6

CHAPTER

# Extrema of functions of several variables

This chapter is devoted to the study of extrema of scalar functions (in other words, functions defined on $\mathbb{R}^n$ with values in $\mathbb{R}$). We consider two types of extrema. Free extrema and bound (or constrained) extrema. The terminology of "bound" (resp. free) extrema results from the presence (resp. absence) of conditions introducing edges into the domain, called constraints.

In both cases, we can define what are called local (or relative) extrema and global (or absolute) extrema. Let us recall that a maximum of the function $f$ can be seen as a minimum of $-f$. Before starting with the study of free extrema, it seems useful to make a reminder on quadratic forms.

## 6.1 Recalls

Let $\mathbb{R}^n$ have any norm $\|\mathrm{cdot}\|$. Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a symmetric bilinear function (in other words for all $x, y \in \mathbb{R}^n$, the applications of $\mathbb{R}^n \to \mathbb{R}$ defined by $B(x, \cdot) : y \mapsto B(x, y)$ and $B(\cdot, y) : x \mapsto B(x, y)$ are linear, and it is symmetric if for any pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ we have $B(x, y) = B(y, x)$.

**Definition 6.1** (Quadratic form). *The quadratic form associated with the symmetric bilinear form* $B$ *is the application*

$$\begin{aligned} q : \mathbb{R}^n &\to \mathbb{R} \\ x &\mapsto B(x, x). \end{aligned}$$

**Proposition 6.1.**

- *A quadratic form on* $\mathbb{R}^n$ *is a homogeneous polynomial of degree* $2$, *i.e., for all* $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *and for all* $t \in \mathbb{R}$.

$$q(tx) = t^2 q(x).$$

- *For all* $x, y \in \mathbb{R}^n$ *and for all* $t \in \mathbb{R}$ *we have*

$$B(x, y) = \frac{1}{2} \left( q(x + y) - q(x) - q(y) \right).$$

*Ainsi, la forme bilinéaire* $B$ *est uniquement déterminée* $q$, *elle s'appelle la forme polaire de* $q$.

**Example 6.1.** *The application*

$$\begin{aligned} q : \mathbb{R}^2 &\to \mathbb{R} \\ (x_1, x_2) &\mapsto x_1^2 + x_2^2 \end{aligned}$$

*is a quadratic form of polar form*

$$\begin{aligned} B : \mathbb{R}^2 \times \mathbb{R}^2 &\to \mathbb{R} \\ ((x_1, x_2), (y_1, y_2)) &\mapsto x_1 y_1 + x_2 y_2. \end{aligned}$$

**Definition 6.2** ((Matrix of quadratic form)). *The matrix of a quadratic form* $q$, *in a* $\mathcal{B}$ *basis of* $\mathbb{R}^n$, *is the matrix of the associated symmetric bilinear form (in the* $\mathcal{B}$ *basis).*

**Proposition 6.2.** *Let* $A = (a_{ij})$ *be the matrix of the quadratic form* $q$ *in a* $\mathcal{B}$ *basis of* $\mathbb{R}^n$. *Then* $q$ *is expressed matrix-wise as*

$$\forall x \in \mathbb{R}^n, q(x) = {}^t x A x.$$

*in other words, of the form*

$$q(x) = \sum_{i=1}^{n} a_{ii} x_i^2 + 2 \sum_{1 \leqslant i < j \leqslant n} a_{ij} x_i x_j.$$

**Exemple 6.2.**

Consider the quadratic form defined in $\mathbb{R}^3$ by

$$q(x, y, z) = x^2 - 2y^2 + 3z^2 - 6xy + 3xz + 2yz.$$

Then the matrix of $q$ in the canonical basis of $\mathbb{R}^3$ is

$$A = \begin{pmatrix} 1 & -3 & \frac{3}{2} \\ -3 & -2 & 1 \\ \frac{3}{2} & 1 & 3 \end{pmatrix}.$$

**Definition 6.3.** *A quadratic form* $q$ *on* $\mathbb{R}^n$ *is said to be*

- *positive definite if and only if for all* $x \in \mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}$, $q(x) > 0$.
- *positive semi-definite if and only if for all* $x \in \mathbb{R}^n$, $q(x) \geqslant 0$.
- *negative definite if and only if for all* $x \in \mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}$, $q(x) < 0$.
- *negative semi-definite if and only if for all* $x \in \mathbb{R}^n$, $q(x) \leqslant 0$.
- *Indefinite if* $q$ *is neither positive semidefinite nor negative semidefinite (i.e., if there exists* $x, y \in \mathbb{R}^n$ *such that* $q(x) > 0$ *and* $q(y) < 0$*).*
- *elliptic on* $\mathbb{R}^n$ *if and only if there exists* $a > 0$, *such that*

$$\forall x \in \mathbb{R}^n, q(x) \geqslant a \|x\|^2.$$

**Remark 6.1.** *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a symmetric matrix and $q$ be its associated quadratic form (i.e., $\forall x \in \mathbb{R}^n, q(x) = {}^t x A x$). Then $A$ is positive definite (resp. positive semidefinite, negative definite, negative semidefinite) if and only if $q$ is.*

