(y-2)^2}}\right| = 0.$$

The function f is therefore differentiable in $(1,2)$.

$$df(1,2) = -4(x-1) + (y-2) = -4x + y + 2.$$

2 If $f(x,y) = xy - 2y^2$, $(2,1)$, we get $f(2,1) = -1$, $\frac{\partial f}{\partial x}(x,y) = y$, $\frac{\partial f}{\partial x}(2,1) = 1$ $\frac{\partial f}{\partial y}(x,y) = x - 6y$ and $\frac{\partial f}{\partial y}(2,1) = -4$. So,

$$\frac{f(x,y) - f(2,1) - \frac{\partial f}{\partial x}(2,1)(x-2) - \frac{\partial f}{\partial y}(2,1)(y-1)}{\sqrt{(x-2)^2+(y-1)^2}} = \frac{xy - 3y^2 + 1 - (x-2) + 4(y-1)}{\sqrt{(x-2)^2+(y-1)^2}}.$$

With the change of variables $x = 4 + r\cos\theta$, $y = 1 + r\sin\theta$, this ratio is rewritten as

$$\frac{xy - 3y^2 + 1 - (x-2) + 4(y-1)}{\sqrt{(x-2)^2+(y-1)^2}} = rsin\theta(\cos\theta - 3\sin\theta).$$

As a consequence,

$$\left|\frac{xy-3y^2+1-(x-2)+4(y-1)}{\sqrt{(x-2)^2+(y-1)^2}}\right| \leq 4r \underset{r\longrightarrow 0}{\longrightarrow} 0\,.$$

Hence,

$$\lim_{(x,y)\to(1,2)}\left|\frac{xy-3y^2+1-(x-2)+4(y-1)}{\sqrt{(x-2)^2+(y-1)^2}}\right| = 0.$$

Therefore f is differentiable at $(2,1)$ and so we have,

$$df(2,1) = (x-2)-4(y-2) = x-4y+6.$$

3 $f(x,y) = y\sqrt{x}$, en $(4,1)$.

$$\begin{cases} f(x,y) = y\sqrt{x} \\ \dfrac{\partial f}{\partial x}(x,y) = \dfrac{y}{2\sqrt{x}} \\ \dfrac{\partial f}{\partial y}(x,y) = \sqrt{x} \end{cases} \Rightarrow \begin{cases} f(4,1) = 2 \\ \dfrac{\partial f}{\partial x}(4,1) = \dfrac{y}{2\sqrt{x}} = \dfrac{1}{4} \\ \dfrac{\partial f}{\partial y}(4,1) = 2 \end{cases}$$

So, we see that,

$$\frac{f(x,y)-f(4,1)-\dfrac{\partial f}{\partial x}(4,1)(x-4)-\dfrac{\partial f}{\partial y}(4,1)(y-1)}{\sqrt{(x-4)^2+(y-1)^2}} = \frac{y\sqrt{x}-2-\frac{1}{4}(x-4)-2(y-1)}{\sqrt{(x-4)^2+(y-1)^2}}.$$

With the change of variables $x = 4+r\cos\theta,\ y = 1+r\sin\theta$, this ratio is rewritten as

$$\begin{aligned} \frac{y\sqrt{x}-2-\frac{1}{4}(x-4)-2(y-1)}{\sqrt{(x-4)^2+(y-1)^2}} &= \frac{(1+r\sin\theta)\sqrt{4+r\cos\theta}-2-\frac{r\cos\theta}{4}-2r\sin\theta}{r} \\ &= \frac{2(1+r\sin\theta)\sqrt{1+\frac{r\cos\theta}{4}}-2-\frac{r\cos\theta}{4}-2r\sin\theta}{r} \\ &= \frac{2(1+r\sin\theta)(1+\frac{r\cos\theta}{8}+o(r))-2-\frac{r\cos\theta}{4}-2r\sin\theta}{r} \\ &= \frac{r}{4}\sin\theta\cos\theta+\frac{o(r)}{r} \underset{r\longrightarrow 0}{\longrightarrow} 0, \end{aligned}$$

where we used the approximation $\sqrt{1+t} \simeq 1+\dfrac{t}{2}+o(t)$, when $t \simeq 0$. We therefore obtain,

$$\lim_{(x,y)\to(4,1)} \frac{f(x,y)-f(4,1)-\dfrac{\partial f}{\partial x}(4,1)(x-4)-\dfrac{\partial f}{\partial y}(4,1)(y-1)}{\sqrt{(x-4)^2+(y-1)^2}} = 0.$$

