University of M'sila	**Correction of Serie: N 01** **(Topology of $\mathbb{R}^n$)**	Faculty: Mathematics and Computer Science
L 2 Mathematics		Module: Analysis 4

Exercise correction 1 ☆

Let $a,\ b \geq 0$. we set $\forall u(x,y) \in \mathbb{R}^2: \ \|u\| = \sqrt{a^2x^2 + b^2y^2}$.

1 Let's show that $\|.\|$ is a norm on $\mathbb{R}^2$. Then, for all $(x,y),\ (x',y') \in \mathbb{R}^2$, and $\lambda \in \mathbb{R}$, we have:

(a) Separation: $\|u(x,y)\| = 0 \Leftrightarrow a^2x^2 + b^2y^2 = 0 \Leftrightarrow (x,y) = (0,0)$.

(b) Homogeneity: $\|(\lambda x, \lambda y) = \sqrt{a^2\lambda^2x^2 + b^2\lambda^2y^2} = |\lambda|\sqrt{a^2x^2 + b^2y^2} = |\lambda|\|(x,y)\|$.

(c) Triangular inequality: Using the Cauchy-Schwarz inequality, we find,

$$\begin{aligned}
\|(x+x', y+y')\|^2 &= a^2(x+x')^2 + b^2(y+y')^2 \\
&= a^2x^2 + b^2y^2 + a^2x'^2 + b^2y'^2 + 2(ax)(ax') + 2(by)(by') \\
&\leq a^2x^2 + b^2y^2 + a^2x'^2 + b^2y'^2 \\
&\qquad +2\sqrt{(a^2x^2 + b^2y^2)(a^2x'^2 + b^2y'^2)} \\
&\leq (\sqrt{a^2x^2 + b^2y^2} + \sqrt{a^2x'^2 + b^2y'^2})^2
\end{aligned}$$

Which gives: $\|(x+x', y+y')\| \leq \sqrt{a^2x^2 + b^2y^2} + \sqrt{a^2x'^2 + b^2y'^2}$.

2 $\overline{B}(0,1) = \{(x,y) \in \mathbb{R}^2 : \ a^2x^2 + b^2y^2 \leq 1\} = \{(x,y) \in \mathbb{R}^2 : \ \frac{x^2}{\frac{1}{a^2}} + \frac{y^2}{\frac{1}{b^2}} \leq 1\}$. The unit ball is therefore the interior (inside) of the ellipse.

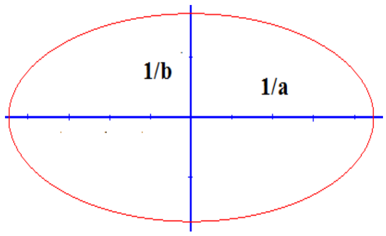

Figure 1: Présentation de la boule unité fermée.

3 It is easy to increase or decrease the given norm.

$$\min(|a|,|b|)\|u(x,y)\| \leq \|u(x,y)\| \leq \max(|a|,|b|)\|u(x,y)\|.$$

Which means that $\|.\|_2 \approx \|.\|$.

Exercise correction 2 ☆

To check the separation condition we must see:

$$x^2 + 2\lambda xy + y^2 > 0 \Leftrightarrow \Delta' = \lambda^2 - 1 < 0 \Leftrightarrow \lambda \in]-1,1[.$$

Exercise correction 3 ☆

Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$. We put

$$\|u\|_1 = \sum_{i=1}^{i=n} |x_i|, \quad \|u\|_2 = \left(\sum_{i=1}^{i=n} |x_i|^2\right)^{\frac{1}{2}} \text{ and } \ \|u\|_\infty = \max_{1\leq i\leq n} |x_i|.$$

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1 We show that $\|.\|_2$ is a norm on $\mathbb{R}^n$.

(a) Separation and homogeneity properties are easy to check.

(b) To show the triangular inequality, consider two points $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ de $\mathbb{R}^2$.
If $x + y = 0$, then the result is clear.
If $x + y \neq 0$, according to the Cauchy-Schwarz inequality, we have:

$$\begin{aligned}
\|x+y\|_2^2 &= \sum_{i=1}^{i=n}(x_i+y_i)^2 \\
&= \sum_{i=1}^{i=n} x_i(x_i+y_i) + \sum_{i=1}^{i=n} y_i(x_i+y_i) \\
&\leq \Big(\sum_{i=1}^{i=n} x_i^2\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{i=n}(x_i+y_i)^2\Big)^{\frac{1}{2}} + \Big(\sum_{i=1}^{i=n} y_i^2\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{i=n}(x_i+y_i)\Big)^{2})^{\frac{1}{2}} \\
&\leq (\|x\|_2 + \|y\|_2)(\|x+y\|_2).
\end{aligned}$$

Since $\|x+y\|_2 neq 0$, dividing by $\|x+y\|_2$, gives the triangular inequality.