**Remark 6.2.** *A real symmetric matrix $A$ is negative definite (resp. negative semidefinite) if and only if its opposite $-A$ (also symmetric) is positive definite (resp. positive semidefinite).*

**Remark 6.3.** *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a real symmetric matrix. Then the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of $A$ are real. Let $\mathrm{sp}(A) = \lambda_1, \lambda_2, \dots, \lambda_n$ be the spectrum of the matrix $A$.*

- *$A$ is positive definite* $\iff \mathrm{sp}(A) \subset \mathbb{R}_+^*$.
- *$A$ is positive semi-definite* $\iff \mathrm{sp}(A) \subset \mathbb{R}_+$.
- *$A$ is negative definite* $\iff \mathrm{sp}(A) \subset \mathbb{R}_-^*$.
- *$A$ is negative semi-definite* $\iff \mathrm{sp}(A) \subset \mathbb{R}_-$.

The following criteria are useful in determining whether a quadratic form is positive definite.

**Proposition 6.3.** *Let $q$ be the quadratic form on $\mathbb{R}^n$ with matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$. The following assertions are equivalent*

- *The quadratic form $q$ is positive definite.*
- *The matrix $A$ is positive definite.*
- *The eigenvalues of $A$ are all strictly positive.*
- *All principal minors of $A$ are strictly positive.*

**Example 6.2.** *The real symmetric matrix*

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

*We have*

$$\Delta_1 = |2| = 2 > 0, \quad \Delta_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 5 > 0, \quad \Delta_3 = \det(A) = 4 > 0,$$

*So the prime minors of $A$ are all positive, so it is positive definite.*

**Example 6.3.** *We define*

$$A = \begin{pmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix}.$$

*We have*

$$\det(A - \lambda I) = -\lambda(\lambda - 6)^2,$$

*so two (real) eigenvalues:* $\lambda_1 = 6$ *and* $\lambda_2 = 0$. *Thus,* $A$ *is not positive definite (it is positive semi-definite).*

**Definition 6.4.** *Let* $f$ *be a function from* $\Omega \subset \mathbb{R}^n$ *to* $\mathbb{R}$. *We say that*

- $f$ *admits a global maximum (resp. minimum) in* $a \in \Omega$ *if*

$$\forall x \in \Omega, \; f(x) \leqslant f(a) \; (\textit{resp.} \;\; f(x) \geqslant f(a)).$$

- $f$ *admits a local maximum (resp. minimum) in* $a \in \Omega$ *if there exists a neighborhood* $V$ *of* $a$ *in* $\mathbb{R}^n$ *such that*

$$\forall x \in \Omega \cap V, \; f(x) \leqslant f(a) \; (\textit{resp.}\; f(x) \geqslant f(a)).$$

- *An extremum is a maximum or a minimum.*

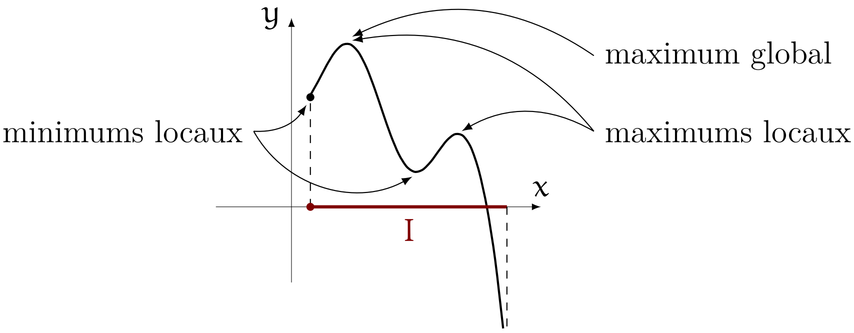



**Fig. 6.1 .** Maximum and minimum of a function

**Remark 6.4.** *It is clear that, if* $f$ *admits in* $a$ *a global maximum (resp. minimum), then* $f$ *admits in* $a$ *a local maximum (resp. minimum).*

The following result gives a sufficient condition for the existence of a global minimum and maximum.