Therefore f is differentiable at $(4,1)$ and so we have,

$$df(4,1) = \frac{1}{4}(x-4)+2(y-1) = \frac{1}{4}x+2y-3.$$

Exercise correction 5 ☆

Let the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & : \ (x,y)\neq(0,0) \\ 0 & : \ (x,y)=(0,0) \end{cases}$

1 We show that f is derivative on $\mathbb{R}^2$.

(a) If $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$, the function f admits partial derivatives, (because f is a quotient of (polynomial) functions which admit partial derivatives where the denominator does not cancel, and we have,

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = \dfrac{x^4 + 3x^2y^2 + 2xy^3}{(x^2+y^2)^2} \\ \dfrac{\partial f}{\partial y}(x,y) = -\dfrac{y^4 + 3y^2x^2 + 2yx^3}{(x^2+y^2)^2} \end{cases}$$

(b) For $(x,y) = (0,0)$, we have,

$$\begin{cases} \dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{x\to 0} \dfrac{f(x,0) - f(0,0)}{x} = 1 \\ \dfrac{\partial f}{\partial y}(0,0) = \lim\limits_{y\to 0} \dfrac{f(0,y) - f(0,0)}{y} = -1. \end{cases}$$

Hence $\dfrac{\partial f}{\partial x}(x,y)$ and $\dfrac{\partial f}{\partial y}(x,y)$ exist at every point of $\mathbb{R}^2$.

2 We study the continuity of f on $\mathbb{R}^2$.

(a) For $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$, the function f is continuous, (because f is a quotient of continuous functions (polynomial), where the denominator does not cancel.

(b) If $(x,y) = (0,0)$, we use polar coordinates, we put

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \end{cases} \qquad \text{where } \ r > 0, \ \ \theta \in [0, 2\pi[.$$

Then, $f(r\cos\theta, r\sin\theta) = \dfrac{r^3(\cos^3\theta - \sin^3\theta)}{r^2\cos^2\theta + r^2\sin^2\theta} = r(\cos^3\theta - \sin^3\theta)$.
So, $|f(r\cos\theta, r\sin\theta)| = r|\cos^3\theta - \sin^3\theta| \leq 2$.

Thus, $\lim\limits_{r\to 0} f(r\cos\theta, r\sin\theta) = 0$, then, we have,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0).$$

Consequently, the function f is continuous on $\mathbb{R}^2$.

3 The differentiability of f on $\mathbb{R}^2$.

(a) For $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$, the function f is differentiable, (because f is a quotient of differential functions (polynomial), where the denominator does not cancel.

(b) If $(x,y) = (0,0)$, we have $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ exist, f is differentiable at $(0,0)$ if and only if

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - h\dfrac{\partial f}{\partial x}(0,0) - k\dfrac{\partial f}{\partial y}(0,0)}{\sqrt{h^2+k^2}} = 0.$$

We have,

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-h\frac{\partial f}{\partial x}(0,0)-k\frac{\partial f}{\partial y}(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{h^3-k^3}{(h^2+k^2)^{\frac{3}{2}}} + \lim_{(h,k)\to(0,0)} \frac{h-k}{\sqrt{h^2+k^2}},$$

changing to polar coordinates, $\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \end{cases}$ where $r>0$, $\theta\in[0,2\pi[$.
Then, $f(r\cos\theta, r\sin\theta) = \cos^3\theta - \sin^3\theta - \cos\theta + \sin\theta$.
So, $\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = \cos^3\theta - \sin^3\theta - \cos\theta + \sin\theta$, is a limit that depends on θ, then f is not differentiable at $(0,0)$. Consequently, the function f is only differentiable on $\mathbb{R}^2 - \{(0,0)\}$.

