University of M'sila | Faculty: Maths-Compt-Science

Derivability and Differentiability

L2 Maths | Module: Analysis 4

Remark 1

The exercise marked by ($\star$) or supplementary will not be corrected in the sience of **TD**.

Exercise 1

Calculate all the first partial derivatives of the following functions.

1. $f(x,y) = x^2 + 3xy^2 - 4y^5$
2. $f(x,y) = y\cos(e^{xy+3y})$.
3. $f(x,y,z) = x\sin(yz) - \ln(3 - e^{x+y})$.

Exercise 2

1. Calculate the directional derivative of the function defined by $f(x,y) = 3x^2y - 4xy$ at the point $(1,2)$ along the direction $\overrightarrow{v} = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.
2. Check the equality $D_v f(1,2) = \nabla f(1,2).v = \frac{\partial f}{\partial x}(1,2).v_1 + \frac{\partial f}{\partial y}(1,2).v_2$.

Exercise 3

Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \dfrac{x^5}{(y-x^2)^2 + x^4} & : \ (x,y) \neq (0,0) \\ 0 & : \ (x,y) = (0,0) \end{cases}$

1. Show that the function f is continuous at $(0,0)$.
2. Prove that the function f has a derivative according to any non-zero vector $h = (h_1, h_2) \in \mathbb{R}^2$ at the point $(0,0)$.

Exercise 4

Let the function $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by, $f(x,y,z) = x + y^2 + z^3$.
Calculate the directional derivative at the point $(1,1,1)$ according to the direction of the following vectors,

1. $\overrightarrow{v} = (2,1,3)$
2. $\overrightarrow{v} = (1,-1,1)$.

Exercise 5

Let f be the function defined on $\mathbb{R}^2$ by $f(x,y) = \begin{cases} xy\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right) & : \ (x,y) \neq (0,0) \\ 0 & : \ (x,y) = (0,0) \end{cases}$

1. Prove that f is continuous on $\mathbb{R}^2(\star)$.
2. Calculate $\nabla f(x,y)$.
3. Show that f has second partial derivatives at all points.

4. What can be deduced from the calculation of $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$, $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$?

Exercise 6

Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$

1. Is f continuous on $\mathbb{R}^2$?
2. Calculate $\nabla f(x,y)$.
3. Is f of class $C^1(\mathbb{R}^2)$?
4. Is f differentiable on $\mathbb{R}^2$?

Exercise 7

Using the definition, check that the following functions $f : \mathbb{R}^2 \to \mathbb{R}$ are differentiable at the given point.

1. $f(x,y) = xy - 3x^2$, $(1,2)$.
2. $f(x,y) = xy - 2y^2$, $(-2,3)$. $(\star)$
3. $f(x,y) = y\sqrt{x}$, $(4,1)$, $(\star)$.

Exercise 8

The function f on $\mathbb{R}^2$ is defined by, $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$

1. Show that f has a partial derivatives in $\mathbb{R}^2$.
2. Study the continuity of f on $\mathbb{R}^2$.
3. Is f differentiable on $\mathbb{R}^2$?

Exercise 9

Let the function $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by, $f(x,y,z) = xy + yz + zx$.
Check that f is differentiable and give it's differential.

Exercise 10

Knowing that the function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and that

$$f(2,5) = 6,\ \partial_x f(2,5) = 1,\ \text{and}\ \ \partial_y f(2,5) = -1,$$

give an approximate value of $f(2.2, 4.9)$.

Exercise 11

Let be the function $f : \mathbb{R}^2 \to \mathbb{R}$ of class $C^1(\mathbb{R}^2)$, such that,

$$f(u,v) = u + v,\ \text{where}\ u(x,y) = e^{x+y}\ \text{and}\ v(x,y) = x^2 + y^2.$$

Calculate the partial derivatives of the first order of the function f.

Exercise 12

Give the limited development of order 2 in $(0,0)$ of the following functions

1. $f(x,y)=\dfrac{\cos x}{\cos y}$.
2. $f(x,y)=\dfrac{e^x}{2+y}$.
3. $f(x,y)=e^{\cos(x+y)}$, $(\star)$.
4. $f(x,y)=e^y\cos x$, $(\star)$.

Exercise 13

We consider the plane curve with equation

$$xe^y+e^x\sin(2y)=0. \tag{1}$$

1. Check that the equation (1) defines one and only one function $y=\varphi(x)$ in the neighbourhood of $(0,0)$.
2. Calculate $\varphi'(0)$ and write the equation of the tangent to the graph of the function φ at the point $(0,\varphi(0))$.
3. Deduce the limit of $\dfrac{y}{x}$ when (x,y) tends to $(0,0)$ on the curve

Supplementary exercise 1

Consider the plane curve with equation

$$2x^3y+2x^2+y^2=0. \tag{2}$$

1. Check that the equation (2) defines one and only one function $y=\varphi(x)$ in the neighbourhood of $(1,1)$.
2. Calculate $\varphi'(0)$ and write the equation of the tangent to the graph of the function φ at the point $(1,\varphi(1))$.
3. Calculate the Taylor expansion of φ for order 1 centred on 1.

Supplementary exercise 2

Consider the equation

$$x^3+4xy+z^2-3yz^2-3=0. \tag{3}$$

1. Check that the equation (3) defines one and only one function $z=\varphi(x,y)$ in the neighbourhood of $(1,1,1)$.
2. Calculate $\dfrac{\partial z}{\partial x}(1,1)$ and $\dfrac{\partial z}{\partial y}(1,1)$.