**Theorem 1** [**(Weierstrass's theorem)**] If $\Omega \subset \mathbb{R}^n$ is compact and if $f$ is continuous, then $f$ admits a global maximum and a global minimum reached at least once, in other words there exists $x_m, x_M \in \Omega$ such that

$$\forall x \in \Omega, \; f(x_m) \leqslant f(x) \leqslant f(x_M).$$

**Remark 6.5.** *If* $q$ *is a quadratic form on* $\mathbb{R}^n$ *associated with a symmetric matrix* $A$ *then* $q$ *has a maximum and a minimum on the bounded closed ball with center* $0$ *and radius* $1$.

**Proposition 6.4.** *If* $q$ *is a quadratic form on* $\mathbb{R}^n$ *associated with a symmetric matrix* $A$ *then*

$$\forall x \in \mathbb{R}^n, \; \alpha \|x\|^2 \leqslant q(x) \leqslant \beta \|x\|^2,$$

*where* $\alpha = \min(sp(A))$ *and* $\beta = \max(sp(A))$.

*Proof.* The matrix $A$ is real symmetric so it is diagonalizable in an orthonormal basis. Let us denote $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ the eigenvalues of $A$. We have

$$A = PDP^\top$$

where $P$ is an orthogonal matrix and $D$ a diagonal matrix

$$q(x) = x^\top PDP^\top x = (Px)^\top D(Px) = x'^\top Dx',$$

with $x' = P^\top x$ of components $(x'_1, \ldots, x'_n)$. Then,

$$q(x) = \sum_{i=1}^{n} \lambda_i x_i'^2,$$

so,

$$\lambda_1 \|x\|^2 \leqslant q(x) \leqslant \lambda_n \|x\|^2.$$

We note that

$$\|x'\|^2 = x'^\top x' = x^\top P^\top Px = x^\top x = \|x\|^2.$$

$\square$

## 6.2 Extrema libres

In this section we study the extrema of a function on an open.

**Definition 6.5** ((Critical point)). *Let $\Omega$ be an open of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ a differentiable application in $a \in \Omega$. We say that $a$ is a critical point of $f$ if all the first partial derivatives of $f$ cancel out at $a$ (or equivalently, if $\nabla f = 0$ the differential of $f$ cancels out at $a$).*

**Example 6.4.** *Consider the function*

$$\begin{aligned} f:\ & \mathbb{R}^2 \to \mathbb{R} \\ & (x, y) \mapsto xy - xy^2 + x^2 y. \end{aligned}$$

*Let's search for the critical points of $f$, which means the pairs $(x, y)$ such that*

$$\begin{cases} \dfrac{\partial f}{\partial x}(x, y) = 0 \\ \dfrac{\partial f}{\partial y}(x, y) = 0 \end{cases} \iff \begin{cases} y(1 - y + 2x) = 0 \\ x(1 - 2y + x) = 0 \end{cases}$$

*Thus,*

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \quad \text{or} \quad \begin{cases} x = 0 \\ y = 1 \end{cases} \quad \text{or} \quad \begin{cases} x = -1 \\ y = 0 \end{cases} \quad \text{or} \quad \begin{cases} 1 - y + 2x = 0 \\ 1 - 2y + x = 0 \end{cases}$$

*By solving the last linear system, we obtain that $f$ admits four critical points which are $(0, 0)$, $(0, 1)$, $(-1, 0)$ and $\left(-\dfrac{1}{3}, \dfrac{1}{3}\right)$.*

In the following, we will give a necessary condition for the existence of local extrema.

**Theorem 2 (Condition nécessaire d'extremum local)** Let $\Omega$ be an open of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ an application **differentiable** at $a \in \Omega$. If $f$ admits a local extremum in $a$ then $d_a f = 0$ (in other words, $a$ is a critical point), which amounts to saying that

$$\forall i \in \{1, \ldots, n\}, \quad \frac{\partial f}{\partial x_i}(a) = 0. \tag{6.1}$$

*Proof.* Without loss of generality, suppose that $f$ admits a local maximum in $a$. then there exists an open neighbourhood $V$ of $a$ in $\Omega$ such that

$$\forall x \in V, \quad f(x) \leqslant f(a).$$

Let be the function of one variable $g(t) = f(a + th)$. Then for $h$ sufficiently small, $a + th \in V$ so that

$$g(t) \leqslant g(0).$$

Thus $g$ has a local maximum at $0$, so its derivative at $0$ is zero. Using the chain derivation rule, for any sufficiently small $h$ we obtain :

$$g'(0) = d_a f(h) = 0,$$

which is equivalent to saying that $d_a f = 0$. □

**Remark 6.6.** *The necessary condition (6.1) only concerns differentiable functions defined on an open set. However, we can have a local extremum where the function is not differentiable, whether or not it is defined on an open set.*