Exercise correction 6

The function $f:\mathbb{R}^3\to\mathbb{R}$ defined by, $f(x,y,z) = xy+yz+zx$, is in $C^\infty(\mathbb{R}^3)$, because it is polynomial, and it's differential given by,

$$df = (y+z)dx + (x+z)dy + (x+y)dz.$$

Exercise correction 7

1 We calculate the directional derivative of the function $f(x,y) = 3x^2y - 4xy$, at point $(1,2)$, along the direction $\vec{v} = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.
According to the definition we have,

$$D_v f(1,1) = \lim_{t\to 0} \frac{f(1+\sqrt{3}2t, 2-\frac{1}{2}t) - f(1,2)}{t} = \lim_{t\to 0}(-\frac{9}{8}t^2 + \frac{9-\sqrt{3}}{2}t + \frac{1+4\sqrt{3}}{2}) = \frac{1+4\sqrt{3}}{2}$$

2 The gradient of f is the vector

$$\nabla f(x,y) = \Big(6xy - 4y, 3x^2 - 4x\Big).$$

The gradient of f at the point $(1,2)$ is,

$$\nabla f(1,2) = \Big(4, -1\Big).$$

Finally, we find

$$\nabla f(1,2).v = \frac{\partial f}{\partial x}(1,2).v_1 + \frac{\partial f}{\partial y}(1,2).v_2 = 4\frac{\sqrt{3}}{2} - (-\frac{1}{2}) = D_v f(1,2).$$

Exercise correction 8

We give an approximate value of $f(2.2, 4.9)$. Knowing that the function $f:\mathbb{R}^2\to\mathbb{R}$ is differentiable and that

$$f(2,5) = 6,\ \partial_x f(2,5) = 1,\ \text{and}\ \ \partial_y f(2,5) = -1$$

Let $L_{(2,5)}(x,y) = 0$ be the equation of the tangent plane to f at point $(2,5)$. Then we have the approximation

$$f(2.2, 4.9) \simeq L_{(2,5)}(2.2, 4.9),$$

and as,

$$L_{(2,5)}(x,y) = f(2,5) + (x-2)\frac{\partial f}{\partial x}(2,5) + (y-5)\frac{\partial f}{\partial y}(2,5) = x - y + 9.$$

So, we see that $f(2.2, 4.9) \simeq 6.3$.

Exercise correction 9 ☆☆

We calculate the partial derivatives of order one of the function f.

$$f(u,v) = u + v, \text{ where } u(x,y) = e^{x+y} \text{ and } v(x,y) = x^2 + y^2.$$

We calculate the partial derivatives of order one of the function f.

1^{st} Method: We use the following formula for the partial derivative of a composite function

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u}.\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}.\dfrac{\partial v}{\partial x} \\ \\ \dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial u}.\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}.\dfrac{\partial v}{\partial y} \end{cases} \quad \text{and}$$

So we get, $\dfrac{\partial f}{\partial x} = e^{x+y} + 2x$, $\dfrac{\partial f}{\partial x} = e^{x+y} + 2y$.

2^{th} Method: Let $F = f \circ g$. We therefore have,

$$\begin{array}{llllll} F : \mathbb{R}^2 & \xrightarrow{g} & \mathbb{R}^2 & - - \overset{f}{-} \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & (u(x,y), v(x,y)) & \longmapsto & f(u,v). \end{array}$$

The Jacobian matrix of the first application g is as follows,

$$J_g = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} e^{x+y} & e^{x+y} \\ 2x & 2y \end{pmatrix}$$

The Jacobian matrix associated with f is

$$J_f = \begin{pmatrix} \dfrac{\partial f}{\partial u}, & \dfrac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} 1, & 1 \end{pmatrix}$$

Which means that,,

$$J_F = \begin{pmatrix} \dfrac{\partial F}{\partial x}, & \dfrac{\partial F}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial u} & , \dfrac{\partial f}{\partial v} \end{pmatrix} \times \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\partial f}{\partial u}.\dfrac{\partial u}{\partial x}.\dfrac{\partial f}{\partial v}.\dfrac{\partial v}{\partial x}, & \dfrac{\partial f}{\partial u}.\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}.\dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} e^{x+y} + 2x, & e^{x+y} + 2y \end{pmatrix}$$

