7 CHAPTER

Multiple Integrals

The multiple integral is a generalization of the definite integral of a function of a single variable to the case of a function of several variables. This notion is often used to calculate volumes, surface areas, masses and centers of gravity.

7.1 Reminder of simple integrals

Let f be a function defined on the interval $[a,b]$. Let us divide this interval into n of subintervals $I_i = [x_{i-1}, x_i]$ by performing a constant-step subdivision $\Delta x = x_i - x_{i-1} = \dfrac{b-a}{n}$. Let us consider in each subinterval $[x_{i-1}, x_i]$ a point x_i^*. The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x,$$

is called the Riemann sum of f over $[a,b]$. If, when n tends to infinity, the Riemann sums tend to a limit that does not depend on the choice of x_i^*, we say that the function f is integrable on the interval $[a,b]$ and we have

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i^*).$$

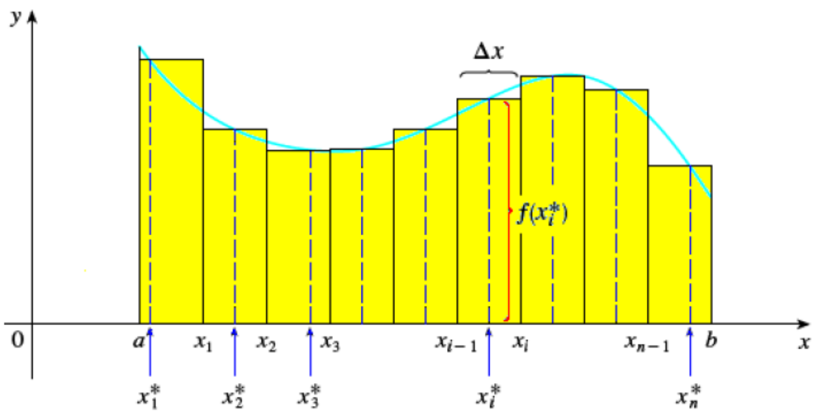


Fig. 7.1 . Graphical representation S_n

$$\int_a^b f(x)\,dx \quad \text{is called the definite integral of } f \text{on the interval } [a,b].$$

Following this, by an analogous procedure, we can define the integral of a function of several variables.

7.2 Riemann integrals and area in the plane

7.2.1 Riemann's integral

We have seen that it is useful to derive functions of several variables with respect to one variable while taking the other constants into account. For example, keeping y constant, we can write

$$\int_1^{3y} 2xy\,dx = \left[x^2y\right]_0^{3y} = (3y)^2y + (1)^2y = 9y^3 - y$$

Note that the integration variable cannot appear in either of the two integration limits. For example, writing

$$\int_0^x ye^x\,dx,$$

Doesn't make sense.

As the previous example shows, $\displaystyle\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx$ is a quantity that depends on y, so it is a function of y

$$A(y) = \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx.$$

If we now integrate the function A with respect to y from $y = c$ to $y = d$, we obtain

$$\int_c^d A(y)\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy.$$

The integral on the right side of the above equality is called the *iterated integral.* The brackets can be omitted. Consequently,

$$\int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy.$$

Similarly, we define the iterated integral

$$\int_a^b \left[\int_{k_1(x)}^{k_2(x)} f(x,y)\,dy\right] dx = \int_a^b \int_{k_1(x)}^{k_2(x)} f(x,y)\,dydx.$$

The bounds of the interior integral can depend on the variable of integration of the exterior integral. However, the bounds of the outer integral must be constant with respect to both variables of integration.

Example 7.1. *Compute the following iterated integrals*

$$(a) \int_1^3 \int_0^2 xy^2 \, dx dy \quad (b) \int_0^2 \int_1^3 xy^2 \, dy dx.$$

a. *If we take* y *to be a constant, we get*

$$A(y) = \int_0^2 xy^2 \, dx = \left[\frac{x^2}{2} y^2\right]_0^2 = \frac{(2)^2}{2} y^2 - \frac{(0)^2}{2} y^2 = 2y^2.$$

We now integrate this function of y *from* 1 *to* 3

$$\int_1^3 A(y) \, dy = \int_1^3 2y^2 \, dy = \left[\frac{2y^3}{3}\right]_1^3 = \frac{2 \cdot 3^3}{3} - \frac{2 \cdot 1^3}{3} = \frac{54}{3} - \frac{2}{3} = \frac{52}{3}.$$

b. *In this case, we first integrate with respect to* y

$$\begin{aligned} \int_0^2 \int_1^3 xy^2 \, dy dx &= \int_0^2 \left[\int_1^3 xy^2 \, dy\right] dx \\ &= \int_0^2 \frac{26}{3} \, dx = \left[\frac{26x^2}{6}\right]_0^2 = \frac{52}{3}. \end{aligned}$$

Note that in this example, we obtained the same result regardless of whether we first integrated with respect to y *or* x*.*

7.2.2 Area of a plane region

Consider the region of the plane R bounded by

$$a \leqslant x \leqslant b \quad \text{and} \quad \varphi_1(x) \leqslant y \leqslant \varphi_2(x),$$

we note $R = \{(x, y) \in \mathbb{R}^2 : \ a \leqslant x \leqslant b, \varphi_1(x) \leqslant y \leqslant \varphi_2(x)\}$.

The area of the region is equal to the integral defined by

$$\text{Area}(R) = \int_a^b [\varphi_2(x) - \varphi_1(x)] \, dx.$$

We can write the integrand $\varphi_2(x) - \varphi_1(x)$ as a definite integral. More precisely, we have,

$$\int_{\varphi_1(x)}^{\varphi_2(x)} dy = [y]_{\varphi_1(x)}^{\varphi_2(x)} = \varphi_2(x) - \varphi_1(x).$$

Combining the above, we can express the area of R in terms of an iterated integral

$$\int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} dy\right) dx = \int_a^b [\varphi_2(x) - \varphi_1(x)] dx.$$

Corollary 7.1.

❶ *Let* $R = \{(x,y) \in \mathbb{R}^2 : \ a \leqslant x \leqslant b, \varphi_1(x) \leqslant y \leqslant \varphi_2(x)\}$ *with* φ_1 *and* φ_2 *two continuous functions on the interval* $[a,b]$. *The area of the region* R *is given by*

$$Area(R) = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} dy \right) dx.$$

❷ *Let* $R = \{(x,y) \in \mathbb{R}^2 : \psi_1(y) \leqslant x \leqslant \psi_2(y), c \leqslant y \leqslant d\}$ *with* ψ_1 *and* ψ_2 *two continuous functions on the interval* $[c,d]$. *The area of the region* R *is given b*

$$Area(R) = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} dx \right) dy.$$

Example 7.2. *Draw the region of integration of the following integral, then calculate it*

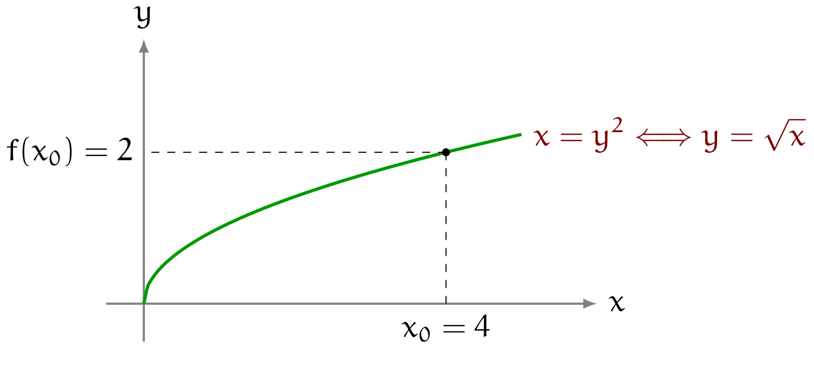
$$\int_0^2 \int_{y^2}^4 dxdy.$$

We have,

$$R = \{(x,y) \in \mathbb{R}^2 : y^2 \leqslant x \leqslant 4, \ 0 \leqslant y \leqslant 2\}.$$

The area of region R *is*

$$\begin{aligned} Area(R) &= \int_0^2 \int_{y^2}^4 dxdy \\ &= \int_0^2 [x]_{y^2}^4 \, dy \\ &= \int_0^2 4 - y^2 dy \\ &= \left[4y - \frac{y^3}{3}\right]_0^2 \\ &= \frac{16}{3}. \end{aligned}$$



We can change the order of integration, but first we have to change the bounds of the integral. Note that the region R *can also be determined as follows*

$$R = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 4, \ 0 \leqslant y \leqslant \sqrt{x}\}.$$

Par conséquent

$$\begin{aligned}
Area(R) &= \int_0^4 \int_0^{\sqrt{x}} dy\,dx \\
&= \int_0^4 [y]_0^{\sqrt{x}}\,dx \\
&= \int_0^4 \sqrt{x}\,dx \\
&= \left[\frac{2}{3}x^{3/2}\right]_0^4 \\
&= \frac{16}{3}.
\end{aligned}$$

7.3 Intégrales doubles

In this section, we introduce the notion of the double integral for continuous functions of two variables. Next, we present how to evaluate the double integral using changes of variables and Fubini's theorem. Finally, we will learn how to apply the double integral to calculate the volume of a solid bounded by graphs of functions of two variables.

7.3.1 Double integral on a rectangle

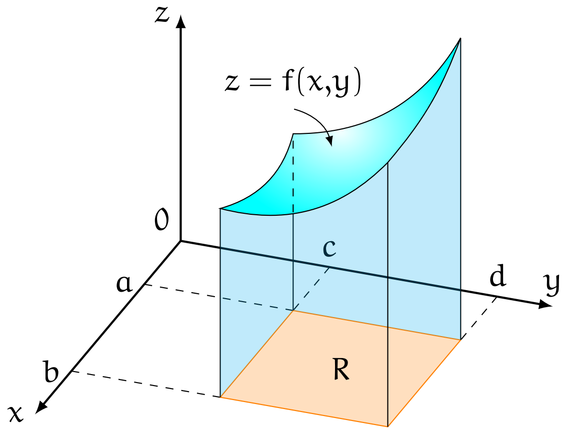
The construction of the double integral on a rectangle is analogous to the construction of the integral of a function of a single variable over a segment. We consider a function f of two continuous variables on a closed rectangle

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 : a \leqslant x \leqslant b, c \leqslant y \leqslant d\}.$$

We assume that $f(x,y) \geqslant 0$ for all $(x,y) \in R$. Our goal is to make sense of the quantity

$$\iint_R f(x,y)\,dx\,dy,$$

which is the volume V of the solid of $\mathbb{R}^3$ delimited by the surface $z = f(x,y)$, the horizontal plane $z = 0$ and the four planes $x = a, x = b, y = c$ et $y = d$.



