University of M'sila		Faculty of Mathematics and Computer Science
L 2 Mathematics	**Correction of Series** $N:02$ **- Limits and Continuity-**	Module: Analysis 4

Remark 1 ☆

1 If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = l$ (finite), at $A(x_0,y_0)$, then the restriction of f to any continuous curve passing through A has the same limit.

2 To show that $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist in $A(x_0,y_0)$, it suffices to look for two restrictions on the continuous curves passing through A which lead to different limits.

3 Here are some continuous curves passing through $A(x_0,y_0)$.

(a) The lines/

$$\begin{aligned} x &= x_0, \quad \text{the vertical line} \\ y &= m(x-x_0)+y_0, \quad m\in\mathbb{R}. \end{aligned}$$

(b) The parabolas

$$\begin{aligned} y &= m(x-x_0)^2+y_0, \quad m\in\mathbb{R}, \\ x &= m(y-y_0)+x_0, \quad m\in\mathbb{R}. \end{aligned}$$

Exercise correction 1 ☆

We determine the domain of definition of the following functions

1 If $f_1(x,y)=\ln(x+y-1)$. So, we have: $\mathcal{D}_1=\left\{(x,y)\in\mathbb{R}^2:\ y>1-x\right\}$.
See, below the presentation of $\mathcal{D}_i$, where $i\in\{1,...,5\}$.

2 For $f_2(x,y)=\sqrt{1-xy}$. So we have,

$$\mathcal{D}_2=\left\{(x,y)\in\mathbb{R}^2:\ xy\le 1\right\}=\left\{(x,y)\in\mathbb{R}^2:\ y\le\frac{1}{x},\ x\neq 0\right\}.$$

the case, $f_3(x,y)=\dfrac{\sqrt{4-x^2-y^2}}{\sqrt{x^2+y^2-1}}$. We have,

$$\begin{aligned} \mathcal{D}_3 &= \left\{(x,y)\in\mathbb{R}^2:\ x^2+y^2\le 4,\ \text{and } x^2+y^2>1\right\} \\ &= \left\{(x,y)\in\mathbb{R}^2:\ 1<x^2+y^2\le 4\right\}. \end{aligned}$$

3 $f_4(x,y)=\dfrac{\ln y}{\sqrt{x-y}}$.

$$\begin{aligned} \mathcal{D}_4 &= \left\{(x,y)\in\mathbb{R}^2:\ y>0,\ \text{and } x-y>0\right\} \\ &= \left\{(x,y)\in\mathbb{R}^2:\ 0<y<x\right\}. \end{aligned}$$

4 $f_5(x,y) = \dfrac{\sqrt{x^2-y}}{\sqrt{y}}$.

$$\begin{aligned}\mathcal{D}_5 &= \left\{(x,y)\in\mathbb{R}^2 :\ y\le x^2,\ \text{and } y>0\right\}\\ &= \left\{(x,y)\in\mathbb{R}^2 :\ 0<y\le x^2\right\}.\end{aligned}$$

5 $f_6(x,y,z) = \dfrac{1+x^2}{xyz}$.

$$\mathcal{D}_6 = \left\{(x,y)\in\mathbb{R}^2 : x\neq 0,\ y\neq 0 \text{ and } z\neq 0\right\} = \mathbb{R}^3 - \{(0,0,0)\}.$$

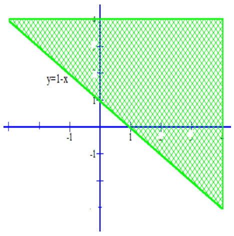


Figure 1: $\mathcal{D}_1$.

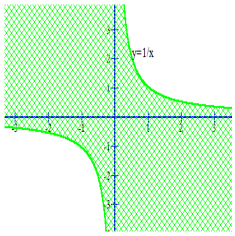


Figure 2: $\mathcal{D}_2$.

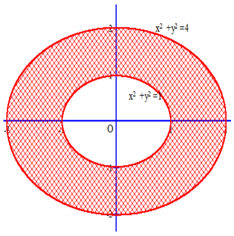


Figure 3: $\mathcal{D}_3$.

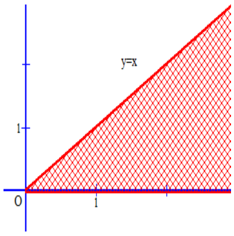


Figure 4: $\mathcal{D}_4$.

