University of M'sila	**Correction of** final exam	Faculty : Mathematics-Computer Sc
L2 Mathematics		Academic year : 2025/2026
Module : Analysis 4		Duration : 1 hour

Scale	**Correction of exercise : 1** (3pts)
	Let $f : \mathbb{R}^2 \setminus \{(1,0)\} \to \mathbb{R}$ be the function defined by : $f(x,y) = \dfrac{(x-1)^2 y}{(x-1)^2+y^2}$.
	(1) Show that f can be extended by continuity at $(1,0)$.
0.5	In polar coordinates, we set : $\begin{cases} x = 1 + r\cos\theta \ : \ r>0,\ \theta \in [0,\ 2\pi[\\ y = r\sin\theta. \end{cases}$
1	Then, $\lvert f(x,y)\rvert = \left\lvert \dfrac{(x-1)^2 y}{(x-1)^2+y^2} \right\rvert = \left\lvert \dfrac{(r\cos\theta)^2 (r\sin\theta)}{(r\cos\theta)^2+(r\sin\theta)^2} \right\rvert$ $= \lvert r\cos^2\theta \sin\theta \rvert \leq r \underset{r\to 0}{\longrightarrow} 0,$
0.5	the limit exists : $\lim\limits_{(x,y)\to(1,0)} f(x,y) = 0.$
	Hence f can be extended by continuity by defining $f(1,0) = 0$.
1	(2) Define $\tilde{f}$, its extended function at $(1,0)$ by : $\tilde{f}(x,y) = \begin{cases} \dfrac{(x-1)^2 y}{(x-1)^2+y^2}, & (x,y) \neq (1,0), \\ 0, & (x,y) = (1,0). \end{cases}$
	The function $\tilde{f}$ is continuous on $\mathbb{R}^2$.

Scale	**Correction of exercise 2:** (7pt)
	Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by : $f(x,y) = \begin{cases} \dfrac{x^3 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$
	(1) Partial derivatives for $(x,y) \neq (0,0)$. we using the standard rules :
1	(a) $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{3x^2y(x^2+y^2) - 2x(x^3y)}{(x^2+y^2)^2} = \dfrac{3x^4y + 3x^2y^3 - 2x^4y}{(x^2+y^2)^2} = \dfrac{x^4y+3x^2y^3}{(x^2+y^2)^2}.$
1	(b) $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{x^3(x^2+y^2) - 2y(x^3y)}{(x^2+y^2)^2} = \dfrac{x^5 + x^3y^2 - 2x^3y^2}{(x^2+y^2)^2} = \dfrac{x^5 - x^3y^2}{(x^2+y^2)^2}.$
	(2) Partial derivatives at $(0,0)$. We use the definition of partial derivatives :
1	(a) $\dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0-0}{h} = 0.$
1	(b) $\dfrac{\partial f}{\partial y}(0,0) = \lim\limits_{k\to 0} \dfrac{f(0,k)-f(0,0)}{k} = \lim\limits_{k\to 0} \dfrac{0-0}{k} = 0.$
	f has partial derivatives at $(0,0)$ and $\nabla f(0,0) = (0,0)$.
	(3) (a) f is differentiable at the point $(0,0)$ if only if :
0.5	$\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y) - f(0,0) - \partial_x f(0,0)(x-0) - \partial_y f(0,0)(y-0)}{\sqrt{x^2+y^2}} = 0.$
0.5	(b) Since $df_{(0,0)}(x,y) = \partial_x f(0,0)(x-0) + \partial_y f(0,0)(y-0) = 0$ (if exists).