We consider a regular grid of the rectangle R obtained by performing a subdivision for each of the two intervals $[a, b]$ and $[c, d]$

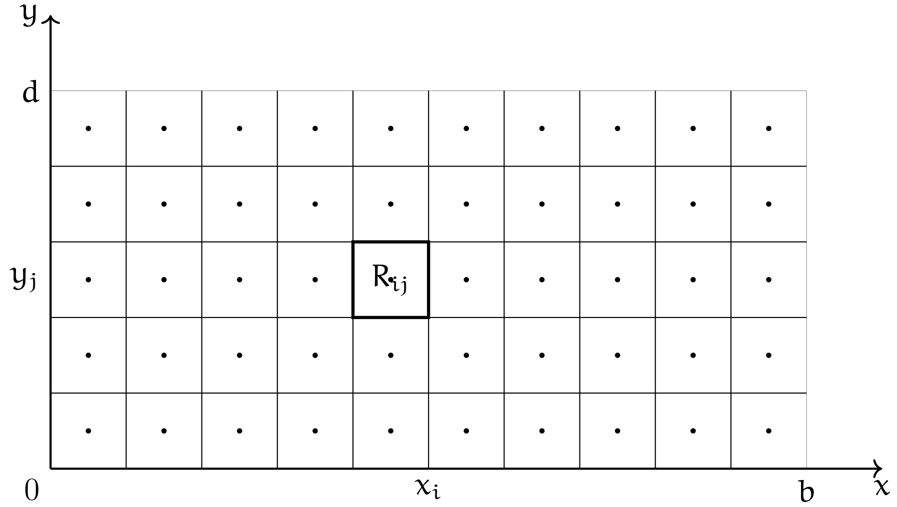
$$a = x_0 < x_1 < \ldots < x_n = b \quad \text{and} \quad c = y_0 < y_1 < \ldots < y_m = d,$$

avec

$$x_i - x_{i-1} = \Delta x = \frac{b-a}{n} \quad \text{and} \quad y_j - y_{j-1} = \Delta y = \frac{d-c}{m}.$$

Thus the resulting grid consists of the $n \times m$ small rectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \quad \text{with} \quad 1 \leqslant i \leqslant n \quad \text{and} \quad 1 \leqslant j \leqslant m.$$



We choose any point (x_{ij}^*, y_{ij}^*) in each R_{ij}. The volume of the rectangular parallelepiped with base R_{ij} is given by the area of the base R_{ij}, multiplied by the height $f(x_{ij}^*, y_{ij}^*)$

$$f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y = f(x_{ij}^*, y_{ij}^*)\frac{(b-a)\,(d-c)}{nm}.$$

By adding the volumes of these $n \times m$ small rectangular parallelepipeds, we obtain an approximation of the volume V

$$V \approx S_{n,m} = \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\frac{(b-a)(d-c)}{nm}.$$

The sum $S_{n,m}$ is called the double Riemann sum, improving as n and m increase. In the limit, we expect that

$$V = \lim_{n,m\to+\infty} \frac{(b-a)(d-c)}{nm} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*).$$

The double integral is defined in a similar way to the simple integral.

Definition 7.1. *If, when $n \to +\infty$ and $m \to +\infty$ the sequence $(S_{n,m})_{n,m}$ tends towards a limit independent of the choice of sample points (x_{ij}^*, y_{ij}^*), we say that f is integrable on the rectangle R. We then call this limit the double integral of f on R and we denote it*

$$\iint_R f(x, y)\, dx\, dy.$$

We write

$$\iint_R f(x,y)\,dx\,dy = \lim_{n,m\to+\infty} \frac{(b-a)(d-c)}{nm} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*).$$

Example 7.3. *Let the rectangle* $R = [1,3]\times[0,1]$ *and the function* $f(x,y) = x - 3y^2$*, we consider the following subdivision*

$$x_i = 1 + \frac{2i}{n}, \qquad i = 1,\dots,n$$
$$y_j = \frac{j}{m}, \qquad j = 1,\dots,m.$$

Let $\bar{x}_i = \dfrac{x_i + x_{i-1}}{2} = 1 + \dfrac{2i-1}{2n}$ *be the midpoint of* $[x_{i-1}, x_i]$ *and* $\bar{y}_j = \dfrac{y_j + y_{j-1}}{2} = \dfrac{2j-1}{2m}$ *is the centre of* $[y_{j-1}, y_j]$*. We want to compute the double Riemann sum with the sample point* $(x_{ij}^*, y_{ij}^*) = (\bar{x}_i, \bar{y}_j)$ *as the centre of* R_{ij}*.*

$$\begin{aligned}
2\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*) &= 2\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\left(1+\frac{2i-1}{n}\right) - 3\left(\frac{2j-1}{2m}\right)^2\right]\\
&= \frac{2}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\left(1+\frac{2i-1}{n}\right) - \frac{3}{4m^2}(2j-1)^2\right]\\
&= \frac{2}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(1+\frac{2i-1}{n}\right) - \frac{3}{4m^2}\sum_{i=1}^{n}\sum_{j=1}^{m}(4j^2-4j+1)\\
&= \frac{2}{nm}\sum_{i=1}^{n}\left[m\left(1+\frac{2i-1}{n}\right) - \frac{3}{4m^2}\left(4\frac{m(m+1)(2m+1)}{6} - 4\frac{m(m+1)}{2} + m\right)\right]\\
&= \frac{2}{nm}\left[2nm - \frac{3}{4m^2}\left(\frac{m(m+1)(2m+1)}{6} - \frac{m(m+1)}{2} + m\right)\right]\\
&= 4 - \frac{2(m+1)(m-1)}{m^2} - \frac{3}{2m^2}
\end{aligned}$$

It follows that

$$\iint_R f(x,y)\,dx\,dy = \lim_{m\to+\infty}\left(4 - \frac{2(m+1)(m-1)}{m^2} - \frac{3}{2m^2}\right) = 2.$$

Corollary 7.2. *If* $f(x,y) \geqslant 0$ *for all* $(x,y) \in R$*, then the volume of the solid* S *above the rectangle* R *and below the surface* $z = f(x,y)$ *is*

$$Vol(S) = \iint_R f(x,y)\,dx\,dy.$$

Sometimes, calculating double integrals using the definition can be complicated. The following theorem allows a double integral to be expressed as an iterated integral.

Theorem 1 (Fubini's theorem) If f is integrable on the rectangle $R = \{(x,y) \in \mathbb{R}^2 : a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$, with

$$\iint_R f(x,y)\,dx\,dy = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy.$$

Example 7.4. *Let's calculate the double integral* $\iint_R (x - 3y^2)\,dx\,dy$ *where* $R = [1,3] \times [0,1]$. *According to Fubini's theorem, we have*

$$\begin{aligned}\iint_R (x-3y^2)\,dx\,dy &= \int_1^3 \left[\int_0^1 (x-3y^2)\,dy\right] dx = \int_1^3 \left[xy - y^3\right]_0^1 dx\\ &= \int_1^3 (x-1)\,dx = \left[\frac{1}{2}x^2 - x\right]_1^3 = 2.\end{aligned}$$

The same result is obtained by first integrating with respect to x. *This is because*

$$\begin{aligned}\iint_R (x-3y^2)\,dx\,dy &= \int_0^1 \left[\int_1^3 (x-3y^2)\,dx\right] dy = \int_0^1 \left[\frac{1}{2}x^2 - 3xy^2\right]_1^3 dy\\ &= \int_0^1 (4-6y^2)\,dy = \left[4y - 2y^3\right]_0^1 = 2.\end{aligned}$$

Example 7.5. *Let* $R = \{(x,y) \in \mathbb{R}^2 : -1 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 1\}$ *and* $f : \mathbb{R}^2 \to \mathbb{R}$ *définie par* $f(x,y) = xe^{x+y}$. *Let's calculate* $I = \iint_R f(x,y)\,dx\,dy$. *First, we integrate with respect to* y, *and we obtain*

$$\begin{aligned}I &= \int_{-1}^2 \left[\int_0^1 xe^{x+y}\,dy\right] dx\\ &= \int_{-1}^2 \left[e^{x+y}\right]_0^1 dx\\ &= \int_{-1}^2 (e^x - 1)\,dx\\ &= \left[e^x - x\right]_{-1}^2\\ &= e^2 - e^{-1} - 3.\end{aligned}$$

Remark 7.1. *By integrating first with respect to* y, *in the previous example, the previous calculation was straightforward. If we reverse the order of integration and integrate first with respect to* x, *we soon realise that the calculation is less straightforward because we have to integrate by parts twice. Consequently, when applying Fubini's Theorem, we need to choose the order of integration carefully to obtain integrals that are easy to calculate.*

Example 7.6. *Calculate the volume* V *of the solid bounded by the surface* $z = 1 + x^2ye^y$ *and the planes* $z = 0, x = \pm 1, y = 0$ *and* $y = 1$.

$$\begin{aligned}V &= \int_0^1 \left[\int_{-1}^1 (1 + x^2ye^y)\,dx\right] dy = \int_0^1 \left[x + \frac{1}{3}x^3ye^y\right]_{-1}^1 dy\\ &= \int_0^1 \left(2 + \frac{2}{3}ye^y\right) dy = \left[2y + \frac{2}{3}(y-1)e^y\right]_0^1 = \frac{8}{3}.\end{aligned}$$

If f *is a function with separate variables, i.e.* $f(x,y) = g(x)h(y)$, *where* $g : [a,b] \to \mathbb{R}$ *and*

$g : [c, d] \to \mathbb{R}$ are continuous functions, then Fubini's theorem gives

$$\begin{aligned}\iint_R f(x,y)\,dx\,dy &= \int_c^d \left[\int_a^b g(x)h(y)\,dx\right] dy \\ &= \int_c^d h(y) \left(\int_a^b g(x)\,dx\right) dy \\ &= \int_a^b g(x)\,dx \int_c^d h(y)\,dy,\end{aligned}$$

where $R = [a, b] \times [c, d]$. In other words, we can write the double integral of f as the product of two simple integrals

$$\iint_R f(x,y)\,dx\,dy = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

Example 7.7. *If $R = [0, 2] \times [0, 1]$, then,*

$$\begin{aligned}\iint\limits_R e^{x+2y}\,dxdy &= \int_0^2 \left(\int_0^1 e^{x+2y}\,dx\right) dy = \int_0^2 \left[\frac{1}{2}e^{x+2y}\right]_0^1 dy = \int_0^2 \frac{1}{2}e^{2y}\,(e-1)\,dy. \\ &= \left[e^{2y}\frac{1}{2}(e-1)\right]_0^2 = (e^2-1)\left(\frac{1}{2}e^2 - \frac{1}{2}\right) = \frac{1}{2}(e^2-1)^2.\end{aligned}$$

7.3.2 Double integral on a bounded domain

The aim of this section is to define the double integral on an arbitrary bounded domain D. Since D is bounded, we can include it in a rectangle $R = [a, b] \times [c, d]$.

