University of M'sila		**Faculty : Mathematics-Computer Sc**
L2 Mathematics	**Final Exam**	**Academic year : 2024/2025**
Module : Analysis 4	**with correction**	**Duration : $1h - 30m$**

Scale	**Exercise : 1** 6pts
	Let f be the function defined on $\mathbb{R}^3$ by : $f(x, y, z) = x^2(y + 1) - \ln(2y + z)$, the point $a(1, 0, 1)$ and the vector $v(-1, 1, 1)$ in $\mathbb{R}^3$.
2	1 Calculate $D_v f(a)$ the directional derivative of f at the point a along the direction v.
2.5	2 a Check the equality : $D_v f(a) = \langle \nabla f(a), v \rangle$.
	b Deduce the value of $df_a(v)$.
1.5	3 Compute an approximate value for $f(-0, 9, 1, 1.1)$.

Scale	**Exercise : 2** 7pts
	Let f be the function defined on $\mathbb{R}^2$ by : $$f(x,y) = \begin{cases} \dfrac{3x^3y}{x^2+(y+1)^2} & : (x,y) \neq (0,-1) \\ 0 & : (x,y) = (0,-1) \end{cases}$$
1.5	1 Show that the function f is continuous at the point $(0, -1)$.
2	2 Study the derivability of f at the point $(0, -1)$.
1	3 Show that f is C^1 class on $\mathbb{R}^2 \setminus \{(0, -1)\}$.
2	4 a Show that f is not differentiable at the point $(0, -1)$.
0.5	b Is f of C^1 class at the point $(0, -1)$?

Scale	**Exercise 3:** 7pts
	Let f be the function defined on $\mathbb{R}^2$ by : $f(x, y) = x^4 - 2x^2 + y^2$.
2.5	1 Find the critical points of f.
3.5	2 Establish their nature (local extremum, saddle point, ...).
1	3 Show that the local minimums obtained are a global minimums for f.

Scale	**Correction of exercise : 1** 6pts
	Let f be the function defined on $\mathbb{R}^2$ by : $f(x,y,z) = x^2(y+1) - \ln(2y+z)$, and the vector $v(-1,1,1)$.
4×0.5	1 The directional derivative of f at point $a(1,0,1)$ along the v direction is $D_v f(a) = \lim\limits_{t\to 0} \dfrac{f(a+tv) - f(a)}{t} = \lim\limits_{t\to 0} \dfrac{f(1-t,t,1+t) - f(1,0,1)}{t}$ $= \lim\limits_{t\to 0}(-1-t+t^2) - \lim\limits_{t\to 0} \dfrac{\ln(3t+1)}{t} = -4$
1	2 a We have, $\nabla f(x,y,z) = \begin{pmatrix} \dfrac{\partial f}{\partial x}(x,y,z) \\ \dfrac{\partial f}{\partial y}(x,y,z) \\ \dfrac{\partial f}{\partial z}(x,y,z) \end{pmatrix} = \begin{pmatrix} 2x(y+1) \\ x^2 - \dfrac{2}{2y+z} \\ -\dfrac{1}{2y+z} \end{pmatrix}$
0.5	Consequently, we obtain, $\nabla f(1,0,1) = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$.
0.5	So, we find, $\langle \nabla f(a), v \rangle = 2 \times (-1) + (-1) \times 1 + (-1) \times 1 = -4 = D_v f(a)$.
0.5	b We know that $df_a(v) = D_v f(a)$. Then, $df_a(v) = -4$.
0.5	3 As $f(-1+h, 1+k, 1+l) = f(-1,1,1) + df_{(1,0,1)}(h,k,l) + o\Big(\\|(h,k,l)\\|\Big)$, here,
0.5	$(h,k,l) = (0.5, 0, 0.1)$. So, $f(-1+h, 1+k, 1+l) \approx f(-1,1,1) + df_{(1,0,1)}(h,k,l)$, if $\\|(h,k,l)\\| \to 0$.
0.5	As a result, we have, $f(-0.9, 1, 1.1) \approx 2 - \ln 3 + 2 \times 0.1 + (-1) \times 0 + (-1) \times 0.1 = 2.1 - \ln 3 = 1.01$.

Scale	**Correction of exercise 2:** 7pt
	1 We study the continuity of the function f on $(0,-1)$.
0.5	a At the point $(0,-1)$. We set : $\begin{cases} x = r\cos\theta & : \ r > 0, \ \theta \in [0, \ 2\pi[\\ y = -1 + r\sin\theta. \end{cases}$
0.5	Then, $\lvert f(x,y)\rvert = \left\lvert \dfrac{3x^3 y}{x^2 + (y+1)^2} \right\rvert = \left\lvert \dfrac{3(r\cos\theta)^3(-1+r\sin\theta)}{(r\cos\theta)^2 + (r\sin\theta)^2} \right\rvert$ $= \lvert 3r\cos^3\theta \sin\theta \rvert \lvert -1 + r\sin\theta \rvert \leq 3r(1+r) \underset{r\to 0}{\to} 0$,
0.5	hence $\lim\limits_{(x,y)\to(0,-1)} f(x,y) = 0 = f(0,-1)$, i.e, f is continuous at the point $(0,-1)$.