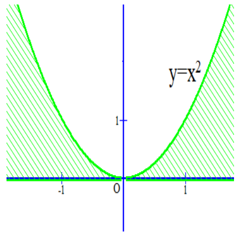
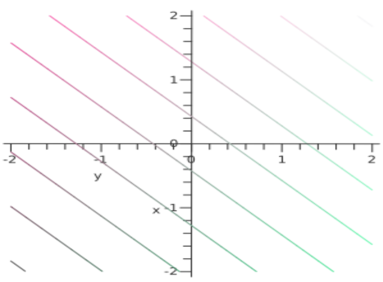


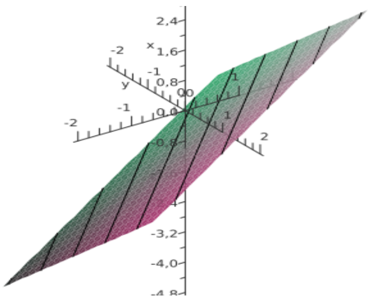
Figure 5: $\mathcal{D}_5$.

Exercise correction 2 ☆

Level curves of functions of two variables. Let k be a real number. Then, we see that:

1 For $f(x,y)=x+y-1$, we have: $f(x,y)=x+y-1=k \Rightarrow y=1-x-k$. So, the level curves are therefore straight lines and the graph of f in a plane is,

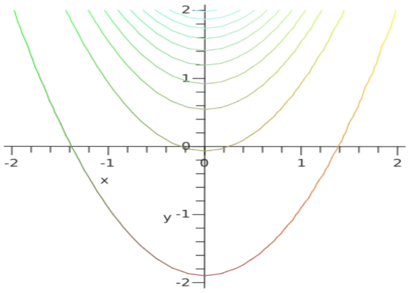


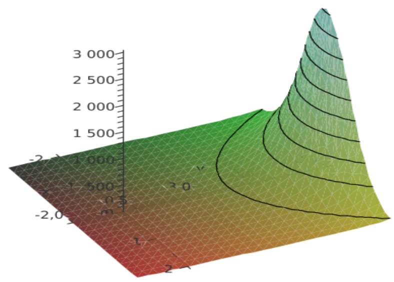


2 If $f(x,y)=e^{y-x^2}$, so, we have $f(x,y)=e^{y-x^2}=k$. There are 2 possible cases

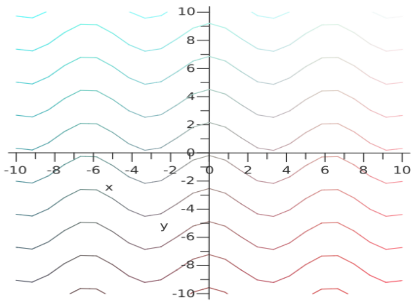
(a) If $k\le 0$, the curve at level k is the empty set $\emptyset$.

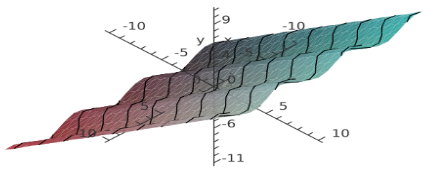
(b) If $k>$, then, we find, $f(x,y)=e^{y-x^2}=k \Rightarrow y=x^2+\ln(k)$. the curve at level k are therefore parabolas.





3 For $f(x,y) = y - \cos x$, we'll have $f(x,y) = y - \cos x = k \Rightarrow y = \cos x + k$.

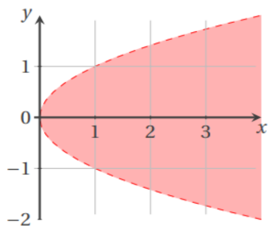


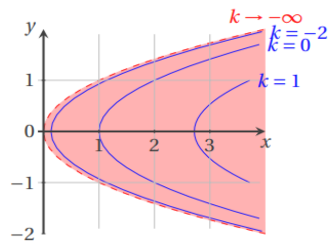


4 If $f(x,y) = \ln(x - y^2)$.

(a) Then the domain of definition of f is $\mathcal{D} = \left\{(x,y) \in \mathbb{R}^2 : \ y < x^2\right\}$.

(b) $f(x,y) = \ln(x - y^2) = k \Leftrightarrow x - y^2 = k \Leftrightarrow x = y^2 + k$.