2 Let's show that for all $x \in \mathbb{R}^n$:

$$\|u\|_\infty \leq \|u\|_2 \leq \|u\|_1 \leq n\|u\|_\infty \text{ and } \|u\|_2\| \leq \sqrt{n}\|u\|_\infty.$$

(a) For all $1 \leq i \leq n$, we have the inequalities,

$$|x_i| \leq \max\{|x_1|, ..., |x_n|\} = \|x\|_\infty. \tag{1}$$

By summing member by member in (1), we find,

$$|x_1| + ... + |x_n| = \|x\|_1 \leq n\|x\|_\infty.$$

(b) If, on the other hand, we take the sum of the squares of the inequalities (1), we obtain

$$x_1^2 + ... + x_n^2 = \|x\|_2^2 \leq n\|x\|_\infty^2.$$

Consequently,

$$\|x\|_2 \leq \sqrt{n}\|x\|_\infty.$$

(c) We know that,

$$x_1^2 + ... + x_n^2 \leq (x_1^2 + ... + x_n)^2,$$

So we'll get:

$$\|x\|_2^2 \leq \|x\|_1^2,$$

Hence,

$$\|x\|_2 \leq \|x\|_1.$$

(d) Since for all $1 \leq i \leq n$ we see that:

$$|x_i| \leq \sqrt{x_1^2 + ... + x_n} = \|x\|_2,$$

Passons donc au maximum on trouve,

$$\max\{|x_1|, ..., |x_n|\} = \|x\|_\infty \leq \|x\|_2.$$

3 (a) The ball $\overline{B}_1(0,1) = \{(x,y) \in \mathbb{R}^2 : \ |x| + |y| \leq 1\}$ in $\mathbb{R}^2$ is a rhombus with respect to $\|x\|_1$. (See the figure below)

(b) The ball $\overline{B}_2(0,1) = \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2$ is a disk centered in $(0,0)$ and of radius 1 with respect to $\|x\|_2$,

(c) The ball $\overline{B}_3(0,1) = \{(x,y) \in \mathbb{R}^2 : \ \max\{||x| + |y| \leq 1\}$ in $\mathbb{R}^2$ is a square with respect to $\|x\|_\infty$.

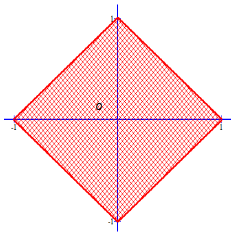

Figure 2: $\overline{B}_1(0,1)$.

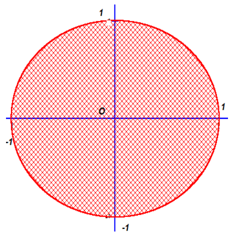

Figure 3: $\overline{B}_2(0,1)$.

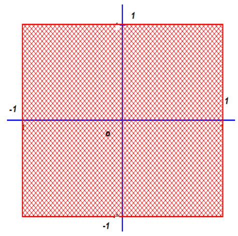

Figure 4: $\overline{B}_3(0,1)$.

Exercise correction 4 ☆

1 Let's show that for all vectors $x, \ y \in \mathbb{R}^n$ we have the inequality

$$\Big| \|x\| - \|y\| \Big| \leq \|x - y\|.$$

From the triangular inequality, we see that

$$\begin{array}{rcl} \|x\| & = & \|(x-y) + y\| \leq \|x - y\| + \|y\| \\ \|y\| & = & \|(y-x) + x\| \leq \|y - x\| + \|x\|. \end{array}$$

Consequently, we have

$$\begin{array}{rcl} \|x\| - \|y\| & \leq & \|x - y\| \\ \|y\| - \|x\| & \leq & \|y - x\|. \end{array}$$

So, we get

$$-\|x - y\| \leq \|x\| - \|y\| \leq \|x - y\|,$$

Hence the result,

$$\Big| \|x\| - \|y\| \Big| \leq \|x - y\|.$$

2 We deduce that the application $\|.\| : \mathbb{R}^n \to \mathbb{R}$ (the norm) is continuous. Which gives, if a sequence $(x_k)_{k\in\mathbb{N}}$ of vectors $x_k \in \mathbb{R}^n$ converges in the normed vector space $(\mathbb{R}^n, \|.\|)$ then,

$$\lim_{k\to+\infty} \|x_k\| = \|\lim_{k\to+\infty} x_k\|.$$

3 2^{iem} Method: Suppose $x_k \to l$ when $k \to +\infty$. Then for all $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that,

$$\forall k \in \mathbb{N} : \ k \geq k_0 \Rightarrow \|x_k - l\| \leq \varepsilon,$$

