Topology of $\mathbb{R}^n$

L2 Maths

Module: Analysis 4

Remark 1

The exercise marked by $(\star)$ or supplementary will not be corrected in the sience of **TD**.

Exercise 1

Let $a,\ b$ be two strictly positive reals. For all $u(x,y) \in \mathbb{R}^2$, we note,

$$\|u(x,y)\| = \sqrt{a^2x^2 + b^2y^2}.$$

1. Show that $\|.\|$ is a norm on $\mathbb{R}^2$.
2. Draw the open unit ball for this norm.
3. Show that $\|.\|$ and $\|.\|_2$ are equivalent norms such that $\|u\|_2 = \sqrt{x^2 + y^2}$.

Exercise 2

Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$. We set

$$\|x\|_1 = \sum_{i=1}^{i=n} |x_i|, \quad \|x\|_2 = \left(\sum_{i=1}^{i=n} |x_i|^2\right)^{\frac{1}{2}} \text{ and } \ \|x\|_\infty = \max_{1\leq i\leq n} |x_i|.$$

1. Show that $\|.\|_2$ is a norm on $\mathbb{R}^n$.
2. Show that for all $x \in \mathbb{R}^n$:
$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq n\|x\|_\infty \text{ and } \ \|x\|_2\| \leq \sqrt{n}\|x\|_\infty.$$
3. Represent the closed unit ball in $\mathbb{R}^2$.
$$\overline{B}(0,1) = \{(x,y) \in \mathbb{R}^2 : \ \|(x,y)\| \leq 1\},$$
for each of a normes $\|.\|_1,\ \|.\|_2$ and $\|.\|_\infty$.

Exercise 3

Let be the normed vector space $\mathbb{R}^n$.

1. Show that for all vectors $x,\ y \in \mathbb{R}^n$ we have the inequality
$$\Big|\|x\| - \|y\|\Big| \leq \|x - y\|.$$
2. Deduce that if a sequence $(x_k)_{k\in\mathbb{N}}$ of vectors $x_k \in \mathbb{R}^n$ converges in the normed vector space $(\mathbb{R}^n, \|.\|)$ then,
$$\lim_{k\to+\infty} \|x_k\| = \|\lim_{k\to+\infty} x_k\|.$$

3. Let (x_k) and (y_k) be two vector sequences of $\mathbb{R}^n$. Show that if $\lim\limits_{k\to+\infty} x_k = l \in \mathbb{R}^n$ and $\lim\limits_{k\to+\infty} \|x_k - y_k\| = 0$, we can see $\lim\limits_{k\to+\infty} y_k = l$.

Exercise 4

For $n \in \mathbb{N}$, we put: $x_n = \left(\dfrac{1}{1+n}, 1+e^{-n}\right)$.

1. Study the convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ equipped with the norm $\|.\|_\infty$.
2. Study the convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ equipped with the norm $\|.\|_1$.

Supplementary exercise 1

Let $E = C([-1,1],\mathbb{R})$ be the vector space of continuous functions on $[-1,1]$ with values in $\mathbb{R}$ with infinite norm $\|f\|_\infty = \sup\limits_{x\in[-1,1]} |f(x)|$, and let be the application φ defined by

$$\begin{array}{rccl} \varphi: & E & \longrightarrow & \mathbb{R} \\ & f & \longrightarrow & \varphi(f) = \displaystyle\int_{-1}^{1} \frac{x}{1+x^2} f(x)dx. \end{array}$$

Show that φ is linear and continuous.

Supplementary exercise 2

Let $C([0,1],\mathbb{R})$ be the space of functions defined and continuous on $[0,1]$ with value in $\mathbb{R}$.

1. Prove that $\|f\|_\infty = \sup\limits_{0\le x\le 1} |f(x)|$, and $\|f\|_1 = \displaystyle\int_0^1 |f(x)|dx$ are normes on $C([0,1],\mathbb{R})$, $(\star)$.
2. The sequence $(f_n)_{n\in\mathbb{N}}$ is defined by $f_n(x) = \begin{cases} 1-(n+1)x & : 0 \le x \le \dfrac{1}{n+1} \\ 0 & : \dfrac{1}{n+1} \le x \le 1 \end{cases}$

   Calculer $\|f_n\|_\infty$ et $\|f_n\|_1$.
3. Check that $\|.\|_\infty$ and $\|.\|_1$ are not equivalent.