Exercise correction 3 ☆☆

In the following cases, limits are calculated (if they exist)

1 We use an increase, and as, $\forall (x,y) \in \mathbb{R}^2 : \ xy \leq 2xy \leq x^2 + y^2$. Then,

$$\lim_{(x,y)\to(0,0)} \left| \frac{xy^2}{x^2+y^2} \right| \leq \lim_{y\to 0} |y| = 0$$

2 We use the restriction of f to the continuous curve $x = 1$, means that it passes through $A(1,1)$.
$\displaystyle\lim_{y\to 1^+} \frac{1}{1-y} = -\infty$, et $\displaystyle\lim_{y\to 1^-} \frac{1}{1-y} = +\infty$. Thus, $\displaystyle\lim_{(x,y)\to(1,1)} \frac{1}{x-y}$ does not exist.

3 By switching to polar coordinates,

$$\begin{cases} x = 1 + r\cos\theta, \\ x = r\sin\theta, \end{cases} \quad \text{where } r > 0, \ \theta \in [0, 2\pi].$$

Then, we obtain,

$$f(1+r\cos\theta, r\sin\theta) = \frac{r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = r\sin^3\theta.$$

Since we have

$$|f(1+r\cos\theta, r\sin\theta)| \leq |r\sin^3\theta| \leq r \underset{r\longrightarrow 0}{\longrightarrow} 0.$$

Thus, $\displaystyle\lim_{(x,y)\to(1,0)} \frac{y^3}{(x-1)^2+y^2} = 0.$

4 The limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist because, for $y = mx,\ m \in \mathbb{R}$, we have,

$$\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}.$$

So the limit is depends on m.

5

$$\lim_{(x,y)\to(0,\pi)} \frac{1}{x}\sin\left(\frac{x}{y}\right) = \lim_{(x,y)\to(0,\pi)} \frac{1}{y} \times \frac{\sin(\frac{x}{y})}{\frac{x}{y}} = \frac{1}{\pi}.$$

Because,

$$\lim_{\alpha\to 0} \frac{\sin\alpha - 1}{\alpha} = 1.$$

6 Using polar coordinates, let us set,

$$\begin{cases} x = r\cos\theta, \\ x = r\sin\theta, \quad \text{where } r > 0,\ \theta \in [0, 2\pi]. \end{cases}$$

So, we find,

$$f(r\cos\theta, r\sin\theta) = r\sin\theta\cos\theta.$$

Subsequently, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0}(r\sin\theta\cos\theta) = 0.$

7

8 By polar coordinates,

$$\begin{cases} x = r\cos\theta, \\ x = r\sin\theta, \quad \text{where } r > 0,\ \theta \in [0, 2\pi]. \end{cases}$$

So, we see that,

$$f(r\cos\theta, r\sin\theta) = \frac{\sin\theta\cos\theta}{1+\sin\theta\cos\theta}.$$

After that, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2} = \lim_{r\to 0} \frac{\sin\theta\cos\theta}{1+\sin\theta\cos\theta} = \frac{\sin\theta\cos\theta}{1+\sin\theta\cos\theta}.$
The limit depends on θ, then it does not exist.

9 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y-3}{x+y+1} = -3.$

10 Let's put $\alpha = xy$, so $\begin{cases} \alpha \longrightarrow 0 \\ \text{if } (x,y) \longrightarrow (0,0). \end{cases}$ Consequently, we will have

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{e^x-1} = \lim_{(x,y)\to(0,0)} \frac{xy\dfrac{e^{xy}-1}{xy}}{x\dfrac{e^x-1}{x}} = \lim_{(x,y)\to(0,0)} y = 0.$$

Car,

$$\lim_{\alpha\to 0} \frac{e^{\alpha}-1}{\alpha} = 1.$$

Exercise correction 4 ☆

1 The function f defined on $\mathbb{R}^2 - \{(x,x) : x \in \mathbb{R}\}$ par $f(x,y) = \dfrac{\sin x - \sin y}{x-y}$.

Then, Using the trigonometric formulas, we get

$$\frac{\sin x - \sin y}{x-y} = \frac{\sin(\frac{x-y}{2})}{\frac{x-y}{2}} \times \cos(\frac{x-y}{2}) = \cos\alpha.$$

Knowing that $\displaystyle\lim_{(x,y)\to(\alpha,\alpha)} \frac{\sin(\frac{x-y}{2})}{\frac{x-y}{2}} = 1$, so we have

$$\lim_{(x,y)\to(\alpha,\alpha)} f(x,y) = \cos\alpha.$$

2 $$\lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} (\sqrt{x}+\sqrt{y}) = 0$$