Definition 7.2. *Let $D \subseteq \mathbb{R}^2$ be a bounded domain and $f : D \to \mathbb{R}$ be a bounded function. Let* tildef *be the extension of f to R by 0*

$$\tilde{f}(x, y) = \begin{cases} f(x, y) & : (x, y) \in D \\ 0 & : (x, y) \in R \setminus D. \end{cases}$$

We say that f is integrable on D if tildef *is integrable on R, and in this case the double integral of f over D is defined as the double integral of* tildef *over R, namely*

$$\iint\limits_D f(x, y)\,dxdy = \iint\limits_R \tilde{f}(x, y)\,dxdy.$$

Theorem 2 Any continuous function on a compact (closed and bounded) set is integrable on this set.

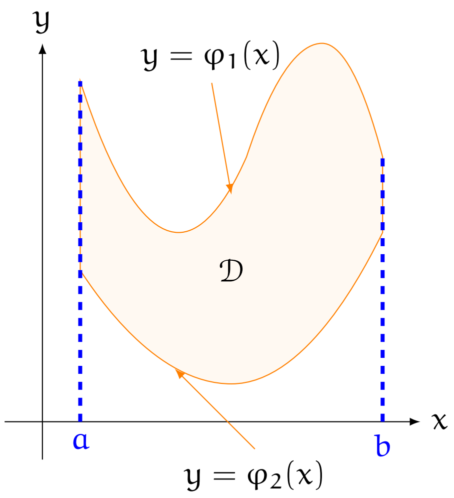
7.3.3 Double integral on an elementary part

The situation that regularly arises is the double integral on an elementary domain. This is the case when the domain is of one of the two types defined below.

Definition 7.3. *We say that a domain* D *of* $\mathbb{R}^2$ *is of type* 1 *if it is included between the graphs of two continuous functions of* x*, that is to say that*

$$D = \{(x, y) \in \mathbb{R}^2 : a \leqslant x \leqslant b, \varphi_1(x) \leqslant y \leqslant \varphi_2(x)\},$$

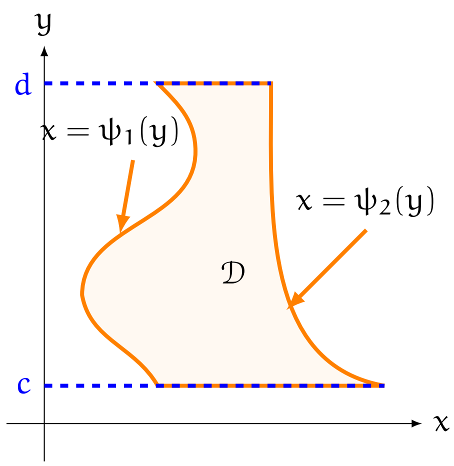
where the functions φ_1 *and* φ_2 *are continuous on the interval* $[a, b]$.



Definition 7.4. *We say that a domain* D *of* $\mathbb{R}^2$ *is of type* 2 *if*

$$D = \{(x, y) \in \mathbb{R}^2 : c \leqslant y \leqslant d, \psi_1(y) \leqslant x \leqslant \psi_2(y)\},$$

where the functions ψ_1 *and* ψ_2 *are continuous on the interval* $[c, d]$.



The following theorem allows us to calculate the double integral as an iterated integral.

Theorem 3 [(**Fubini's theorem for elementary domains**)]

❶ If f is continuous on a domain D of type 1, then,

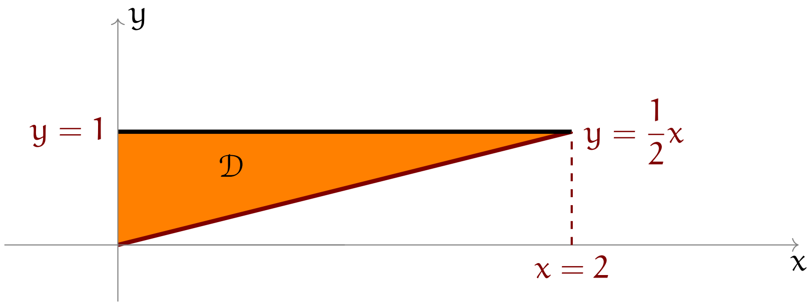
$$\iint_D f(x,y)\,dxdy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right] dx.$$

❷ If f is continuous on a domain D of type 2, then,

$$\iint_D f(x,y)\,dxdy = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\right] dy.$$

Example 7.8. *Let's compute* $\iint_D xy^2\,dxdy$, *where* D *is shown in the figure opposite. According to the figure,* D *is a type* 1 *domain and*

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 2, \frac{1}{2}x \leqslant y \leqslant 1\}.$$



Consequently, we have

$$\begin{aligned}
\iint_D xy^2\,dxdy &= \int_0^2 \left(\int_{\frac{1}{2}x}^1 xy^2\,dy\right) dx\\
&= \int_0^2 \left[\frac{1}{3}xy^3\right]_{\frac{1}{2}x}^1 dx\\
&= \int_0^2 \left(\frac{1}{3}x \cdot 1^3 - \frac{1}{3}x\left(\frac{1}{2}x\right)^3\right) dx\\
&= \int_0^2 \left(\frac{1}{3}x - \frac{1}{24}x^4\right) dx\\
&= \left[\frac{1}{6}x^2 - \frac{1}{120}x^5\right]_0^2 = \left(\frac{1}{6}\cdot 4 - \frac{1}{120}\cdot 32\right) = \frac{4}{6} - \frac{32}{120} = \frac{2}{5}.
\end{aligned}$$

We can also write D *as a domain of type* 2

$$D = \{(x,y) \in \mathbb{R}^2 :\ 0 \leqslant y \leqslant 1, 0 \leqslant x \leqslant 2y\}.$$

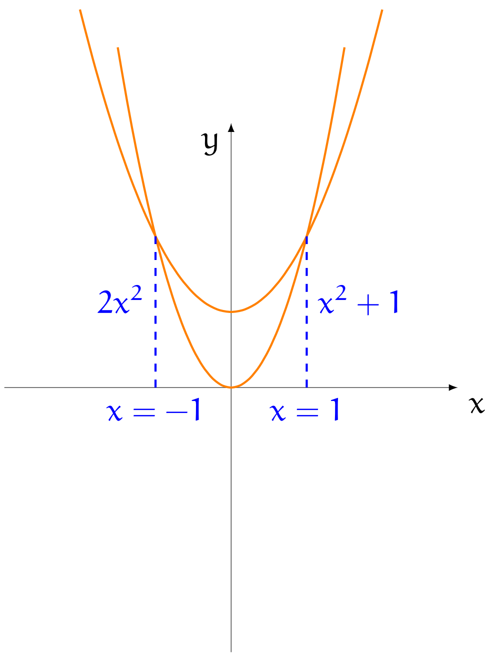
Then,

$$\begin{aligned}\iint\limits_D xy^2\,dxdy &= \int_0^1\left(\int_0^{2y} xy^2\,dx\right)dy\\ &= \int_0^1\left[\frac{1}{2}x^2y^2\right]_0^{2y}dy\\ &= \int_0^1 2y^4\,dy\\ &= \left[\frac{2}{5}y^5\right]_0^1 = \frac{2}{5}.\end{aligned}$$

Remark 7.2. *It is essential to draw the domain of integration if we want to calculate a double integral.*

Example 7.9. *Let us calculate* $\iint\limits_D (2x+y)\,dxdy$, *where* D *is the domain bounded by the parabolas* $y = 2x^2$ *and* $y = x^2 + 1$.
The parabolas $y = 2x^2$ *and* $y = x^2+1$ *intersect when* $x^2 + 1 = 2x^2$, *that is, when* $x = \pm 1$. *We note that the domain* D, *represented in the figure opposite, is a type* 1 *domain, but not a type* 2domain. *We can write*

$$D = \{(x,y) \in \mathbb{R}^2 : -1 \leqslant x \leqslant 1, 2x^2 \leqslant y \leqslant x^2+1\}.$$



Consequently we have,

$$\begin{aligned}
\iint\limits_{D}(2x+y)\,dxdy &= \int_{-1}^{1}\left(\int_{2x^2}^{x^2+1}(2x+y)\,dy\right)dx\\
&= \int_{-1}^{1}\left[2xy+\frac{1}{2}y^2\right]_{2x^2}^{x^2+1}dx\\
&= \int_{-1}^{1}\left(2x^3+2x+\frac{1}{2}(x^4+2x^2+1)-4x^3-2x^4\right)dx\\
&= \int_{-1}^{1}\left(-\frac{3}{2}x^4-2x^3+x^2+2x+\frac{1}{2}\right)dx\\
&= \left[-\frac{3}{10}x^5-\frac{2}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2+\frac{1}{2}x\right]_{-1}^{1} = \frac{16}{15}.
\end{aligned}$$