So, according to question 1, for any integer $k \in \mathbb{N}$ such that $k \geq k_0$, we obtain,

$$\Big| \|x_k\| - \|l\| \Big| \leq \|x_k - l\| \leq \varepsilon,$$

which means,

$$\lim_{k\to+\infty} \|x_k\| = \|l\| = \|\lim_{k\to+\infty} x_k\|.$$

4 Suppose that $\lim_{k\to+\infty} x_k = l \in \mathbb{R}^n$ and $\lim_{k\to+\infty} \|x_k - y_k\| = 0$. So, according to the triangular inequality

$$|y_k - l\|leq\|y_k - x_k\| + \|x_k - l\|.$$

Passing to the limit, we find,

$$\lim_{k\to+\infty} \|y_k - l\| \leq \lim_{k\to+\infty} \|y_k - x_k\| + \lim_{k\to+\infty} \|x_k - l\| = 0.$$

Hence $\lim_{k\to+\infty} y_k = l$.

0.0.1 Corrections of supplementary exercises

Remark 1 ☆

As we know, all norms are equivalent in finite dimensional space. This example explains the non-equivalence of norms, even though it is defined in infinite dimensional space.

Remark 2 ☆

Recall that two norms $\|f\|_1$ and $\|f\|_2$ on a vector space E are not equivalent, if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to+\infty} \frac{\|f_n\|_1}{\|f_n\|_2} = 0, \quad \text{ou} \quad \lim_{n\to+\infty} \frac{\|f_n\|_1}{\|f_n\|_2} = +\infty.$$

Exercise correction 5 ☆

1 Let $f,\ g \in E$ and $\lambda, \mu \in \mathbb{R}$, then we have:

$$\begin{aligned}
\varphi(\lambda f + \mu g) &= \int_{-1}^{1} \frac{x}{1+x^2}[\lambda f(x) + \mu g(x)]dx \\
&= \lambda \int_{-1}^{1} \frac{x}{1+x^2} f(x)dx + \mu \int_{-1}^{1} \frac{x}{1+x^2} g(x)dx \\
&= \lambda\varphi(f) + \mu\varphi(g).
\end{aligned}$$

So, φ is linear.

2 φ is continuous if and only if

$$\exists C > 0,\ \forall f \in E : |\varphi(f)| \leq C\|f\|_\infty.$$

3 Let $f \in E$, then we have:

$$\begin{aligned}
|\varphi(f)| &= \left| \int_{-1}^{1} \frac{x}{1+x^2} f(x)dx \right| \\
&\leq \|f\|_\infty \int_{-1}^{1} \frac{|x|}{1+x^2} dx \\
&= \|f\|_\infty \int_{0}^{1} \frac{2x}{1+x^2} dx = \ln 2 \|f\|_\infty. \qquad (2)
\end{aligned}$$

We take $C = \ln 2$ so that φ is continuous.

Exercise correction 6 ☆

Let the sequence $x_n = \left(\frac{1}{1+n}, 1+e^{-n}\right), \quad n \in \mathbb{N}$. We notice the sequence $(x_n)_{n\in\mathbb{N}}$ converges only to the limit $(0,1)$, because,

$$\lim_{n\to+\infty} \frac{1}{1+n} = 0, \text{ and } \lim_{n\to+\infty}(1+e^{-n}) = 1.$$

Then, we see that:

1 $(x_n)_{n\in\mathbb{N}}$ converges uniformly (or by the norm $\|.\|_\infty$) if and only if,

$$\forall \varepsilon > 0, \ \exists n_0, \ \forall n \in \mathbb{N}: \ n \geq n_0 \Rightarrow \|x_n - (0,1)\|_\infty < \varepsilon.$$

Let $\varepsilon > 0$, and for all $n \in \mathbb{N}$, such that $(n \geq n_0)$, so we get,

$$\|x_n - (0,1)\|_\infty = \left\|\left(\frac{1}{1+n}, e^{-n}\right)\right\|_\infty = \sup_{n\in\mathbb{N}}\left\{\frac{1}{1+n}, e^{-n}\right\} \leq \frac{1}{1+n} < \varepsilon.$$

This implies that $n > \frac{1}{\varepsilon} - 1$. So, we just need to choose $n_0 = \max\left\{0, [\frac{1}{\varepsilon} - 1] + 1\right\}$.