3 We have $|\sin(2xy)| \leq 2xy$. So, let's pose

$$\begin{cases} x = r\cos\theta, \\ x = r\sin\theta, \quad \text{where } r > 0,\ \theta \in [0, 2\pi]. \end{cases}$$

So we see that,

$$|e^{-(x^2+y^2)}\sin(2xy)| \leq 2e^{-(x^2+y^2)}|x||y| = 2e^{-r}r^2|\sin\theta\cos\theta|.$$

Hence,

$$0 \leq \lim_{r\to+\infty} 2e^{-r}r^2|\sin\theta\cos\theta| \leq 2r^2e^{-r} \underset{r\longrightarrow+\infty}{\longrightarrow} 0\ .$$

Exercise correction 5 ☆

Let f be a function defined on $\mathbb{R}^2 - \{(0,0)\}$ by $f(x,y) = \dfrac{2xy^2}{x^2+y^2}$.

Show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

1 1^{st} Method: By increase. We have

$$\forall (x,y) \in \mathbb{R}^2 - \{(0,0)\} :\ 2|xy| \leq x^2+y^2.$$

Then, consequently, we find,

$$\forall (x,y) \in \mathbb{R}^2 - \{(0,0)\} :\ 2|xy^2| \leq |y|(x^2+y^2).$$

So, it follows that

$$\lim_{(x,y)\to(0,0)} \left|\frac{2xy^2}{x^2+y^2}\right| \leq \lim_{y\to 0} |y| = 0$$

2 2^{th} Method: Using polar coordinates.

$$\begin{cases} x = r\cos\theta, \\ x = r\sin\theta, \quad \text{where } r > 0,\ \theta \in [0, 2\pi]. \end{cases}$$

So we can see that,

$$\lim_{(x,y)\to(0,0)} f(r\cos\theta, r\sin\theta) = \lim_{r\to 0} 2r\sin^2\theta\cos\theta = 0.$$

3 3^{th} Method: We need to show that,

$$\forall \varepsilon > 0,\ \exists \delta(\varepsilon)\ \|(x,y)-(0,0)\|_2 \leq \delta(\varepsilon) \Rightarrow \left|\frac{2xy^2}{x^2+y^2} - 0\right| < \varepsilon.$$

Let $\varepsilon > 0$. Bearing in mind that $x^2 + y^2 > y^2$, then, we find

$$\left|\frac{2xy^2}{x^2+y^2}\right| < \left|\frac{2xy^2}{x^2}\right| = 2|x| < 2\sqrt{x^2+y^2} < \varepsilon.$$

Just take $\delta(\varepsilon) = \dfrac{\varepsilon}{2}$.

Exercise correction 6 ☆

We study the continuity of functions f, g on $\mathbb{R}^2$ such that,

1 $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0). \end{cases}$

We know that f is continuous on $\mathbb{R}^2 - \{(0,0)\}$ because it is the quotient of two continuous functions, $(x,y) \longmapsto xy$, (polynomial), and $(x,y) \longmapsto \sqrt{x^2+y^2}$, (square root) of polynomial). So, it remains to prove the continuity at point $(0,0)$.
Using polar coordinates, let us set, so,

$$\begin{cases} x = r\cos\theta, \\ x = r\sin\theta, \quad \text{where } r > 0,\ \theta \in [0, 2\pi]. \end{cases}$$

So, we find,

$$f(r\cos\theta, r\sin\theta) = r\sin\theta\cos\theta.$$

Subsequently, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0}(r\sin\theta\cos\theta) = 0.$

2 $g(x,y) = \begin{cases} \dfrac{x+y}{\sqrt{x^2+y^2}} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$

The continuity of g on $\mathbb{R}^2 - \{(0,0)$ is clear.
The limit $_{(x,y)\to(0,0)} g(x,0)$ does not exist, Indeed, if $y = mx$, $m \in \mathbb{R}$, we find,
$\displaystyle\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{x\to 0} \frac{(1+m)x}{\sqrt{1+m^2}|x|} = \pm\frac{1+m}{\sqrt{1+m^2}}$. This means that the limit does not exist.
Hence g is continuous only on $\mathbb{R}^2 - \{(0,0)\}$.