Example 7.10. *Calculate the volume* V *of the solid below the plane* $z = 3x + 2y$ *and above the region bounded by the parabolas* $y = x^2$ *and* $x = y^2$*. The parabolas* $y = x^2$ *and* $x = y^2$ *intersect when* $x = y = 1$*. According to the figure opposite, the region* D *can be seen as a* 1 *region.*

$$D = \{(x,y)\in\mathbb{R}^2 : 0 \leqslant x \leqslant 1, x^2 \leqslant y \leqslant \sqrt{x}\}$$

We have,

$$\begin{aligned}
V = \iint\limits_{D}(3x+2y)\,dx\,dy &= \int_{0}^{1}\left(\int_{x^2}^{\sqrt{x}}(3x+2y)\,dy\right)dx\\
&= \int_{0}^{1}\left[3xy+y^2\right]_{x^2}^{\sqrt{x}}dx\\
&= \int_{0}^{1}\left(x\sqrt{x}+\frac{6}{5}x^{5/2}-\frac{3}{4}x^4-\frac{1}{5}x^5\right)dx\\
&= \left[\frac{1}{2}x^2+\frac{6}{5}\frac{x^5}{5}-\frac{3}{4}\frac{x^4}{4}-\frac{1}{5}\frac{x^5}{5}\right]_{0}^{1} = \frac{3}{4}
\end{aligned}$$

7.4 Properties of double integrals

Theorem 4 Let D be a bounded domain of $\mathbb{R}^2$, $\lambda \in \mathbb{R}$ and $f, g : D \to \mathbb{R}$ be two integrable functions on D and let D_1 and D_2 be two non-overlapping domains except, possibly, on their boundaries. Then,

❶ $\displaystyle\iint_D (f(x,y) + g(x,y))\, dx\, dy = \iint_D f(x,y)\, dx\, dy + \iint_D g(x,y)\, dx\, dy.$

❷ $\displaystyle\lambda \iint_D f(x,y)\, dx\, dy = \lambda \iint_D f(x,y)\, dx\, dy.$

❸ If $f(x,y) \leqslant g(x,y)$ for all $(x,y) \in D$: $\displaystyle\iint_D f(x,y)\, dx\, dy \leqslant \iint_D g(x,y)\, dx\, dy.$

❹ $\displaystyle\iint_{D_1 \cup D_2} f(x,y)\, dx\, dy = \iint_{D_1} f(x,y)\, dx\, dy + \iint_{D_2} f(x,y)\, dx\, dy.$

❺ The area of the domain D is given by $\displaystyle\text{Area}(D) = \iint_D 1\, dx\, dy.$

❻ If $f(x,y) \geqslant g(x,y)$ for all $(x,y) \in D$, then the volume of the solid S bounded by the surfaces of equations $z = f(x,y)$ and $z = g(x,y)$ is given by

$$\text{Vol}(S) = \iint_D (f(x,y) - g(x,y))\, dx\, dy.$$

Example 7.11. *Let's calculate* $\displaystyle\iint_D y\, dx\, dy$, *with* D *shown in the opposite illustration.*

Note that D *is not an elementary domain, but can be expressed as the union* $D = D_1 \cup D_2 \cup D_3$ *with*

$$D_1 = \{(x,y) \in \mathbb{R}^2 : -1 \leqslant y \leqslant 0, -1 \leqslant x \leqslant y - y^3\} \textit{of type 2}$$
$$D_2 = \{(x,y) \in \mathbb{R}^2 : -1 \leqslant x \leqslant 0, 0 \leqslant y \leqslant (x+1)^2\} \textit{ of type 1}$$
$$D_3 = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant y \leqslant 1, 0 \leqslant x \leqslant y - y^3\} \textit{of type 2}$$

We have,

$$\iint_{D_1} y\, dx\, dy = \int_{-1}^{0} \left(\int_{y^3-y}^{y} y\, dx \right) dy = \int_{-1}^{0} (y^2 + y - y^4)\, dy = \left[\frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{5}y^5 \right]_{-1}^{0} = -\frac{11}{30}$$

$$\iint_{D_2} y\, dx\, dy = \int_{-1}^{0} \left(\int_{0}^{(x+1)^2} y\, dy \right) dx = \int_{-1}^{0} \left[\frac{1}{2}y^2 \right]_{0}^{(x+1)^2} dx = \left[\frac{1}{6}(x+1)^3 \right]_{-1}^{0} = \frac{1}{10}$$

$$\iint_{D_3} y\, dx\, dy = \int_{0}^{1} \left(\int_{0}^{y-y^3} y\, dx \right) dy = \int_{0}^{1} y^2\, dy = \left[\frac{1}{3}y^3 - \frac{1}{5}y^5 \right]_{0}^{1} = \frac{2}{15}$$

Then,

$$\iint_D y\,dx\,dy = \iint_{D_1\cup D_2\cup D_3} y\,dx\,dy = \iint_{D_1} y\,dx\,dy + \iint_{D_2} y\,dx\,dy + \iint_{D_3} y\,dx\,dy$$
$$= -\frac{11}{30} + \frac{1}{10} + \frac{2}{15} = -\frac{2}{15}$$

Example 7.12. *Calculate the volume* V *of the solid bounded by the paraboloid* $z = x^2 + y^2 + 1$ *and the planes* $z = 0$, $x = 0$, $y = 0$ *and* $x + y = 2$. *Let* D *be the area bounded by the lines* $x = 0$, $y = 0$ *and* $x + y = 2$. *According to the figure, we have,*

$$D = \{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 2 - x\}$$

and so,

$$V = \iint_D (x^2 + y^2 + 1)\,dx\,dy = \int_0^2 \left(\int_0^{2-x} (x^2 + y^2 + 1)\,dy\right)\,dx$$
$$= \int_0^2 \left[x^2 y + \frac{1}{3}y^3 + y\right]_0^{2-x} dx$$
$$= \int_0^2 \left(\frac{14}{3} - 5x + 4x^2 - \frac{4}{3}x^3\right) dx$$
$$= \left[\frac{14}{3}x - \frac{5}{2}x^2 + \frac{4}{3}x^3 - \frac{1}{3}x^4\right]_0^2 = \frac{14}{3}$$

7.4.1 Change of variables in double integrals

The formula for the change of variables in the case of simple integrals tells us that if $t \mapsto x(t)$ is a function of class C^1 of an interval J in the interval I and if f is a continuous function on I, then for all $a, b \in J$ we have,

$$\int_{x(a)}^{x(b)} f(x)dx = \int_a^b f(x(t))x'(t)dt.$$

This result extends to the case of integrals as follows

Theorem 5 [**(Change of variables)**] Let D et D' deux ouverts bornés de $\mathbb{R}^2$ and

$$\begin{aligned} \phi : D' &\to D, \\ (u, v) &\mapsto (x(u, v), y(u, v)), \end{aligned}$$

a diffeomorphism of class C^1. Then,

$$\iint_{D'} f(x, y)\,dx\,dy = \iint_D f(x(u, v), y(u, v))|\det(J_\phi(u, v))|\,du\,dv.$$

Example 7.13. *Find the area of the region* D *bounded by the four curves*

$$y = ax^2, \quad y = bx^2, \quad xy = c \quad \text{and} \quad xy = d,$$

where a, b, c, d *are four constants satisfying* $0 < a < b$ *and* $0 < c < d$.

Solution. *The area is given by* $\iint_D dx\,dy$*, which is difficult to calculate. Instead, we perform the change of variables*

$$u = \frac{y}{x^2}, \quad v = xy.$$

According to the proposition ??, we have,

$$\det(J_\phi(u,v)) = \frac{1}{\det(J_{\phi^{-1}}(x,y))} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{-2y}{x^3} & \frac{1}{x} \\ \frac{1}{x^2} & 0 \end{vmatrix}$$
$$= \frac{1}{-3\frac{y}{x^2}} = \frac{1}{-3u}.$$

As a result,

$$\iint_D dx\,dy = \iint_{D'} |\det(J_\phi(u,v))|\,du\,dv = \iint_{D'} \left|\frac{1}{-3u}\right| du\,dv = \iint_{D'} \frac{1}{3u}\,du\,dv$$
$$= \frac{1}{3}\int_a^b \frac{1}{y}du \int_c^d dv = \frac{1}{3}(b-a)\ln(b-a).$$

7.4.2 Change of variables in polar coordinates

This is a C^1-diffeomorphism

$$\phi : D' \subseteq]0,+\infty[\times[0,2\pi[\to D = \phi(D')$$
$$(r,\theta) \mapsto (a + r\cos\theta, b + r\sin\theta).$$

The Jacobian of ϕ is

$$\det(J_\phi(r,\theta)) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

The formula for changing variables is therefore written in the case of polar coordinates

$$\iint_D f(x,y)\,dxdy = \iint_{D'} f(a + r\cos\theta, b + r\sin\theta)r\,drd\theta.$$

Example 7.14. *Let's evaluate the double integral*

$$\iint_D \frac{xy}{x^2+y^2}\,dxdy,$$

where;

$$D = \{(x,y) \in \mathbb{R}^2 : x \geqslant 0, y \geqslant 0, 1 \leqslant x^2 + y^2 \leqslant 4\}.$$

With a domain of this form, we must convert to polar coordinates. We set $x = r\cos\theta$ *and* $y = r\sin\theta$*. The inequalities* $1 \leqslant x^2 + y^2 \leqslant 4$ *translate to* $1 \leqslant r \leqslant 2$*, while* $x \geqslant 0$ *and* $y \geqslant 0$ *mean that* $\cos\theta \geqslant 0$ *and* $\sin\theta \geqslant 0$ *i.e.* $0 \leqslant \theta \leqslant \frac{\pi}{2}$*. Then we can represent* D *in polar coordinates by*

$$D' = \{(r,\theta) \in \mathbb{R}^2 : 0 \leqslant \theta \leqslant \frac{\pi}{2}, 1 \leqslant r \leqslant 2\}.$$

The formula for changing variables gives

$$\begin{aligned}\iint_{\mathcal{D}} f(x,y)dxdy &= \int_1^2\int_0^{\pi/2}\frac{r^2\cos\theta\sin\theta}{r^2}\times rd\theta dr\\ &= \int_1^2\int_0^{\pi/2}\frac{1}{2}r\sin(2\theta)d\theta dr\\ &= \frac{1}{4}\int_1^2 r[-\cos(2\theta)]_0^{\pi/2}dr\\ &= \frac{1}{4}\int_1^2 2rdr\\ &= \frac{1}{4}\left[r^2\right]_1^2\\ &= \frac{3}{4}\end{aligned}$$

7.5 Triple integrals

The process used to define the triple integral is similar to that of the double integral..