Exercise correction 10 ☆

We give the limited expansion of order 2 in $(0,0)$ of the following functions,

1 We use the usual limited developments in the neighbourhood of 0 as follows,

$$\cos t = 1 - \frac{1}{2}t^2 + o(t^2), \ \frac{1}{1-t} = 1 + t + t^2 + o(t^2), \text{ and } e^t = 1 + t + \frac{1}{2}t^2 + o(t^2).$$

So, in the neighbourhood of 0 we see that

$$\frac{1}{1-\frac{1}{2}y^2+o(y^2)} = 1+\frac{1}{2}y^2+o(y^2).$$

Therefore, the limited expansion of f to order 2 in $(0,0)$ is given by

$$\begin{aligned} f(x,y) &= \frac{\cos x}{\cos y} = \left(1-\frac{1}{2}x^2+o(x^2)\right)\left(1+\frac{1}{2}y^2+o(y^2)\right) \\ &= 1-\frac{1}{2}x^2+\frac{1}{2}y^2+o(x^2+y^2). \end{aligned}$$

2 On a au voisinage de 0,

$$\begin{aligned} \frac{1}{2+y} &= \frac{1}{2}\frac{1}{1+\frac{y}{2}} \\ &= \frac{1}{2}\left(1-\frac{1}{2}y+\frac{1}{4}y^2+o(x^2+y^2)\right). \end{aligned}$$

Consequently, we have

$$\begin{aligned} f(x,y) &= \frac{e^x}{2+y} = \frac{1}{2}\left(1+x+\frac{1}{2}x^2+o(x^2)\right)\left(1-\frac{1}{2}y+\frac{1}{4}y^2+o(y^2)\right) \\ &= \frac{1}{2}+\frac{1}{2}x-\frac{1}{4}y+\frac{1}{2}x^2+\frac{1}{8}y^2-\frac{1}{4}xy+o(x^2+y^2). \end{aligned}$$

3 If $x,\ y$ is in the neighbourhood of 0, then $x+y$ is also in the neighbourhood of 0, and we have

$$\cos(x+y) = 1-\frac{1}{2}(x^2+y^2)+o(x^2+y^2).$$

Hence,

$$\begin{aligned} e^{\cos(x+y)} &= e\times\left(e^{-\frac{1}{2}(x^2+y^2)+o(x^2+y^2)}\right) \\ &= e\times\left(1-\frac{1}{2}(x^2+y^2)+o(x^2+y^2)\right). \end{aligned}$$

4

$$\begin{aligned} e^y\cos x &= \left(1+y+\frac{1}{2}y^2+o(y^2)\right)\times\left(1-\frac{1}{2}x^2+o(x^2)\right) \\ &= 1+y-\frac{1}{2}x^2+\frac{1}{2}y^2+o(x^2+y^2). \end{aligned}$$

Exercise correction 11 ☆

1 We want to show that

$$f(x,y) = \frac{x^5}{(y-x^2)^2+x^4} \underset{\|(x,y)\|=\sqrt{x^2+y^2}\to 0}{\longrightarrow} f(0,0)=0.$$

Let $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Since, $(y-x^2)^2 + x^4 \geq 0$ and $(y-x^2)^2 + x^4 \geq 0$. So,

$$\begin{aligned} 0 \leq |f(x,y)| &= \left|\frac{x^5}{(y-x^2)^2+x^4}\right| = \left|\frac{|x||x|^4}{(y-x^2)^2+x^4}\right| \\ &\leq \left|\frac{|x||x|^4}{x^4}\right| = |x| \leq \sqrt{x^2+y^2} = \|(x,y)\|. \end{aligned}$$

Ainsi, par Théorème d'encadrement

$$f(x,y) \underset{\|(x,y)\|=\sqrt{x^2+y^2}\to 0}{\longrightarrow} f(0,0) = 0.$$

2 Let $h = (h_1, h_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$. So at least one of the numbers $h_1,\ h_2$ is non-zero. Let $t \in \mathbb{R}^* be$. Then we have

$$\begin{aligned} \frac{f((0,0)+t(h_1,h_2)) - f(0,0)}{t} &= \frac{1}{t}\frac{t^5h_1^5}{(th_2 - t^2h_1^2)^2 + t^4h_1^4} \\ &= \frac{t^2h_1^5}{h_2^2 - 2th_1^2h_2 + 2t^2h_1^4} \end{aligned}$$

There are two possible cases.