Exercise correction 7 ☆☆

We study the extension by continuity at point a of the following functions

1 $f(x,y)=\dfrac{x^2y^2}{x^2+y^2}$, $a(0,0)$. We study the limit of f at $(0,0)$. By switching to polar coordinates, we get, $\begin{cases} x=r\cos\theta, \\ x=r\sin\theta, \quad \text{where } r>0,\ \theta\in[0,2\pi[, \end{cases}$ so, we obtain,

$$\lim_{(x,y)\to(0,0)}|f(x,y)|=\lim_{r\to0}|\frac{r^4\sin^2\theta\cos^2\theta}{r^2}|\leq\lim_{r\to0}r^2=0.$$

Consequently $\lim\limits_{(x,y)\to(0,0)}f(x,y)=0$. This proves that f has a boundary at $(0,0)$, so it can be extended by continuity at $(0,0)$ by $\tilde{f}$, so that

$$\tilde{f}(x,y)=\begin{cases} f(x,y), & : (x,y)\neq(0,0) \\ 0, & : (x,y)=(0,0). \end{cases}$$

On remarque que $\tilde{f}$ est continue sur $\mathbb{R}^2$. Car la fonction f est continue sur $\mathbb{R}^2-\{(0,0)\}$ comme fraction rationnelle dont le dénominateur ne s'annule pas, and at point $a(0,0)$ we have $\lim\limits_{(x,y)\to(0,0)}\tilde{f}(x,y)=0=\tilde{f}(0.0)$.

2 For the function f which defines by $f(x,y)=\dfrac{x-2}{(x-2)^2+y^2}+1$, $a(2,0)$. We put, $\begin{cases} x=r\cos\theta+2, \\ x=r\sin\theta, \quad \text{where } r>0,\ \theta\in[0,2\pi[, \end{cases}$ we will therefore have,

$$\lim_{(x,y)\to(0,0)}|f(x,y)|=\lim_{r\to0}\frac{|\cos\theta|}{r}=+\infty.$$

Which means that f is not extendable by continuity in $a(2,0)$.

3 $f(x,y)=\dfrac{(x+y)^2}{x^2+y^2}$, $a(0,0)$. $(\star)$. Remains as an exercise.

Supplementary exercise correction 1 ☆☆

Let's show the continuity of the function f on $\mathbb{R}^2$ such that

$$f(x,y)=\begin{cases} \frac{1}{2}x^2+y^2-1, & : x^2+y^2>1, \\ -\frac{1}{2}x^2, & : \text{otherwise}. \end{cases}$$

Let D be the exterior of the unit disk and E its interior, that is to say

$$D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2>1\},\quad E=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}.$$

Let $(x_0,y_0)\in\mathbb{R}^2$. So, we distinguish two possible cases.

1 If the point (x_0,y_0) is in D or E, then f is continuous, because it is polynomial.

2 If the point (x_0,y_0) is on the boundary of D (or E), which means $x_0^2+y_0^2=1$, then we find

(a) Quand $(x,y)\in D$,

$$\lim_{(x,y)\to(x_0,y_0)}f(x,y)=\lim_{(x,y)\to(x_0,y_0)}(\frac{1}{2}x^2+y^2-1)=-\frac{1}{2}x_0^2=f(x_0,y_0).$$

(b) Lorsque $(x,y) \in E$,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} (-\frac{1}{2}x^2) = -\frac{1}{2}x_0^2 = f(x_0,y_0).$$

Hence the continuity of f on $\mathbb{R}^2$.

Supplementary exercise correction 2

Show that the following sets are compact. Firstly, recall that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

1 $A = \left\{(x,y) \in \mathbb{R}^2 : \; x^2+y^2 \leq 9, \right\}$. Let the application f be defined by

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ x &\mapsto x^2+y^2. \end{aligned}$$

Then, f is continuous, because it is a polynomial, as, we have

$$A = f^{-1}([0,9]),$$

and the set $[0,9]$ is closed in $(\mathbb{R}, |.|)$, so A is also closed in $(\mathbb{R}^2, \|.\|)$. We also have $A \subseteq \overline{B}((0,0),6)$, i.e. A is bounded. Hence the compactness of A.

2 $B = \left\{(x,y,z) \in \mathbb{R}^3 : \; \frac{x^2}{4}+\frac{y^2}{9}+z^2 = 1\right\}$. In the same way as above, we can define the application f by

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R} \\ x &\mapsto \frac{x^2}{4}+\frac{y^2}{9}+z^2. \end{aligned}$$

So, we have,

(a) $B = f^{-1}(\{1\})$, f is continuous (a polynomial), and $\{1\}$ is closed. So B is too.

(b) B is bounded, because $B \subseteq \overline{B}((0,0,0),6)$. Hence the compactness of B.