7.5.1 Triple integral on a rectangular parallelepiped

As in the previous section, we are given a function of three variables f defined on a rectangular parallelepiped

$$\mathcal{P} = [a,b]\times[c,d]\times[r,s].$$

Our aim is to describe the main steps leading to the definition. We make a subdivision for each of the three intervals $[a,b]$, $[c,d]$ and $[r,s]$,

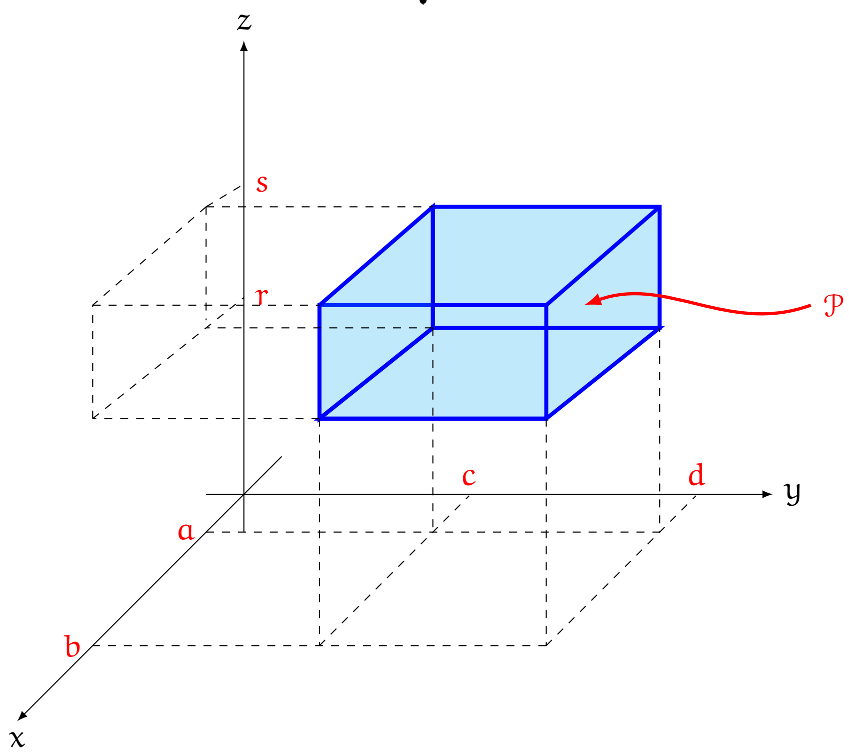
$$a = x_0 < x_1 < \ldots < x_n = b,\quad c = y_0 < y_1 < \ldots < y_m = d \quad \text{and} \quad r = z_0 < z_1 < \ldots < z_p = s.$$

with,

$$x_i - x_{i-1} = \Delta x = \frac{b-a}{n},\quad y_j - y_{j-1} = \Delta y = \frac{d-c}{m} \quad \text{and} \quad z_k - z_{k-1} = \Delta z = \frac{s-r}{p}.$$

This gives a partition of $\mathcal{P}$ into $n\times m\times p$ elementary parallelepipeds

$$\mathcal{P}_{ijk} = [x_{i-1},x_i]\times[y_{j-1},y_j]\times[z_{k-1},z_k] \quad \text{with} \quad 1\leqslant i\leqslant n,\quad 1\leqslant j\leqslant m \quad \text{and} \quad 1\leqslant k\leqslant p,$$



whose volume is

$$\Delta x \Delta y \Delta z = \frac{(b-a)(d-c)(s-r)}{nmp}$$

$$S_{n,m,p} = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta x \Delta y \Delta z.$$

where the sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ belongs to $\mathcal{P}_{ijk}$. The triple integral is defined in the same way as the double integral.

Definition 7.5. *If, when* $n \to +\infty$, $m \to +\infty$ *and* $p \to +\infty$ *the sequence* $(S_{n,m,p})_{n,m,p}$ *tends to a limit independent of the choice of sample points* $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$*, we say that* f *is integrable on the parallelepiped* $\mathcal{P}$*. We then call this limit the triple integral of* f *on* $\mathcal{P}$ *and we denote it* $\iiint_{\mathcal{P}} f(x,y,z)\,dx\,dy\,dz$*. We write*

$$\iiint_{\mathcal{P}} f(x,y,z)\,dx\,dy\,dz = \lim_{n,m,p\to+\infty} \frac{(b-a)(d-c)(s-r)}{nmp}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}).$$

This limit does not always exist. However, we have the following theorem

Theorem 6 Any continuous function on a rectangular parallelepiped $\mathcal{P}$ is integrable on $\mathcal{P}$.

As with the double integral, it is possible to evaluate a triple integral using iterated integrals.

Theorem 7 [(**Fubini's theorem for triple integrals**)] If f is integrable on the rectangular parallelepiped

$$\mathcal{P} = \{(x,y,z) \in \mathbb{R}^3 : a \leqslant x \leqslant b,\ c \leqslant y \leqslant d \text{ et } r \leqslant z \leqslant s\},$$

then,

$$\iiint_{\mathcal{P}} f(x,y,z)\,dx\,dy\,dz = \int_r^s \left(\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy \right) dz.$$

Example 7.15. *If* $\mathcal{P} = [0,2] \times [0,1] \times [-1,1]$, *then,*

$$\begin{aligned}
\iiint_{\mathcal{P}} (1+x-2y+3z)\,dx\,dy\,dz &= \int_{-1}^{1} \left(\int_0^1 \left(\int_0^2 (1+x-2y+3z)\,dx \right) dy \right) dz \\
&= \int_{-1}^{1} \left(\int_0^1 \left[x + \frac{1}{2}x^2 - 2xy + 3xz \right]_0^2 dy \right) dz \\
&= \int_{-1}^{1} \left(\int_0^1 (4-4y+6z)\,dy \right) dz \\
&= \int_{-1}^{1} \left[4y - 2y^2 + 6yz \right]_0^1 dz \\
&= \int_{-1}^{1} (2+6z)\,dz \\
&= \left[2z + 3z^2 \right]_{-1}^{1} = 4
\end{aligned}$$

7.5.2 Triple integral on a bounded domain

We define the triple integral over any bounded domain $\Omega \subset \mathbb{R}^3$ using a technique analogous to that used in the case of the double integral.

Definition 7.6. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain and a rectangular parallelepiped* $\mathcal{P}$ *containing* Ω. *We will say that* $f:\ \Omega \to \mathbb{R}$ *is integrable on* Ω *if the function* $\tilde{f}$ *defined by*

$$\tilde{f}(x,y,z) = \begin{cases} f(x,y,z) & : (x,y,z) \in \Omega \\ 0 & : (x,y,z) \in \mathcal{P} \text{ and } (x,y,z) \notin \Omega \end{cases}$$

is integrable on $\mathcal{P}$, *and in this case*

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iiint_{\mathcal{P}} \tilde{f}(x,y,z)\,dx\,dy\,dz$$

Fubini's theorem reduces the calculation of a triple integral to a superposition of a single integral and a double integral.

Definition 7.7. *A domain* Ω *of* $\mathbb{R}^3$ *is said to be of type 1 if it is included between the graphs of two continuous functions* g_1 *and* g_2 *of* x *and* y, *namely*

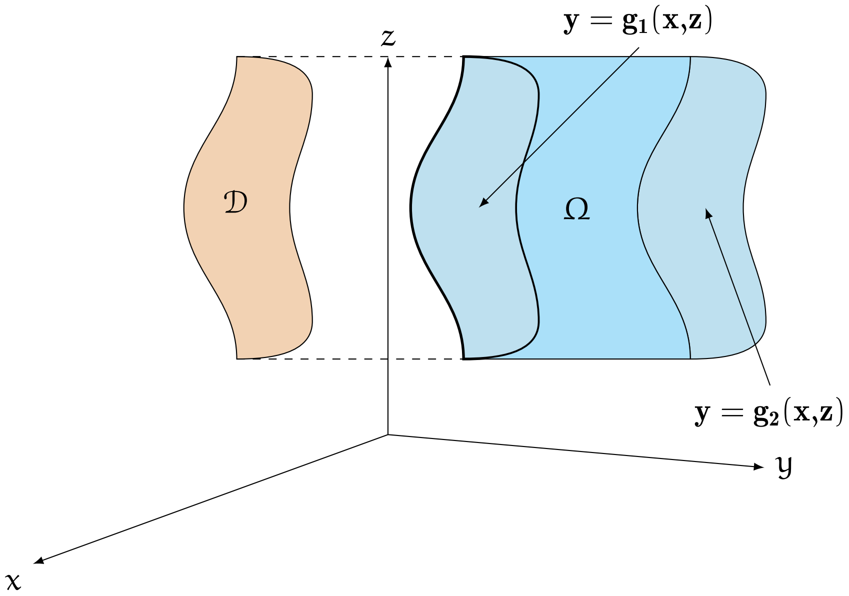
$$\Omega = \{(x,y,z) \in \mathbb{R}^3 :\ (x,y) \in \mathcal{D},\ g_1(x,y) \leqslant z \leqslant g_2(x,y)\}$$

where $\mathcal{D}$ *is the orthogonal projection of* Ω *in the* xOy *plane, as shown in the figure opposite.*

Definition 7.8. *A domain Ω of $\mathbb{R}^3$ is said to be of **type 2** if it lies between the graphs of two continuous functions g_1 and g_2 of x and z, which means that*

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : (x,z) \in D,\ g_1(x,z) \leqslant y \leqslant g_2(x,z)\},$$