(a) If $h_2 \neq 0$, therefore,

$$\frac{f((0,0)+t(h_1,h_2)) - f(0,0)}{t} \underset{t\to 0}{\sim} \frac{t^2h_1^5}{h_2^2} \underset{t\to 0}{\longrightarrow} 0.$$

So f is differentiable at the point $(0,0)$ along the vector $h = (h_1, h_2)$ and

$$D_h f(0,0) = 0.$$

(b) If $h_2 = 0$, then, as, $h = (h_1, h_2) \neq (0,0)$ this means that $h_1 neq 0$, and we have

$$\frac{f((0,0)+t(h_1,h_2)) - f(0,0)}{t} = \frac{t^2h_1^5}{2t^2h_1^4} = \frac{h_1}{2} \underset{t\to 0}{\longrightarrow} \frac{h_1}{2}.$$

So f is differentiable at $(0,0)$ following the vector $h = (h_1, h_2)$ and

$$D_h f(0,0) = \frac{h_1}{2}.$$

Supplementary exercise correction 1

Let the function f be of class C^1 on $\mathbb{R}^2$.We calculate the partial derivatives of the following functions.

1 (a)

$$\begin{array}{rcccl} F : \mathbb{R}^2 & \xrightarrow{g} & \mathbb{R}^2 & \overset{f}{- - - \longrightarrow} & \mathbb{R} \\ (x,y) & \longmapsto & (y,x) & \longmapsto & f(y,x). \end{array}$$

The Jacobian matrix of the first application g is as follows,

$$J_g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The Jacobian matrix associated with f is as follows,

$$J_f = \begin{pmatrix} \frac{\partial f}{\partial x}, & \frac{\partial f}{\partial y} \end{pmatrix}$$

Ce qui signifie que

$$\begin{aligned} J_F = \begin{pmatrix} \frac{\partial F}{\partial x}, & \frac{\partial F}{\partial y} \end{pmatrix} &= \begin{pmatrix} \frac{\partial f}{\partial x}, & \frac{\partial f}{\partial y} \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial f}{\partial x}, & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} y, & x \end{pmatrix} \end{aligned}$$

Finally we find,

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial y}(y,x) \quad \text{and,} \quad \frac{\partial F}{\partial y} = \frac{\partial f}{\partial x}(y,x).$$

(b) Application:

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial y}(y,x) = 2xy \quad \text{and,} \quad \frac{\partial F}{\partial y} = \frac{\partial f}{\partial x}(y,x) = 3y^2 + x^2.$$

2 (a)

$$\begin{array}{rcccl} F : \mathbb{R} & \xrightarrow{h} & \mathbb{R}^2 & \overset{f}{- - - \longrightarrow} & \mathbb{R} \\ x & \longmapsto & (x,x) & \longmapsto & f(x,x). \end{array}$$

The Jacobian matrix of the first application h is as follows,

$$J_h = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The Jacobian matrix associated with f is as follows,

$$J_f = \begin{pmatrix} \frac{\partial f}{\partial x}, & \frac{\partial f}{\partial y} \end{pmatrix}$$

This gives us,

$$\begin{aligned} F'(x) = J_F = \begin{pmatrix} \frac{\partial F}{\partial x}, & \frac{\partial F}{\partial y} \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} \frac{\partial F}{\partial x}, & \frac{\partial F}{\partial y} \end{pmatrix} \begin{pmatrix} x, & x \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial f}{\partial x} + & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} x, & x \end{pmatrix} \end{aligned}$$