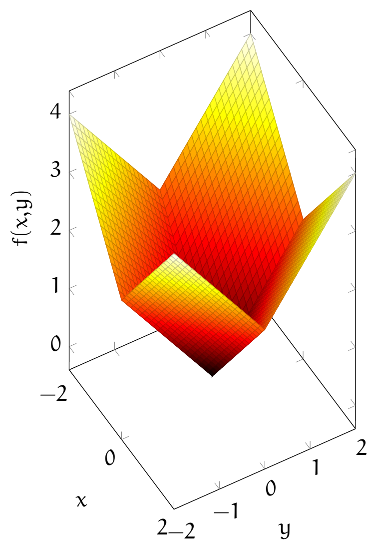
**Example 6.5.** *The function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$f(x, y) = |x| + |y|,$$

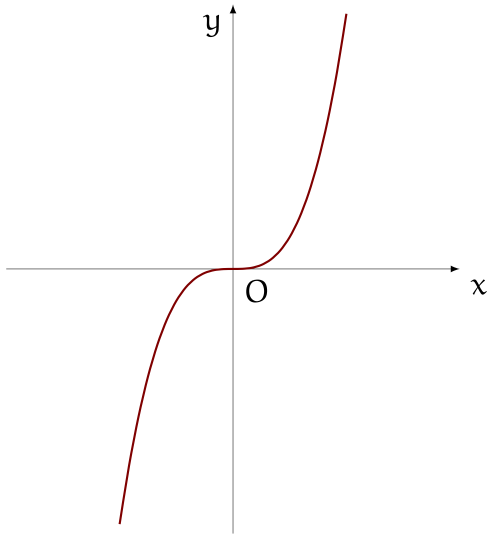
*has a global minimum at* $(0, 0)$. *Indeed*

$$\forall (x, y) \in \mathbb{R}^2, \quad f(x, y) \geqslant 0 = f(0, 0).$$

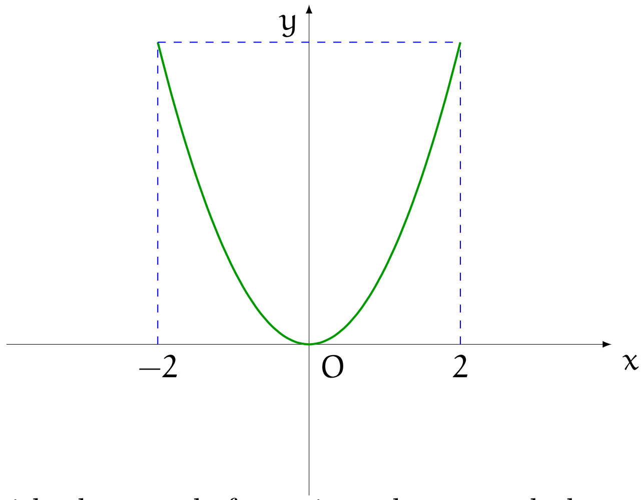
*However, the function is not differentiable at* $(0, 0)$.



The reciprocal of the previous theorem is false. A critical point may not be an extremum. In this case, it is said to be a saddle point or a Col point. For example for the function defined on $\mathbb{R}$ by $f(x) = x^3$, the point $a = 0$ is a critical point of $f)sincef'(0) = 0$, but $f$ does not admit a local extremum at $0$.




The open $\Omega$ assumption is essential. For example, the function $f(x) = x^2$ has two global maxima on the closed interval $[-2, 2]$ at $-2$ and at $2$, and yet these points are not critical points since $f'(2) = 4$ and $f'(-2) = -4$.




The determination of the local extrema begins with the search for points that cancel the gradient. This will not be sufficient, but it will allow us to reduce the set of possibilities and to proceed to the second step, which involves the development limited to the order $2$.

Recall that if $f : \Omega \to \mathbb{R}$ is a function differentiable twice on an open $\Omega \subset \mathbb{R}^n$ and $a \in \Omega$ is a critical point of $f$. Then the Taylor-Young expansion of order $2$ in the neighbourhood of $a$ is

$$\begin{aligned} f(a+h) &= f(a) + \frac{1}{2} d_a^2 f(h,h) + \|h\|^2 \varepsilon(h) \\ &= f(a) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a) h_i h_j + \|h\|^2 \varepsilon(h) \\ &= f(a) + \frac{1}{2} h^\top H_f(a) h + \|h\|^2 \varepsilon(h), \end{aligned}$$

where $\lim_{h \to 0} \varepsilon(h) = 0$ and $H_f(a) = \left( \frac{\partial^2 f}{\partial x_i \partial x_j}(a) \right)_{1 \leqslant i,j \leqslant n}$ is the Hessian matrix of $f$ in $a$ which is symmetrical.