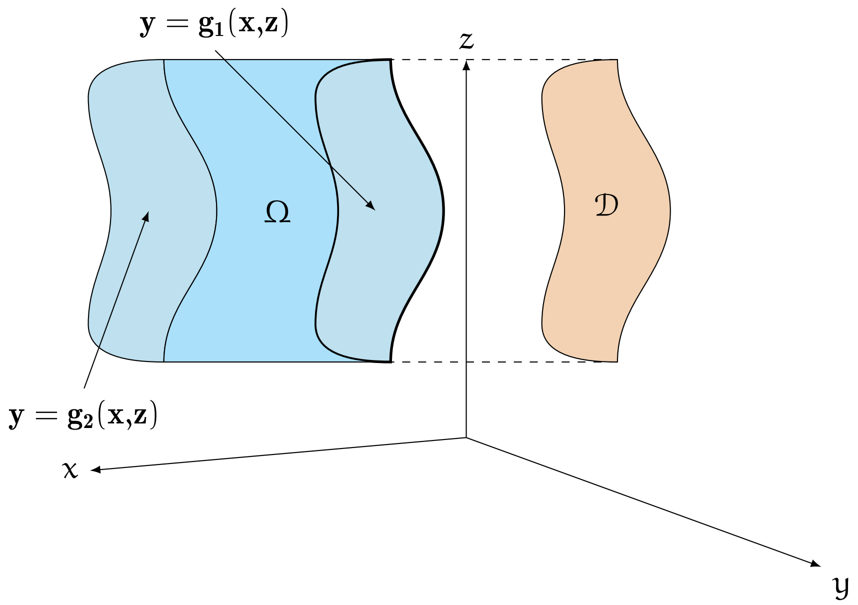
where D is the orthogonal projection of Ω in the xOz, plane, as shown in the figure opposite.



Definition 7.9. *A domain Ω of $\mathbb{R}^3$ is said to be of **type 3** if it lies between the graphs of two continuous functions g_1 and g_2 of y and z, namely*

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : (y,z) \in D,\ g_1(y,z) \leqslant x \leqslant g_2(y,z)\},$$

where D is the orthogonal projection of Ω in the yOz, plane, as shown in the diagram opposite.



A calculation method for evaluating a triple integral by calculating a single integral followed by a double integral is given by the following theorem.

Theorem 8 [(**Fubini's theorem for triple integrals**)]

❶ If f is continuous on a domain Ω of type 1, then,

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iint_{D}\left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right)dx\,dy.$$

❷ If f is continuous on a domain Ω of type 2, then,

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iint_{D}\left(\int_{g_1(x,z)}^{g_2(x,z)} f(x,y,z)\,dy\right)dx\,dz.$$

❸ If f is continuous on a domain Ω of type 3, then,

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iint_{D}\left(\int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)\,dx\right)dy\,dz.$$

Example 7.16. *Let us calculate* $I = \iiint_{\Omega} x\,dx\,dy\,dz$ *where* Ω *is the solid tetrahedron bounded by the four planes* $x = 0$, $y = 0$, $z = 0$ *et* $x + y + z = 1$.

Solution. *It is always useful to draw the two domains* Ω *and its orthogonal projection* D. *Figure 7.2 represents the tetrahedron* Ω. *We see graphically that* Ω *is included between two surfaces* $z = 0$ *and* $z = 1 - x - y$.

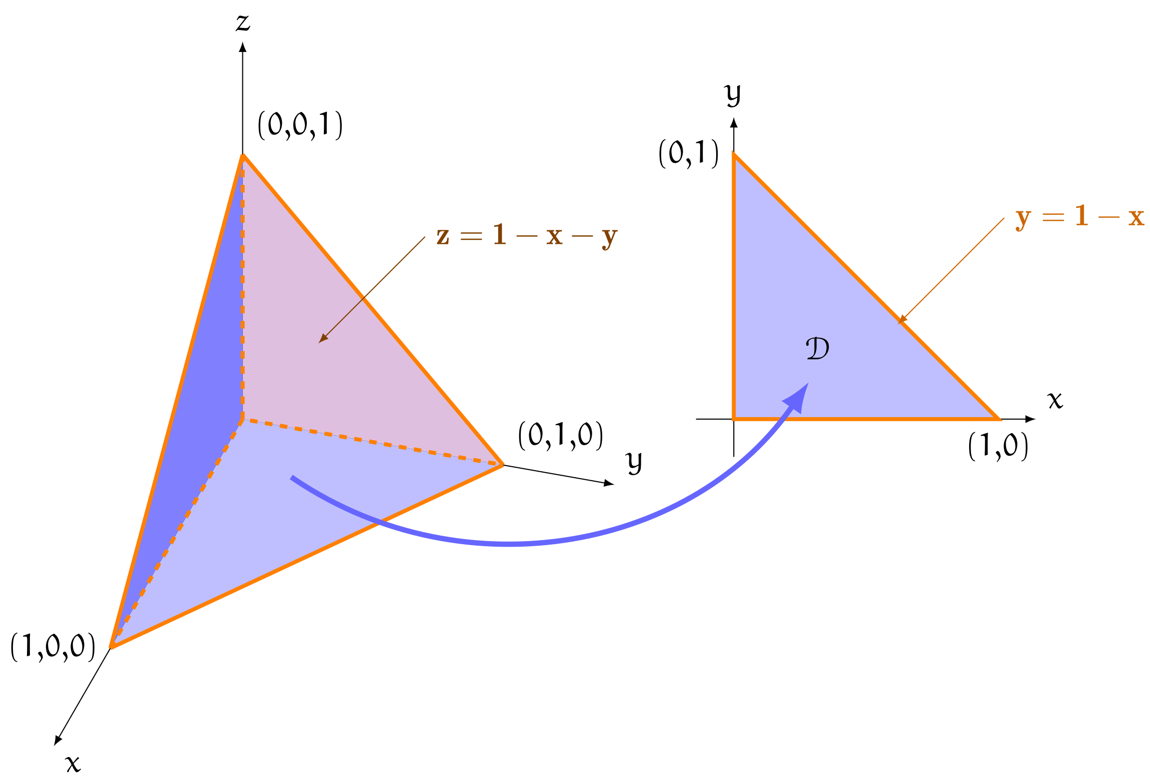


Fig. 7.2 . The integration domain Ω.

For this we describe Ω *as a domain of type* 1

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D,\ 0 \leqslant z \leqslant 1-x-y\},$$

with,

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant y \leqslant 1-x \textit{ and } 0 \leqslant x \leqslant 1\}.$$

Subsequently,

$$\begin{aligned}
\iiint_{\Omega} y\,dxdydz &= \iint_{\mathcal{D}} \left[\int_0^{1-x-y} y\,dz\right] dxdy = \iint_{\mathcal{D}} [yz]_0^{1-x-y} dxdy \\
&= \iint_{\mathcal{D}} y(1-x) - y^2 dxdy = \int_0^1 \left[\frac{1}{2}y^2(1-x) - \frac{1}{3}y^3\right]_0^{1-x} dx \\
&= \int_0^1 \frac{1}{6}(1-x)^3 dx = \left[\frac{1}{24}(1-x)^4\right]_0^1 = \frac{1}{24}.
\end{aligned}$$

7.5.3 Properties of the triple integral

The following properties follow directly from those of the simple integral and the double integral.

Theorem 9 Let Ω be a bounded domain of $\mathbb{R}^3$, $\lambda \in \mathbb{R}$ and $f, g : \Omega \to \mathbb{R}$ be two integrable functions on Ω and let Ω_1 and Ω_2 be two non-overlapping domains except possibly at their boundaries. Then,

❶
$$\iiint_{\Omega} \left(f(x,y,z) + g(x,y,z)\right)\, dx\, dy\, dz = \iiint_{\Omega} f(x,y,z)\, dx\, dy\, dz + \iiint_{\Omega} g(x,y,z)\, dx\, dy\, dz.$$

❷
$$\iiint_{\Omega} \lambda f(x,y,z)\, dx\, dy\, dz = \lambda \iiint_{\Omega} f(x,y,z)\, dx\, dy\, dz.$$

❸ If $f(x,y,z) \leqslant g(x,y,z)$ for all $(x,y,z) \in \Omega$, then,
$$\iiint_{\Omega} f(x,y,z)\, dx\, dy\, dz \leqslant \iiint_{\Omega} g(x,y,z)\, dx\, dy\, dz.$$

❹ $$\iiint_{\Omega_1 \cup \Omega_2} f(x,y,z)\, dx\, dy\, dz = \iiint_{\Omega_1} f(x,y,z)\, dx\, dy\, dz + \iiint_{\Omega_2} f(x,y,z)\, dx\, dy\, dz.$$

7.6 Change of variables in triple integrals

The formula for the change of variables is identical to that for the case of the double integral.

Theorem 10 [**(Change of variables)**] Let Ω and Ω' be two bounded open sets of $\mathbb{R}^3$ and

$$\begin{aligned} \phi : \Omega &\to \Omega \\ (u,v,w) &\mapsto (x(u,v,w), y(u,v,w), z(u,v,w)), \end{aligned}$$

a diffeomorphism of class C^1. Then,

$$\iiint_{\Omega'} f(x,y,z)\, dx\, dy\, dz = \iiint_{\Omega} f(x(u,v,w), y(u,v,w), z(u,v,w))\, |\det(J_\phi(u,v,w))|\, du\, dv\, dw, \tag{7.1}$$

where J_ϕ is the Jacobian matrix

$$J_\phi(u,v,w) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}.$$

Example 7.17. *Let us calculate* $\iiint_\Omega e^x\, dx\, dy\, dz$ *where* Ω *is given by*

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 :\ 1 - e^x \leqslant y \leqslant 3 - e^x,\ 1 - y \leqslant 2z \leqslant 2 - y,\ y \leqslant e^x \leqslant y + 4\}.$$

By changing the variables

$$u = e^x + y,\ v = y + 2z,\ w = e^x - y.$$

We have,

$$\det(J_\phi(u,v,w)) = \frac{1}{\det(J_{\phi^{-1}}(x,y,z))},$$

with,

$$\det(J_{\phi^{-1}}(x,y,z)) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} e^x & 1 & 0 \\ 1 & 1 & 2 \\ e^x & -1 & 0 \end{vmatrix} = 4e^x.$$

So,

$$\det(J_{\phi(u,v,w)}) = \frac{1}{4e^x} = \frac{1}{2(u+w)}.$$

It is easy to check that the reciprocal image of Ω *by* ϕ *is*

$$\phi^{-1}(\Omega) = \Omega' = \{(x,y,z) \in \mathbb{R}^3 :\ 1 \leqslant u \leqslant 3,\ 1 \leqslant v \leqslant 2,\ 0 \leqslant x \leqslant 4\}.$$

Consequently,

$$\begin{aligned} \iiint_\Omega 2e^x\, dx\, dy\, dz &= \iiint_{\Omega'} (u+w) |\det(J_{\phi(u,v,w)})|\, du\, dv\, dw \\ &= \iint_{\Omega'} \frac{(u+w)}{4(u+w)}\, du\, dv\, dw = \frac{1}{4} \int_1^3 \int_1^2 \int_0^4 1\, du\, dv\, dw = \frac{8}{4} = 2. \end{aligned}$$

7.6.1 Cylindrical coordinates

Cylindrical coordinates are often useful when the integration domain is symmetrical about an ordinate axis. Cylindrical coordinates are defined by applying the change of variables

$$\begin{aligned} \phi:\ \Omega' \subset \mathbb{R}_+^* \times [\theta_0, \theta_0 + 2\pi[\times \mathbb{R} &\to \Omega = \phi(\Omega') \\ (r, \theta, z) &\mapsto (r\cos\theta, r\sin\theta, z), \end{aligned}$$

where $\theta_0 \in \mathbb{R}$.