(b) Application:

$$f(x,y) = x^3 + xy^2 \Rightarrow F(x) = f(x,x) = 2x^3 \Rightarrow F'(x) = (3x^2 + y^2 + 2xy)_{(x,x)} = 6x^2.$$

Supplementary exercise correction 2

The function f defined on $\mathbb{R}^3$ by $f(x,y,z) = x + y^2 + z^3$. Calculate the directional derivative at point $(1,1,1)$ according to the direction of the following vectors,

1 According to $\overrightarrow{v} = (2,1,3)$, we have

$$\nabla f(x,y,z) = \Big(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z)\Big) = (1,2y,3z^2).$$

$$\nabla f(1,1,1) = \Big(\frac{\partial f}{\partial x}(1,1,1), \frac{\partial f}{\partial y}(1,1,1), \frac{\partial f}{\partial z}(1,1,1)\Big) = (1,2,3).$$

As a result, we get,

$$d_v f(1,1,1) = \nabla f(1,1,1).\frac{\overrightarrow{v}}{\| \overrightarrow{v} \|} = (1,2,3).\frac{(2,1,3)}{\sqrt{14}} = \frac{12}{\sqrt{14}}.$$

2 If $\overrightarrow{v} = (1,-1,1)$, we obtain

$$d_v f(1,1,1) = \nabla f(1,1,1).\frac{\overrightarrow{v}}{\| \overrightarrow{v} \|} = (1,2,3).\frac{(1,-1,1)}{\sqrt{3}} = \frac{2}{\sqrt{3}}.$$

Exercise correction 12 ☆

Let's put $f(x,y) = xe^y + e^x \sin(2y)$.

1 Note that $(0,0)$ is a solution of the equation $f(x,y) = 0$, so we have

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = e^y + e^x \sin(2y) \\ \dfrac{\partial f}{\partial y}(x,y) = xe^y + 2e^x \cos(2y) \end{cases} \Rightarrow \begin{cases} \dfrac{\partial f}{\partial x}(0,0) = 1 \\ \dfrac{\partial f}{\partial y}(0,0) = 2 \end{cases}$$

Since $\dfrac{\partial f}{\partial y}(0,0) \neq 0$ there is one and only one function $y = \varphi(x)$ defined in the neighbourhood of 0 such that $f(x,\varphi(x)) = 0$.

2 When it was,

$$\varphi'(0) = -\frac{\dfrac{\partial f}{\partial x}(0,0)}{\dfrac{\partial f}{\partial y}(0,0)} = -\frac{1}{2}.$$

Then the equation of the tangent line at $x = 0$ is $y = -\dfrac{1}{2}x$.

3 We have,

$$\lim_{(x,y)\to(0,0)} = \lim_{x\to 0} \frac{\varphi(x)}{x} = \lim_{x\to 0} \frac{\varphi(x) - \varphi(0)}{x - 0} = \varphi'(0) = -\frac{1}{2}.$$

Supplementary exercise correction 3 ☆

We put $f(x,y) = 2x^3 y + 2x^2 + y^2$.

1 Note that $(1,1)$ is a solution of the equation $f(x,y) = 0$. Then we have,

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = 6x^2 y + 4x \\ \dfrac{\partial f}{\partial y}(x,y) = 2x^3 + 2y \end{cases} \Rightarrow \begin{cases} \dfrac{\partial f}{\partial x}(1,1) = 10 \\ \dfrac{\partial f}{\partial y}(1,1) = 4 \end{cases}$$

Since $\dfrac{\partial f}{\partial y}(1,1) \neq 0$ there is one and only one function $y = \varphi(x)$ defined in the neighbourhood of 1 such that $f(x,\varphi(x)) = 1$.

