University of M'sila | Faculty: Maths-Compt-Science

Limits and Continuity

L2 Maths | Module: Analysis 4

Exercise series n°2

Remark 1

The exercise marked by $(\star)$ or supplementary will not be corrected in the sience of **TD**.

Exercise 1

In each case, determine and represent the domain of definition of the given functions if possible.

1. $f_1(x,y) = \ln(x+y-1).(\star)$
2. $f_2(x,y) = \sqrt{1-xy}$.
3. $f_3(x,y) = \dfrac{\sqrt{4-x^2-y^2}}{\sqrt{x^2+y^2-1}}$.
4. $f_4(x,y,z) = \dfrac{1+x^2}{xyz}$.
5. $f_5(x,y) = \dfrac{\ln y}{\sqrt{x-y}}$.
6. $f_6(x,y) = \dfrac{\sqrt{x^2-y}}{\sqrt{y}}.(\star)$

Exercise 2

In each case, determine the level curves of the given two-variable functions.

1. $f(x,y) = x+y-1$
2. $f(x,y) = e^{y-x^2}$.
3. $f(x,y) = y - \cos x.(\star)$
4. $f(x,y) = \ln(x-y^2)$.

Exercise 3

Calculate the limit if it exists in the following cases

1. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$.
2. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{1}{x-y}$.
3. $\displaystyle\lim_{(x,y)\to(1,0)} \frac{y^3}{(x-1)^2+y^2}$.
4. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$, $(\star)$.
5. $\displaystyle\lim_{(x,y)\to(0,\pi)} \frac{1}{x}\sin\left(\frac{x}{y}\right)$.
6. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$
7. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2}$.
8. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y-3}{x+y+1}$, $(\star)$.
9. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{e^x-1}$.

Exercise 4

1. Let the function f be defined on $\mathbb{R}^2$ except the points of the first bisector $(y=x)$, such that $f(x,y) = \dfrac{\sin x - \sin y}{x-y}$. Prove that, $\displaystyle\lim_{(x,y)\to(\alpha,\alpha)} \frac{\sin x - \sin y}{x-y} = \cos\alpha$.
2. Calculate the following limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x}-\sqrt{y}}$.
3. Calculate the following limit $\displaystyle\lim_{\|(x,y)\|_2\to+\infty} e^{-(x^2+y^2)}\sin(2xy)$.

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Exercise 5

Let f be the function of $\mathbb{R}^2 - \{(0,0)\}$ with a value in $\mathbb{R}$ defined by $f(x,y) = \dfrac{2xy^2}{x^2+y^2}$.
Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$ in three different ways

1. By increase.
2. Using polar coordinates.
3. According to definition.

Exercise 6

Let the following two functions $f,\ g$ be defined on $\mathbb{R}^2$ by

1. $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & : \ (x,y) \neq (0,0) \\ 0 & : \ (x,y) = (0,0) \quad (\star). \end{cases}$
2. $g(x,y) = \begin{cases} \dfrac{x+y}{\sqrt{x^2+y^2}} & : \ (x,y) \neq (0,0) \\ 0 & : \ (x,y) = (0,0). \end{cases}$

are they continuous on $\mathbb{R}^2$?.

Exercise 7

Study whether the following functions can be extended by continuity at point a

1. $f(x,y) = \dfrac{x^2y^2}{x^2+y^2},\ a(0,0).$
2. $f(x,y) = \dfrac{x-2}{(x-2)^2+y^2},\ a(2,0).$
3. $f(x,y) = \dfrac{(x+y)^2}{x^2+y^2},\ a(0,0)\ \ (\star).$

Supplementary exercise 1

Show that the following sets are compact

1. $A = \Big\{(x,y) \in \mathbb{R}^2 : \ x^2+y^2 \leq 9, \Big\}.$
2. $B = \Big\{(x,y,z) \in \mathbb{R}^3 : \ \dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1\Big\}.$

Supplementary exercise 2

Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \dfrac{1}{2}x^2 + y^2 - 1, & : \ x^2+y^2 > 1, \\ -\dfrac{1}{2}x^2, & : \ \text{otherwise}. \end{cases}$

Show that f is continuous on $\mathbb{R}^2$.