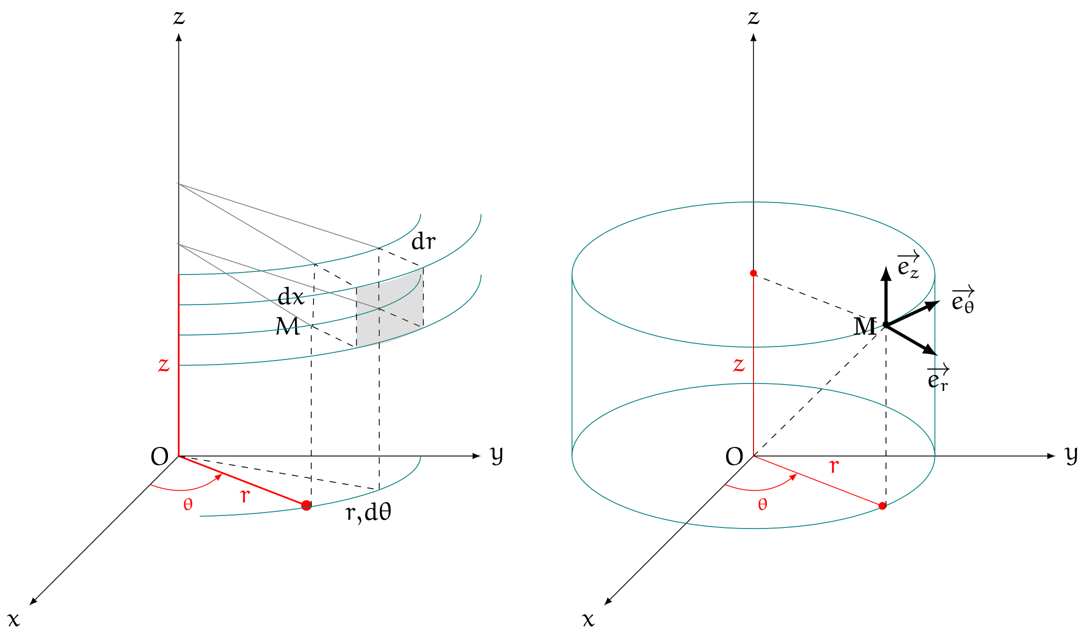


Fig. 7.3 . Cylindrical coordinates

The Jacobian of ϕ is

$$J_{\phi(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Thus, the change of variables formula (7.1) takes the following form

$$\iiint_\Omega f(x, y, z)\, dx\, dy\, dz = \iiint_{\Omega'} f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz.$$

Example 7.18. *Let's calculate the integral* $\iiint_\Omega x^{2+y^2}\, dx\, dy\, dz$ *where,*

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leqslant 1,\ 0 \leqslant z \leqslant 1\}.$$

We use the change of variables in cylindrical coordinates. We have,

$$\Omega' = \phi^{-1}(\Omega) = \{(r, \theta, z) \in \mathbb{R}^3 :\ 0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant z \leqslant 1\}.$$

and so,

$$\begin{aligned}
\iiint_{\Omega} z^{x^2+y^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \iiint_{\Omega'} z^{r^2} r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z \\
&= \int_0^{2\pi}\int_0^1\left[\int_0^1 z^{r^2} r\mathrm{d}z\right]\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{2\pi}\left[\int_0^1 \frac{r}{1+r^2}\mathrm{d}r\right]\mathrm{d}\theta \\
&= \frac{\ln 2}{2}\int_0^{2\pi}\mathrm{d}\theta \\
&= \pi\ln 2.
\end{aligned}$$

7.6.2 Spherical coordinates

Spherical coordinates are often useful when the domain of integration is symmetrical about the origin of the reference frame. Spherical coordinates are defined by applying the change of variables

$$\begin{aligned}
\varphi : \Omega' \subset \mathbb{R}_+^* \times]0,\pi[\times[0,2\pi[&\longrightarrow \Omega = \varphi(\Omega') \subset \mathbb{R}^3 \setminus \mathbb{R}_- \times \{0\} \times \mathbb{R} \\
(r,\theta,\varphi) &\mapsto (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta).
\end{aligned}$$

The Jacobian of φ is

$$\det(J_{\varphi(r,\theta,\varphi)}) = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

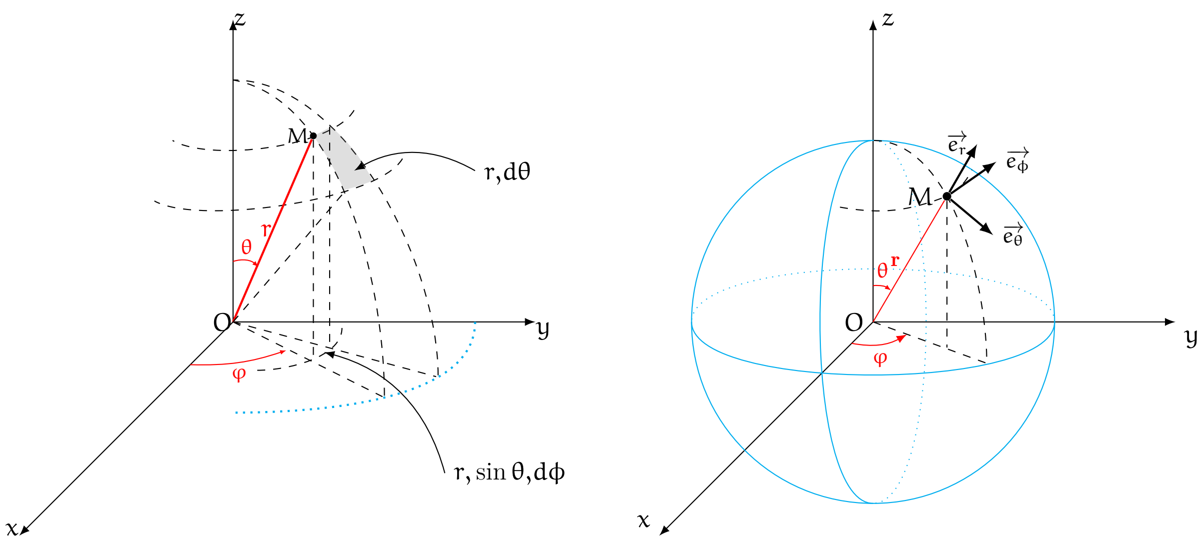


Fig. 7.4 . Coordonnées sphériques

Substituting (7.1), we obtain the formula for changing to spherical coordinates

$$\iiint_{\Omega} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{\Omega'} f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

Example 7.19. *Let's calculate the integral* $\iiint_\Omega x^2+y^2+z^2\,dx\,dy\,dz$ *where* Ω *is the half-ball*

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : z \geqslant 0,\ x^2+y^2+z^2 \leqslant 1\}.$$

Using the change of variables in spherical coordinates

$$\varphi : \Omega' = [0,1] \times [0,\frac{\pi}{2}] \times [0,2\pi[\longrightarrow \Omega = \varphi(\Omega')$$

$$(r,\theta,\varphi) \mapsto (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta).$$

We perform the change of variable in the integral

$$\begin{aligned}\iiint_\Omega x^2+y^2+z^2\,dx\,dy\,dz &= \iiint_{\Omega'} r^4\sin\theta\,dr\,d\theta\,d\varphi\\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left[\int_0^1 r^4\sin\theta\,dr\right]d\theta\,d\varphi\\ &= \frac{1}{5}\int_0^{2\pi}\left[\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\right]d\varphi\\ &= \frac{1}{5}\int_0^{2\pi}d\varphi\\ &= \frac{2\pi}{5}.\end{aligned}$$

7.6.3 Applications of the multiple integral

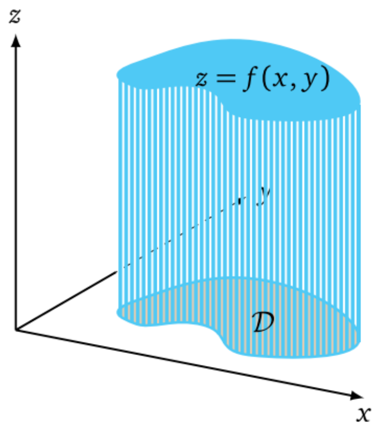
There are a large number of applications of the multiple integral in mathematics, physics and science in general. In this course, we will only discuss the calculation of areas and volumes. We have seen in the corollary 7.2 that the double integral of a positive function f on a rectangle R is equal to the volume of the solid located above R and below the graph of f. This result generalizes to the case of any bounded domain D of $\mathbb{R}^2$.