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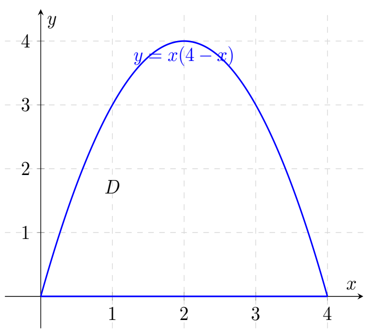
0.5	(c) Then, We have : $\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y)}{\sqrt{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y}{(x^2+y^2)^{\frac{3}{2}}}$. By using polar coordinate, we have $\left\lvert \dfrac{x^3y}{(x^2+y^2)^{\frac{3}{2}}} \right\rvert \leq \dfrac{r^4\cos^3\theta\sin\theta}{r^3} = r\lvert\cos^3\theta\sin\theta\rvert \leq r \underset{r\to0}{\longrightarrow} 0.$ The limit is indeed 0. Then f is differentiable into $(0,0)$. (4) For $(x,y)\neq(0,0)$, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous. Let's verify this at $(0,0)$. by using the polars again for $\dfrac{\partial f}{\partial x}$: $\left\lvert \dfrac{\partial f}{\partial x}(r\cos\theta, r\sin\theta) \right\rvert = \left\lvert \dfrac{r^5\cos^4\theta\sin\theta + 3r^5\cos^2\theta\sin^3\theta}{r^4} \right\rvert = r\lvert cos^4\theta\sin\theta + 3\cos^2\theta\sin^3\theta\rvert$ $\leq 4r.$ Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{\partial f}{\partial x}(x,y) = 0 = \dfrac{\partial f}{\partial x}(0,0)$. The partial derivative $\dfrac{\partial f}{\partial x}$ is continuous. A similar argument shows that $\dfrac{\partial f}{\partial y}$ is continuous at $(0,0)$. Because $\left\lvert \dfrac{\partial f}{\partial y}(r\cos\theta, r\sin\theta) \right\rvert = \left\lvert \dfrac{r^5\cos^5\theta - r^5\cos^3\theta\sin^2\theta}{r^4} \right\rvert = r\lvert cos^5\theta - \cos^3\theta\sin^2\theta\rvert \leq 2r \underset{r\to0}{\longrightarrow} 0.$ Then f is of class C^1 on $\mathbb{R}^2$.
Scale	**Exercise : 3** 7pt
	f defined on $\mathbb{R}^2$ by : $f(x,y) = x^3 + y^3 - 3xy.$
	(1) Let us calculate the critical points of f. We have
0.5	(a) $\nabla f(x,y) = \begin{pmatrix} \partial_x f(x,y) \\ \partial_y f(x,y) \end{pmatrix} \Rightarrow \nabla f(x,y) = \begin{pmatrix} 3x^2-3y \\ 3y^2-3x \end{pmatrix}.$
0.5	(b) So, $\nabla f(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 4x(x^2-1)=0, \\ y=0 \end{cases} \Rightarrow \begin{cases} y=x^2, \\ 3x(x^3-1)=0 \end{cases}$
0.5	(c) Therefore, the critical points are $A\Big(0,0\Big)$, and $B\Big(1,1\Big)$.
	(2) Nature of critical points :
0.5	(a) The hessian is given by $\mathrm{H}_f(x,y) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(x,y) & \dfrac{\partial^2 f}{\partial x\partial y}(x,y) \\ \dfrac{\partial^2 f}{\partial x\partial y}(x,y) & \dfrac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$
	(b) The determinant of the Hessian matrix at the point $A\Big(0,0\Big)$ is
0.5	$\left\lvert H_f\Big(0,0\Big) \right\rvert = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = -9 < 0.$ Then, $A\Big(0,0\Big)$ is saddle point.

0.5	c For the point $B\left(1,1\right)$, we have : $\left\|H_f\left(1,0\right)\right\| = \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 27 > 0.$
0.5	Since $\frac{\partial^2 f}{\partial x^2}(B) = 6 > 0,$ the function f has a local minimum at $B(1,0)$.
0.5	3 We have $f(1,1) = -1.$
0.5	So, for $y = 0,$ we have : $f(x,0) = x^3 \underset{x\to-\infty}{\longrightarrow} -\infty < -1.$
	Then, $B(1,1)$ doesn't a global minima for f.
0.5	Also, $f(x,0) = x^3 \underset{x\to+\infty}{\longrightarrow} +\infty$
0.5	Since the function is not bounded above, on $\mathbb{R}^2$, it does not have a global maximum.
Scale	**Exercise : 4** 4pt

	Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ be the function defined by : $f(x,y) = xy,$
0.5	1 We drawing the domain of integration $D = \{(x,y) \in \mathbb{R}^2 : \ x \geq 0,\ 0 \leq y \leq x(4-x)\}.$



2 $\mu(D)$, the area of D given by : $\mu(D) = \int_{x=0}^{4}\int_{y=0}^{x(4-x)} dy\,dx.$

First, integrate with respect to y then to x, we obtain

0.5 + 1
$$\mu(D) = \int_{x=0}^{x=4}\left[\int_{y=0}^{x(4-x)} dy\right]dx = \int_{x=0}^{x=4}[y]_0^{x(4-x)}\,dx = \int_{x=0}^{x=4} x(4-x)dx.$$
$$= \left|2x^2 - \frac{x^3}{3}\right|_0^4 = \boxed{\frac{32}{3}}.$$

1
3 $$I = \int_{x=0}^{4}\int_{y=0}^{x(4-x)} xy\,dy\,dx. = \int_{x=0}^{4}\left[\frac{1}{2}xy^2\right]_{y=0}^{y=x(4-x)} dx = \frac{1}{2}\int_0^4 x^3(4-x)^2\,dx$$

1
$$= \frac{1}{2}\int_0^4 (16x^3 - 8x^4 + x^5)\,dx = \frac{1}{2}\left[4x^4 - \frac{8}{5}x^5 + \frac{1}{6}x^6\right]_0^4 = \boxed{\frac{512}{15}}.$$

End — **Dr. Bouafia Dahmane** — **Good luck**