**Theorem 3** [**(Necessary condition for local extremum)**]
Let $f : \Omega \to \mathbb{R}$ be a two differentiable application on an open $\Omega \subset \mathbb{R}^n$ and $a \in \Omega$ a critical point of $f$. Let $q$ be the quadratic form defined on $\mathbb{R}^n$ by

$$\forall h = (h_1, \dots, h_n) \in \mathbb{R}^n, \quad q(h) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a) h_i h_j.$$

- If $q$ is positive, then $f$ has a local minimum in $a$.
- If $q$ is negative, then $f$ has a local maximum in $a$.
- If $q$ is indefinite, then $f$ has no extremum at $a$.
- If $q$ is positive semidefinite (or negative semidefinite) such that there exists $h \in \mathbb{R}^n$ nonzero such that $q(h) = 0$, then this theorem does not allow us to conclude.

*Proof.* According to the Taylor-Young formula of order 2 and since $a$ is a critical point of $f$, we have,

$$f(a+h) - f(a) = \frac{1}{2}q(h) + \|h\|^2 \epsilon(h).$$

- Suppose $q$ is positive definite. By proposition 6.4, we have

$$f(a+h) - f(a) > \frac{1}{2}\alpha\|h\|^2 + \|h\|^2 \epsilon(h),$$

where $\alpha = \min(\mathrm{sp}(H_f(a))) > 0$. Since $\lim\limits_{h \to 0} \epsilon(h) = 0$, then there exists $\eta > 0$ such that $B(a, \eta) \subset \Omega$ and that

$$\forall h \in B(a, \eta), \quad |\epsilon(h)| \leqslant \frac{1}{4}\alpha,$$

and we conclude

$$f(a+h) - f(a) > \frac{1}{4}\alpha\|h\|^2. \geqslant 0$$

This shows that $f$ admits a local minimum in $a$.

- Suppose $q$ is negative definite. By proposition 6.4, we have,

$$f(a+h) - f(a) \leqslant \frac{1}{2}\beta\|h\|^2 + \|h\|^2 \epsilon(h),$$

where $\beta = \max(\mathrm{sp}(H_f(a))) < 0$. Since $\lim\limits_{h \to 0} \epsilon(h) = 0$, then there exists $\eta > 0$ such that $B(a, \eta) \subset \Omega$ and that

$$\forall h \in B(a, \eta), \quad |\epsilon(h)| \leqslant \frac{1}{4}\beta,$$

and we conclude

$$f(a+h) - f(a) \leqslant \frac{1}{4}\beta\|h\|^2 \leqslant 0.$$

This shows that $f$ has a local maximum at $a$.

▶ Suppose $q$ is undefined. There exist $x, y \in \mathbb{R}^n$ such that $q(x) > 0$ and $q(y) < 0$.

$$\begin{cases} f(a+\lambda x) - f(a) = \frac{1}{2}\lambda^2 q(x) + \lambda^2 \|x\|^2 \epsilon(\lambda) \sim_{\lambda \to 0} \frac{1}{2}\lambda^2 q(x) \\ f(a+\lambda y) - f(a) = \frac{1}{2}\lambda^2 q(y) + \lambda^2 \|y\|^2 \epsilon(\lambda) \sim_{\lambda \to 0} \frac{1}{2}\lambda^2 q(y). \end{cases}$$

Which shows that, on any neighborhood of $a$, $f$ takes values less than $f(a)$ and values greater than $f(a)$. We conclude that $f$ does not admit a local extremum in $a$.

□

**Remark 6.7.** *If $q$ is positive semi-definite (or negative semi-definite) such that there exists $h \in \mathbb{R}^n$ non-zero such that $q(h) = 0$, then we cannot conclude anything directly. We then return to the definition by studying the sign of $f(x) - f(a)$ in the neighborhood of $a$.*

**Example 6.6.** *The point $(0,0)$ is a critical point of the function $f$ defined on $\mathbb{R}^2$ by*

$$f(x,y) = x^2 - 2xy + y^2 + x^4 + y^4.$$

*In effect,*

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 2x - 2y + 4x^3 \\ \frac{\partial f}{\partial y}(x,y) = -2x + 2y + 4y^3 \end{cases} \Longrightarrow \begin{cases} \frac{\partial f}{\partial x}(0,0) = 0 \\ \frac{\partial f}{\partial y}(0,0) = 0 \end{cases}$$