	2 f is derivable at the point $(0,-1)$ if only if $\dfrac{\partial f}{\partial x}(0,-1)$ and $\dfrac{\partial f}{\partial y}(0,-1)$ are exist. So,
1	a $\dfrac{\partial f}{\partial x}(0,-1) = \lim\limits_{x\to 0} \dfrac{f(x,-1)-f(0,-1)}{x} = \lim\limits_{x\to 0}(-3) = -3.$
1	b $\dfrac{\partial f}{\partial y}(0,-1) = \lim\limits_{y\to -1} \dfrac{f(0,y)-f(0,-1)}{y+1} = \lim\limits_{y\to -1} 0 = 0.$
	Thus, f is derivable at the point $(0,-1)$, and $\nabla f(0,-1) = (-3,0)$.
1	3 f is continuous and derivable on $\mathbb{R}^2 \setminus \{(0,-1)\}$, its partial derivatives are continuous on $\mathbb{R}^2 \setminus \{(0,-1)\}$ because they are quotients of continuous functions whose denominators do not cancel. Then f is of class $C^1(\mathbb{R}^2 \setminus \{(0,-1)\})$.
	4 a f is not differentiable at the point $(0,-1)$ if only if :
0.5	$\lim\limits_{(x,y)\to(0,-1)} \dfrac{f(x,y)-f(0,-1)-\partial_x f(0,-1)(x-0)-\partial_y f(0,-1)(y+1)}{\sqrt{x^2+(y+1)^2}} \neq 0$, or not exists.
	We have, $\lim\limits_{(x,y)\to(0,-1)} \dfrac{f(x,y)-f(0,-1)-\partial_x f(0,-1)x-\partial_y f(0,-1)(y+1)}{\sqrt{x^2+(y+1)^2}}$
0.5	$= \lim\limits_{(x,y)\to(0,-1)} \dfrac{3x^3y+3x^3+3x(x^2+(y+1)^2)}{(x^2+(y+1)^2)^{\frac{3}{2}}}$. By using polar coordinate, we have
0.5	$\left\lvert \dfrac{3x^3y+3x^3+3x(x^2+(y+1)^2)}{(x^2+(y+1)^2)^{\frac{3}{2}}} \right\rvert = 3r\lvert\cos^3\theta\sin\theta\rvert + 3\lvert\cos^3\theta\rvert + 3\lvert\cos\theta\sin^2\theta\rvert$
	$\underset{r\to 0}{\longrightarrow} 3\lvert\cos^3\theta\rvert + 3\lvert\cos\theta\sin^2\theta$, the limit depending to θ.
0.5	Consequently, f is not differentiable at the point $(0,-1)$.
0.5	b Since f is not differentiable in $(0,-1)$, it is not of class C^1 at $(0,-1)$.
Scale	**Exercise : 3** 7pt
	f defined on $\mathbb{R}^2$ by : $f(x,y) = x^4 - 2x^2 + y^2$.
	1 Let us calculate the critical points of f. We have
1	a $\nabla f(x,y) = \begin{pmatrix} \partial_x f(x,y) \\ \partial_y f(x,y) \end{pmatrix} \Rightarrow \nabla f(x,y) = \begin{pmatrix} 4x^3-4x \\ 2y \end{pmatrix}$.
1	b So, $\nabla f(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 4x(x^2-1)=0, \\ y=0 \end{cases} \Rightarrow \begin{cases} x=0 \vee x=1 \vee x=-1, \\ y=0 \end{cases}$
0.5	Therefore, the critical points are $A\Big(0,0\Big)$, $B\Big(1,0\Big)$ and $C\Big(-1,0\Big)$.
	2 Nature of critical points :
0.5	a The hessian is given by $\mathrm{H}_f(x,y) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(x,y) & \dfrac{\partial^2 f}{\partial x \partial y}(x,y) \\ \dfrac{\partial^2 f}{\partial x \partial y}(x,y) & \dfrac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix}$

1	As a result, we find, $\mathrm{H}_{f}(x, y) = \begin{pmatrix} 12x^2 - 4 & 0 \\ 0 & 2 \end{pmatrix}$
	(b) The determinant of the Hessian matrix(Jacobian) at the point $A\Big(0, 0\Big)$ is
0.5	$\left\vert H_f\Big(0,0\Big)\right\vert = \begin{vmatrix} -4 & 0 \\ 0 & 2 \end{vmatrix} = -8 < 0.$ Then, $A\Big(0,0\Big)$ is saddle point.
0.5	(c) For the point $B\Big(1,0\Big),$ we have : $\left\vert H_f\Big(1,0\Big)\right\vert = \begin{vmatrix} 8 & 0 \\ 0 & 2 \end{vmatrix} = 16 > 0.$
0.5	As $\dfrac{\partial^2 f}{\partial x^2}(B) = 8 > 0,$ we have a local minimum at $B(1,0).$
0.5	(d) At $C\Big(-1,0\Big),$ we have : $\left\vert H_f\Big(-1,0\Big)\right\vert = \begin{vmatrix} 8 & 0 \\ 0 & 2 \end{vmatrix} = 16 > 0.$ Since
	$\dfrac{\partial^2 f}{\partial x^2}(B) = 8 > 0,$ so, the point $C(-1,0),$ is a local minimum.
0.5	(3) Since $f(1,0) = f(-1,0) = -1.$ So, we have
0.5	$\forall (x,y) \in \mathbb{R}^2 : \begin{cases} f(x,y) - f(1,0) = (x^2-1)^2 + y^2 \geq 0, \\ f(x,y) - f(-1,0) = (x^2-1)^2 + y^2 \geq 0. \end{cases}$
	Then, $B(1,0)$ and $C(-1,0)$ are a global minimums for $f.$
End	**Dr. Bouafia Dahmane** **Good luck**