2 When it was,

$$\varphi'(1) = -\frac{\frac{\partial f}{\partial x}(1,1)}{\frac{\partial f}{\partial y}(1,1)} = -\frac{5}{2}.$$

So, the equation of the tangent line to φ at $x = 1$ is,

$$y = -\frac{5}{2}(x-1) + 1 = -\frac{5}{2}x + \frac{7}{2}.$$

3 The Taylor expansion of φ to order 1 at point 1 is,

$$\varphi(x) = \varphi(1) + \varphi'(1)(x-1) + o(x) = \frac{7}{2} - \frac{5}{2}x + o(x).$$

Supplementary exercise correction 4

1 Soit f la fonction définie sur $\mathbb{R}^3$ par $f(x,y,z) = x^3 + 4xy + z^2 - 3yz^2 - 3$. Alors, f possède des dérivées partielles continues,

$$\frac{\partial f}{\partial x}(x,y,z) = 3x^2 + 4y \quad \frac{\partial f}{\partial y}(x,y,z) = 4x - 3z^2 \text{ et } \frac{\partial f}{\partial z}(x,y,z) = 2z - 6yz,$$

et au point $(1,1,1)$ on a,

$$\frac{\partial f}{\partial x}(1,1,1) = 7 \quad \frac{\partial f}{\partial y}(1,1,1) = 1 \text{ et } \frac{\partial f}{\partial z}(1,1,1) = -4,$$

Puisque $f(1,1,1) = 0$ et $\frac{\partial f}{\partial z}(x,y,z) \neq 0$, le théorème des fonctions implicites nous assure qu'on peut exprimer z en fonction de (x,y) dans un voisinage de $(1,1,1)$.

2 D'après le théorème des fonctions implicites, on obtient,

$$\frac{\partial z}{\partial x}(1,1) = -\frac{\frac{\partial f}{\partial x}(1,1,1)}{\frac{\partial f}{\partial z}(x,y,z)} = \frac{7}{4} \text{ et } \frac{\partial z}{\partial y}(1,1) = -\frac{\frac{\partial f}{\partial y}(1,1,1)}{\frac{\partial f}{\partial z}(x,y,z)} = \frac{1}{4}.$$

Supplementary exercise correction 5

1 Let f be the function defined on $\mathbb{R}^3$ by $f(x,y,z) = x^3 + 4xy + z^2 - 3yz^2 - 3$. Then, f has continuous partial derivatives,

$$\frac{\partial f}{\partial x}(x,y,z) = 3x^2 + 4y \quad \frac{\partial f}{\partial y}(x,y,z) = 4x - 3z^2 \text{ and } \frac{\partial f}{\partial z}(x,y,z) = 2z - 6yz,$$

and at the point $(1,1,1)$ we have,

$$\frac{\partial f}{\partial x}(1,1,1) = 7 \quad \frac{\partial f}{\partial y}(1,1,1) = 1 \text{ and } \frac{\partial f}{\partial z}(1,1,1) = -4,$$

Since $f(1,1,1) = 0$ and $\frac{\partial f}{\partial z}(x,y,z) \neq 0$, the implicit function theorem assures us that we can express z as a function of (x,y) in a neighbourhood $(1,1,1)$.

2 According to the implicit function theorem, we obtain,

$$\frac{\partial z}{\partial x}(1,1) = -\frac{\frac{\partial f}{\partial x}(1,1,1)}{\frac{\partial f}{\partial z}(x,y,z)} = \frac{7}{4} \text{ and } \frac{\partial z}{\partial y}(1,1) = -\frac{\frac{\partial f}{\partial y}(1,1,1)}{\frac{\partial f}{\partial z}(x,y,z)} = \frac{1}{4}.$$

Supplementary exercise correction 6

For all $(x,y) \in \mathbb{R}^2$ such that $1+x-y \neq 0$,

$$f(x,y) = \frac{2+x+y}{1+x-y} = (2+x+y) \times \frac{1}{1+x-y}.$$

If (x,y) is in the neighbourhood of $(0,0)$, then $x-y$ is also in the neighbourhood of 0. Using the following Taylor limited expansion to order 2,

$$\frac{1}{1+t} = 1 - t + t^2 + o(t^2).$$

Consequently,

$$\begin{aligned} f(x,y) &= (2+x+y) \times \frac{1}{1+x-y} \\ &= (2+x+y) \times (1-(x-y)+(x-y)^2+o(x^2+y^2) \\ &= 2-x+3y+x^2-y^2-4xy+o(x^2+y^2). \end{aligned}$$