*We have,*

$$\begin{cases} \frac{\partial^2 f}{\partial x^2}(x,y) = 2 + 12x^2 \\ \frac{\partial^2 f}{\partial y^2}(x,y) = 2 + 12y^2 \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) = -2 \end{cases} \Longrightarrow \begin{cases} \frac{\partial^2 f}{\partial x^2}(0,0) = 2 \\ \frac{\partial^2 f}{\partial y^2}(0,0) = 2 \\ \frac{\partial^2 f}{\partial x \partial y}(0,0) = -2 \end{cases}$$

*The quadratic form*

$$q(h_1, h_2) = 2h_1^2 - 4h_1 h_2 + 2h_2^2 = 2(h_1 - h_2)^2 \geqslant 0.$$

*is positive semi-definite with $q(h_1, h_1) = 0$ for all $h_1 \in \mathbb{R}$, and $f$ admits a local minimum at $(0,0)$, indeed*

$$\forall (x,y) \in \mathbb{R}^2, \quad f(x,y) - f(0,0) = (x-y)^2 + x^4 + y^4 \geqslant 0.$$

**Example 6.7.** *The point $(0,0)$ is a critical point of the function $f$ defined on $\mathbb{R}^2$ by*

$$f(x,y) = x^2 - 2xy + y^2 - x^4 - y^4.$$

*Indeed,*

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 2x - 2y - 4x^3 \\ \frac{\partial f}{\partial y}(x,y) = -2x + 2y - 4y^3 \end{cases} \Longrightarrow \begin{cases} \frac{\partial f}{\partial x}(0,0) = 0 \\ \frac{\partial f}{\partial y}(0,0) = 0 \end{cases}$$

*We have,*

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2}(x,y) = 2 - 12x^2 \\ \dfrac{\partial^2 f}{\partial y^2}(x,y) = 2 - 12y^2 \\ \dfrac{\partial^2 f}{\partial x \partial y}(x,y) = -2 \end{cases} \implies \begin{cases} \dfrac{\partial^2 f}{\partial x^2}(0,0) = 2 \\ \dfrac{\partial^2 f}{\partial y^2}(0,0) = 2 \\ \dfrac{\partial^2 f}{\partial x \partial y}(0,0) = -2 \end{cases}$$

*The quadratic form*

$$q(h_1, h_2) = 2h_1^2 - 4h_1 h_2 + 2h_2^2 = 2(h_1 - h_2)^2 \geqslant 0.$$

*is positive semi-definite with* $q(h_1, h_1) = 0$ *for all* $h_1 \in \mathbb{R}$, *and* $f$ *does not admit a local extremum at* $(0,0)$, *indeed*

$$\forall x \in \mathbb{R}^*, \quad f(x, -x) - f(0,0) = -2x^4 < 0.$$

$$\forall x \in ]-\sqrt{2}, \sqrt{2}[, \quad f(x, -x) - f(0,0) = 2x^2(2 - x^2) > 0.$$

The following corollary provides the equivalent sufficient conditions of local extremum depending on the sign of the eigenvalues of the Hessian matrix at a critical point.

**Theorem 4** Let $f : \Omega \to \mathbb{R}$ be a two-differentiable application on an open $\Omega \subset \mathbb{R}^n$ and $a \in \Omega$ be a critical point of $f$. Let $H_f(a)$ be the Hessian matrix of $f$ at $a$.

- If the eigenvalues of $H_f(a)$ are strictly positive, then $f$ has a local minimum at point $a$.
- If the eigenvalues of $H_f(a)$ are strictly negative, then $f$ has a local maximum at point $a$.
- If $H_f(a)$ has at least one strictly positive eigenvalue and one strictly negative eigenvalue, then $f$ has neither a maximum nor a local minimum at point $a$ (a is a saddle point).
- If all the eigenvalues of $H_f(a)$ have the same sign and if $0$ is an eigenvalue, this theorem does not allow us to conclude.

**Example 6.8.** *We consider the function*

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R} \\ (x,y,z) &\mapsto 4x^2 + y^2 + 3z^2 - 4xz - 2yz - 4x + 2. \end{aligned}$$

*Let's look for the critical points of* $f$, *i.e., the triplets (the coordinates)* $(x,y,z)$ *such that,*

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y,z) = 0 \\ \dfrac{\partial f}{\partial y}(x,y,z) = 0 \\ \dfrac{\partial f}{\partial z}(x,y,z) = 0 \end{cases} \iff \begin{cases} 8x - 4z - 4 = 0 \\ 2y - 2z = 0 \\ 6z - 4x - 2y = 0 \end{cases}$$

*By solving the linear system, we obtain that* $f$ *has a single critical point which is* $a = (1, 1, 1)$.