Theorem 11 Let D be a bounded domain of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a positive and integrable function on D. We denote by S the solid located above D and below the surface $z = f(x,y)$, that is,

$$S = \{(x,y,z) \in \mathbb{R}^3 :\ (x,y) \in D \text{ and } 0 \leqslant z \leqslant f(x,y)\}.$$

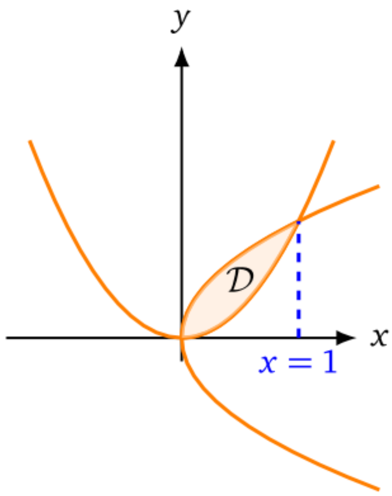
The volume of S is given by

$$\mathrm{Vol}(S) = \iint_D f(x,y)\,dx\,dy.$$



Example 7.20. *Calculate the volume* V *of the solid under the plane* $z = 3x + 2y$ *and above the region bounded by the parabolas* $y = x^2$ *and* $x = y^2$*. The parabolas* $y = x^2$ *and* $x = y^2$ *intersect when* $x = y = 1$*. According to the figure opposite, the region* D *can be seen as a type* 1 *region*

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1, x^2 \leqslant y \leqslant \sqrt{x}\}.$$



We have,

$$\begin{aligned} V = \iint_D 3x + 2y \, dx \, dy &= \int_0^1 \left[\int_{x^2}^{\sqrt{x}} (3x + 2y) \, dy \right] dx \\ &= \int_0^1 \left[3xy + y^2 \Big|_{x^2}^{\sqrt{x}} \right] dx \\ &= \int_0^1 \left(x + 3x^{\frac{3}{2}} - 3x^3 - x^4 \right) dx \\ &= \left[\frac{1}{2}x^2 + \frac{6}{5}x^{\frac{5}{2}} - \frac{3}{4}x^4 - \frac{1}{5}x^5 \right]_0^1 = \frac{3}{4}. \end{aligned}$$

By applying the previous theorem in the particular case where the function f is constant equal to 1. In this case, the graph of f is none other than the plane $z = 1$. The solid S is therefore a cylinder with base D and height 1. Consequently,

$$\text{Vol}(S) = \text{Area}(D) \times 1 = \text{Area}(D).$$

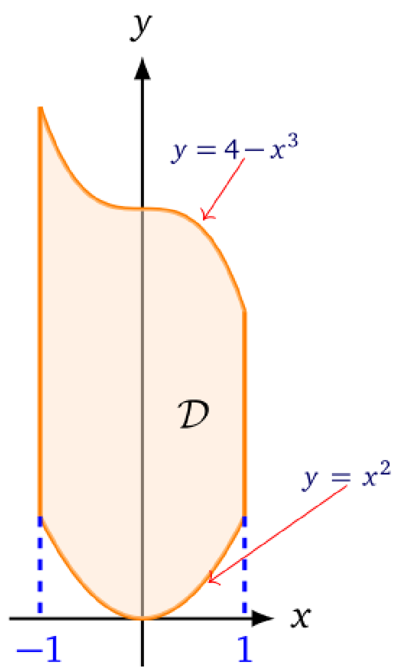
We then obtain the following corollary

Corollary 7.3. *The area of a bounded domain* D *of* $\mathbb{R}^2$ *is given by*

$$\textit{Area}(D) = \iint_D dx \, dy.$$

Example 7.21. *Let* $D = \{(x,y) \in \mathbb{R}^2 : -1 \leqslant x \leqslant 1, x^2 \leqslant y \leqslant 4 - x^3\}$, *is represented in the figure opposite. The form of the domain* D *allows its area to be calculated using an iterated integral.*

$$\begin{aligned} Area(D) &= \int_{-1}^{1} \left[\int_{x^2}^{4-x^3} dy \right] dx \\ &= \int_{-1}^{1} (4 - x^3 - x^2) dx \\ &= \left[4x - \frac{x^4}{4} - \frac{x^3}{3} \right]_{-1}^{1} = \frac{22}{3}. \end{aligned}$$



Example 7.22. *Let's calculate the area of the disk*

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant R\}.$$

The description of this domain in polar coordinates is

$$D' = \{(r,\theta) \in \mathbb{R}^2 : 0 \leqslant r \leqslant R, 0 \leqslant \theta \leqslant 2\pi\}.$$

The area is given by

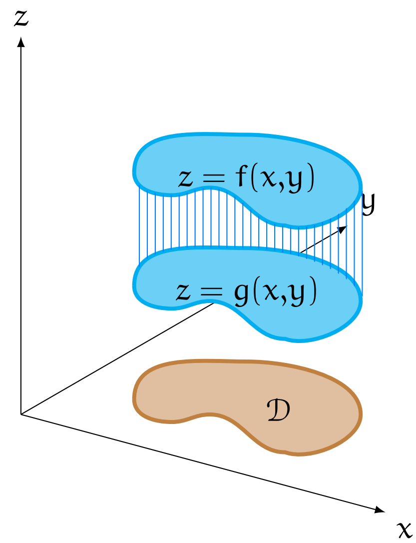
$$Area(D) = \iint_D dx\, dy = \int_0^{2\pi} \int_0^R r\, dr\, d\theta = \frac{1}{2} R^2 \int_0^{2\pi} d\theta = \pi R^2.$$

Corollary 7.4. *Let* D *be a bounded domain of* $\mathbb{R}^2$ *and* $f, g : D \to \mathbb{R}$ *be two functions integrable on* D *such that* $f(x,y) \geqslant g(x,y)$ *for all* $(x,y) \in D$. *Let* S *be the solid bounded by two surfaces* $z = f(x,y)$ *and* $z = g(x,y)$, *in other words,*

$$S = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D \text{ and } g(x,y) \leqslant z \leqslant f(x,y)\}.$$

The volume of S *is given by*

$$Vol(S) = \iint_D (f(x,y) - g(x,y))\, dx\, dy.$$



Example 7.23. *Calculate the volume of the solid enclosed by the paraboloids $z = 27 - x^2$ and $z = 2x^2 + 3y^2$. Let D be the projection of S onto the xOy plane. Then D is bounded by the projection of the intersection of these two paraboloids onto the xOy plane. The equation of the intersection is*

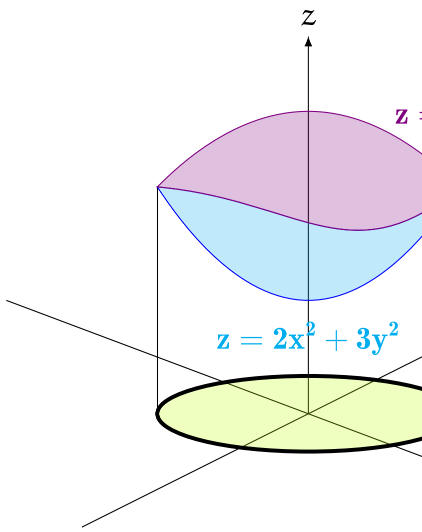
$$27 - x^2 = 2x^2 + 3y^2,$$

in other words

$$x^2 + y^2 = 9.$$

As a result,

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 9\}.$$



The volume of S is given by

$$Vol(S) = \iint_D (27 - x^2 - 2x^2 - 3y^2)\, dx\, dy = 3 \iint_D (9 - x^2 - y^2)\, dx\, dy.$$

In polar coordinates, D can be represented by

$$D' = \{(r, \theta) \in \mathbb{R}^2 : 0 \leqslant r \leqslant 3, 0 \leqslant \theta \leqslant 2\pi\}.$$

So,

$$Vol(S) = 3 \int_0^{2\pi} \left(\int_0^3 (9 - r^2) r\, dr \right) d\theta = \frac{243}{4} \int_0^{2\pi} d\theta = \frac{243\pi}{2}.$$

One of the benefits of triple integrals is that they can be used to calculate the volume of regions of space $\mathbb{R}^3$. In fact, we have a result analogous to the consequence 7.5

Corollary 7.5. *The volume of a bounded domain Ω of $\mathbb{R}^3$ is given by*

$$Vol(\Omega) = \iiint_\Omega dx\, dy\, dz.$$

Example 7.24. *Let us calculate the volume of the ball S with center O and radius R, defined by*

$$B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leqslant R^2\}.$$

We have

$$Vol(B) = \iiint_B dx\, dy\, dz.$$

As usual in the case of the ball we use the passage in spherical coordinates which transforms *the ball into a parallelepiped. We obtain*

$$\begin{aligned}
Vol(B) &= \int_0^{2\pi} \int_0^{\pi} \left[\int_0^R r^2 \sin\theta \, dr \right] d\theta \, d\varphi \\
&= \frac{1}{3} R^3 \int_0^{2\pi} \left(\int_0^{\pi} \sin\theta \, d\theta \right) d\varphi \\
&= \frac{2}{3} R^3 \int_0^{2\pi} d\varphi \\
&= \frac{4\pi R^3}{3}.
\end{aligned}$$

Example 7.25. *Let* $(a, b, c) \in (\mathbb{R}_+^*)^3$. *Let us calculate the volume of the ellipsoid*

$$E = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1 \right\}.$$

We consider the following linear change of variables

$$\begin{aligned}
&\phi : \mathcal{E}' \to \mathcal{E} \\
&(u, v, w) \mapsto (x(u, v, w), y(u, v, w), z(u, v, w)) = (au, bv, cw),
\end{aligned}$$

We have,

$$\det\left(J_\phi(u, v)\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc,$$

and

$$\mathcal{E}' = \phi^{-1}(\mathcal{E}) = \left\{ (u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 \leqslant 1 \right\}.$$

$\mathcal{E}'$ *is the sphere with center* 0 *and radius* $R = 1$. *Therefore,*

$$\begin{aligned}
\iiint_{\mathcal{E}} dxdydz &= \iiint_{\mathcal{E}'} \left|\det\left(J_\phi(u, v, w)\right)\right| \, dudvdw \\
&= abc \iiint_{\mathcal{E}'} dudvdw \\
&= abc \cdot \mathrm{Vol}\left(\mathcal{E}'\right) = \frac{4\pi}{3} abc
\end{aligned}$$