2 $(x_n)_{n\in\mathbb{N}}$ converges by the norm $\|.\|_1$ if and only if,

$$\forall \varepsilon > 0, \ \exists n_0, \ \forall n \in \mathbb{N}: \ n \geq n_0 \Rightarrow \|x_n - (0,1)\|_1 < \varepsilon.$$

Let $\varepsilon > 0$, and for all $n \in \mathbb{N}$, such that $(n \geq n_0)$. Then we have,

$$\|x_n - (0,1)\|_1 = \left\|\left(\frac{1}{1+n}, e^{-n}\right)\right\|_1 = \left|\frac{1}{1+n}\right| + \left|e^{-n}\right| \leq \frac{2}{1+n} < \varepsilon.$$

As a result, $n > \frac{2}{\varepsilon} - 1$. Then, we just need to take $n_0 = \max\left\{0, [\frac{2}{\varepsilon} - 1] + 1\right\}$.

Exercise correction 7 ☆

Let $C([0,1], \mathbb{R})$ be the space of functions defined and continuous on $[0,1]$ with value in $\mathbb{R}$.

1 Remains as an exercise.

2 We have $\|f_n\|_\infty = 1$ and $\|f_n\|_1 = \frac{1}{n+1}$.

3 $\|f_n\|_\infty$ and $\|f_n\|_1$ are not equivalent, because,

$$\lim_{n\to+\infty} \frac{\|f_n\|_\infty}{\|f_n\|_1} = \lim_{n\to+\infty}(n+1) = +\infty.$$

Exercise correction 8 ☆

Let $N : \mathbb{R}^2 \to \mathbb{R}$ an application defined by $N(x,y) = \sup_{t\in\mathbb{R}} \frac{|tx+y|}{\sqrt{1+t^2}}$.

1 Let's show that N is a norm on $\mathbb{R}^2$. Let $(x,y), \ (x',y') \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. Then,

(a) Separation:

$$N(x,y) = 0 \Leftrightarrow |tx+y| = 0, \ \forall t \in \mathbb{R} \Rightarrow x = y = 0.$$

Conversely, if $x = y = 0 \Rightarrow N(x,y) = 0$.

(b) Homogeneity: $N(\lambda x, \lambda y) = \sup_{t\in\mathbb{R}} \frac{|t\lambda x + \lambda y|}{\sqrt{1+t^2}} = |\lambda| N(x,y)$.

(c)

$$\begin{aligned} N(x+x', y+y') &= \sup_{t\in\mathbb{R}} \frac{|t(x+x') + (y+y')|}{\sqrt{1+t^2}} = \sup_{t\in\mathbb{R}} \frac{|(tx+y) + (tx'+y')|}{\sqrt{1+t^2}} \\ &\leq \sup_{t\in\mathbb{R}} \frac{|tx+y|}{\sqrt{1+t^2}} + \sup_{t\in\mathbb{R}} \frac{|tx'+y'|}{\sqrt{1+t^2}} = N(x,y) + N(x',y'). \end{aligned}$$

2 Using the Cauchy-Schwarz inequality, we therefore see,

$$|tx+y| = |\langle (t,1), (x,y) \rangle| \leq \sqrt{1+t^2}\sqrt{x^2+y^2},$$

which give $N \leq \|.\|_2$.

3 (a) 1^{st} Method: We have

$$d(M_0, \Delta) = \frac{|tx_0 + y_0|}{\sqrt{1+t^2}}.$$

Let A be the orthogonal projection of the point M_0 on the line (Δ). Then, we have

$$d(A, \Delta) = \frac{|tx_0 + y_0|}{\sqrt{1+t^2}} = d(M_0, \Delta).$$

Since the line (Δ) passes through the origin, this distance $d(A, \Delta)$ will be maximal when A coincides with the origin. We therefore see,

$$d(O, M_0) = \sqrt{x_0^2 + y_0^2} = \sup_{t\in\mathbb{R}} \frac{|tx_0 + y_0|}{\sqrt{1+t^2}}.$$

Hence the equivalence of the two norms N and N_2.

(b) 2^{th} Method: Let the function $f(t) = \frac{(tx+y)^2}{1+t^2}$, then, we know that

$$f'(t) = \frac{2(tx+y)(x-ty)}{(1+t^2)^2}.$$

Therefore, the function f admits a maximum for $t = \frac{x}{y}$, it's value is

$$f\left(\frac{x}{y}\right) = x^2 + y^2 = \sup_{t\in\mathbb{R}} \frac{(tx+y)^2}{1+t^2}.$$

So, $N \approx N_2$.