*The Hessian matrix of* $f$ *in* $a$ *is*

$$H_f(1,1,1) = \begin{pmatrix} 8 & 0 & -4 \\ 0 & 2 & -2 \\ -4 & -2 & 6 \end{pmatrix},$$

*whose characteristic polynomial*

$$p(\lambda) = \det(H_f(1,1,1) - \lambda I) = (4-\lambda)(\lambda^2 - 12\lambda + 8),$$

*Thus, the eigenvalues of* $H_f(1,1,1)$ *are all strictly positive:*

$$4, \quad 6+2\sqrt{7}, \quad 6-2\sqrt{7},$$

*and therefore* $f$ *has a local minimum at* $(1,1,1)$*, this minimum being worth* $f(1,1,1) = 0$*.*

**Example 6.9.** *Consider the function*

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R} \\ (x,y,z) &\mapsto \frac{1}{2}x^2 + xyz - z + y. \end{aligned}$$

*Search for the critical points of* $f$*, namely, the triplets* $(x,y,z)$ *such that*

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y,z) = 0 \\ \dfrac{\partial f}{\partial y}(x,y,z) = 0 \\ \dfrac{\partial f}{\partial z}(x,y,z) = 0 \end{cases} \iff \begin{cases} x + yz = 0 \\ xx + 1 = 0 \\ xy - 1 = 0 \end{cases}$$

*Solving the system, we obtain that* $f$ *has a single critical point which is* $a = (1,1,-1)$*. The Hessian matrix of* $f$ *at* $a$ *is*

$$H_f(x,y,z) = \begin{pmatrix} 1 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix} \implies H_f(1,1,-1) = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

*which has a characteristic polynomial,*

$$p(\lambda) = \det(H_f(1,1,1) - \lambda I) = (1-\lambda)(\lambda^2 + 3),$$

*Thus, the matrix* $H_f(1,1,1)$ *has two positive eigenvalues, and one negative.*

$$1, \quad \sqrt{3}, \quad -\sqrt{3}.$$

*So it's neither a maximum nor a minimum, it's a saddle point.*

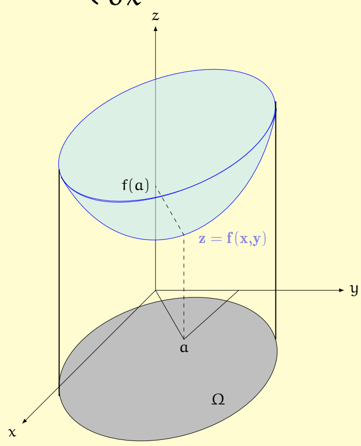
### 6.2.1 Two-variable function case

**Theorem 5** Let $f : \Omega \to \mathbb{R}$ be a two differentiable application on an open $\Omega \subset \mathbb{R}^2$ and $a \in \Omega$ a critical point of $f$. We note

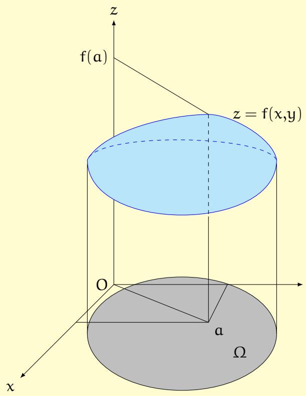
$$H_f(a) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(a) & \dfrac{\partial^2 f}{\partial x \partial y}(a) \\ \dfrac{\partial^2 f}{\partial x \partial y}(a) & \dfrac{\partial^2 f}{\partial y^2}(a) \end{pmatrix},$$

the Hessian matrix of $f$ at $a$.

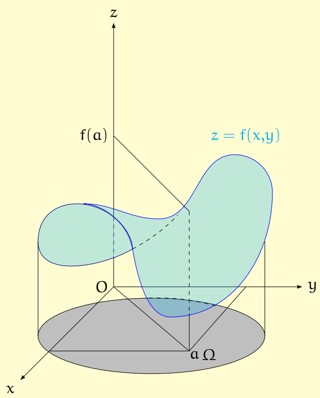
❶ Si $\begin{cases} \det(H_f(a)) > 0 \\ \dfrac{\partial^2 f}{\partial x^2}(a) > 0, \end{cases}$ then $f$ has a local minimum at $a$.




❷ Si $\begin{cases} \det(H_f(a)) > 0 \\ \dfrac{\partial^2 f}{\partial x^2}(a) < 0, \end{cases}$ alors $f$ admet un maximum local en $a$.




