University of M'sila	**Interrogation**	Faculty : Mathematics-Computer Sc
L2 Mathematics- G : $1-2-3$		Academic year : 2024/2025
Module : Analysis 4	Fourth semester	Duration : 1 hour

Scale	**Exercise : 1** 8pts
	Let f be the function defined on $\mathbb{R}^2$ by : $f(x,y) = \begin{cases} \dfrac{3y^4}{(x-1)^2+y^2} & : (x,y) \neq (1,0) \\ 0 & : (x,y) = (1,0) \end{cases}$
2	1 Show that the function f is continuous on $\mathbb{R}^2$.
2	2 Study the derivability of f at the point $(1,0)$.
2	3 Show that f is differentiable at point $(1,0)$.
2	4 Is f of C^1 class at the point $(1,0)$?

Scale	**Correction of exercise 1:** 8pt
	1 We study the continuity of the function f on $\mathbb{R}^2$.
0.5	a f is clearly continuous on $\mathbb{R}^2 \setminus \{(1,0)\}$. Because it is a rational function.
0.5	b At the point (1.0). We set : $\begin{cases} x = 1 + r\cos\theta & : r > 0,\ \theta \in [0,\ 2\pi[\\ y = r\sin\theta. \end{cases}$
0.5	Then, $\lvert f(x,y)\rvert = \left\lvert \dfrac{3y^4}{(x-1)^2+y^2} \right\rvert = \left\lvert \dfrac{3(r\sin\theta)^4}{(r\cos\theta)^2+(r\sin\theta)^2} \right\rvert = \lvert 3r^2\sin^4\theta\rvert \leq 3r^2 \underset{r\to 0}{\longrightarrow} 0,$
0.5	hence $\lim\limits_{(x,y)\to(1.0)} f(x,y) = 0 = f(1.0)$, in other words, f is continuous at the point (1.0).
	2 f is derivable at the point (1.0) if only if $\dfrac{\partial f}{\partial x}(1.0)$ and $\dfrac{\partial f}{\partial y}(1.0)$ are exist. So,
1	a $\dfrac{\partial f}{\partial x}(1.0) = \lim\limits_{x\to 1} \dfrac{f(x,0)-f(1.0)}{x-1} = \lim\limits_{x\to 0} 0 = 0.$
1	b $\dfrac{\partial f}{\partial y}(1.0) = \lim\limits_{y\to 0} \dfrac{f(1,y)-f(1.0)}{y-0} = \lim\limits_{y\to 0} 3y = 0.$
	Thus, f is derivable at point (1.0), and $\nabla f(1.0) = (0,0)$.
	3 f is differentiable at the point (1.0) if only if
0.5	$\lim\limits_{(x,y)\to(1.0)} \dfrac{f(x,y) - f(1.0) - \partial_x f(1.0)(x-1) - \partial_y f(1.0)(y-0)}{\sqrt{(x-1)^2+y^2}} = 0.$
	We have, $\lim\limits_{(x,y)\to(1.0)} \dfrac{f(x,y) - f(1.0) - \partial_x f(1.0)(x-1) - \partial_y f(1.0)y}{\sqrt{(x-1)^2+y^2}}$
0.5	$= \lim\limits_{(x,y)\to(1.0)} \dfrac{3y^4}{((x-1)^2+y^2)^{\frac{3}{2}}} = 0.$
0.5	Because, $\left\lvert \dfrac{3y^4}{((x-1)^2+y^2)^{\frac{3}{2}}} \right\rvert = \left\lvert \dfrac{3(r\sin\theta)^4}{((r\cos\theta)^2+(r\sin\theta)^2)^{\frac{3}{2}}} \right\rvert = 3r^{\frac{3}{2}}\lvert\sin^4\theta\rvert \leq 3r^{\frac{3}{2}} \underset{r\to 0}{\longrightarrow} 0.$

0.5	Consequently, f is differentiable at the point (1.0), and we have :
	Pour tout $(x,y)\in\mathbb{R}^2 : df_{(1.0)}(x,y)=\partial_x f(1.0)(x-1)+\partial_y f(1.0)(y-0)=0.$
	4 f is of C^1 class at the point $(1,0)$. Indeed :
0.5	a According to question 2, f is differentiable at the point $(1,0)$.
	b It remains to demonstrate that the partial derivatives of f are continuous at $(1,0)$.
	For all $(x,y)\neq(1,0)$, we have :
	$\lvert\partial_x f(x,y)\rvert=\left\lvert\dfrac{-6(x-1)y^4}{((x-1)^2+y^2)^2}\right\rvert=6\left\lvert\dfrac{r\cos\theta(r\sin\theta)^4}{((r\cos\theta)^2+(r\sin\theta)^2)^2}\right\rvert\leq 6r\underset{r\to0}{\longrightarrow}0,$ and,
1	$\lvert\partial_y f(x,y)\rvert=\left\lvert\dfrac{6y^3+(12(x-1)^2y^3)}{((x-1)^2+y^2)^2}\right\rvert=\left\lvert\dfrac{6(r\sin\theta)^5+(12(r\cos\theta)^2(r\sin\theta)^3)}{((r\cos\theta)^2+(r\sin\theta)^2)^2}\right\rvert\leq 18r\underset{r\to0}{\longrightarrow}0.$
	So, we have, $\lim\limits_{(x,y)\to(1,0)}\partial_x f(x,y)=0=\partial_x f(1,0),$
	and $\lim\limits_{(x,y)\to(1,0)}\partial_y f(x,y)=0=\partial_y f(1,0),$
0.5	Thus, the function f is C^1 classe at the point $(1,0)$.