❸ Si $\det(H_f(a)) < 0$, alors $f$ n'admet pas d'extremum local en $a$. Dans ce cas, on dit que $f$ admet un point-selle en $a$.




❹ If $\det(H_f(a)) = 0$, we cannot conclude a priori.

**Example 6.10.** *Consider the function*

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto x^3 + y^3 - 6(x^2 - y^2). \end{aligned}$$

$f$ *is of class* $C^2$ *in its domain of definition, the open. Let's look for the critical points of* $f$*, in other words the pairs* $(x, y)$ *such that*

$$\begin{cases} \dfrac{\partial f}{\partial x}(x, y) = 0 \\ \dfrac{\partial f}{\partial y}(x, y) = 0 \end{cases} \iff \begin{cases} 3x^2(x-4) = 0 \\ 3y^2(y+4) = 0 \end{cases}$$

*By solving the system, we obtain that* $f$ *has four critical points which are*

$$(0, 0), \quad (0, -4), \quad (4, 0), \quad (4, -4).$$

*Let us look for the nature of these critical points, the Hessian matrix of* $f$ *at* $(x, y)$ *is*

$$H_f(x, y) = \begin{pmatrix} 6(x-2) & 0 \\ 0 & 6(y+2) \end{pmatrix},$$

- $\det(H_f(0, 0)) < 0$ *so* $(0, 0)$ *is a saddle point.*
- $\det(H_f(0, -4)) > 0$ *and* $\dfrac{\partial^2 f}{\partial x^2}(0, -4) < 0$ *so* $(0, -4)$ *is a local maximum for* $f$.
- $\det(H_f(4, 0)) > 0$ *and* $\dfrac{\partial^2 f}{\partial x^2}(4, 0) > 0$ *so* $(4, 0)$ *is a local minimum for* $f$.
- $\det(H_f(4, -4)) < 0$ *so* $(0, 0)$ *is a saddle point.*

**Example 6.11.** *We consider the function*

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto x^4 + y^4 - 4(x^2 - y^2). \end{aligned}$$

$f$ *is of class* $C^2$ *in its domain of definition, the open set. Let us look for the critical points of* $f$*, that is to say the pairs* $(x, y)$ *such that*

$$\begin{cases} \dfrac{\partial f}{\partial x}(x, y) = 0 \\ \dfrac{\partial f}{\partial y}(x, y) = 0 \end{cases} \iff \begin{cases} 4(x^3 - 2x + 2y) = 0 \\ 4(4^3 + 2x - 2y) = 0 \end{cases}$$

*By solving the system, we obtain that* $f$ *admits three critical points which are*

$$(0, 0), \quad (2, -2), \quad (-2, 2).$$

*Research the nature of these critical points; the Hessian matrix of* $f$ *in* $(x, y)$ *is*

$$H_f(x, y) = \begin{pmatrix} 12x^2 - 8 & 8 \\ 8 & 12y^2 - 8 \end{pmatrix},$$

- $\det(H_f(2,-2)) = 1536 > 0$ *and* $\frac{partial^2 f}{partial x^2}(2,-2) = 40 > 0$ *so* $(2,-2)$ *is a local minimum for* $f$.
- $\det(H_f(-2,2)) = 1536 > 0$ *and* $\frac{\partial^2 f}{\partial x^2}(-2,2) = 40 > 0$ *so,* $(-2,2)$ *is a local minimum for* $f$.
- $\det(H_f(0,0)) = 0$, *we cannot conclude using the Hessian matrix. To determine the nature of the point* $(0,0)$ *we study the sign of* $\delta(x,y) = f(x,y) - f(0,0)$ *for* $(x,y)$ *neighbours of* $(0,0)$

$$\delta(x,y) = x^4 + y^4 - 4(x-y)^2.$$

*since* $\delta(x,y) = x^2(x^2-4) < 0$ *when* $x$ *is neighbouring* $0$ *(e.g.* $x \in ]-2,2[$*) but* $\delta(x,x) = 2x^4 > 0$, *then* $(0,0)$ *is a saddle point. Note that we can rewrite the function* $f$ *in the form*

$$\forall (x,y) \in \mathbb{R}^2, f(x,y) = (x^2-4)^2 + (y^2-4)^2 + 4(x+y)^2 - 32.$$

*Since* $f(2,-2) = f(-2,2) = -32$, *then the points* $(2,-2)$ *and* $(-2,2)$ *are global minima.*

We can also give a result on the global extrema

**Proposition 6.5.** *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a continuous function such that* $\lim_{\|x\| \to +\infty} f(x) = +\infty$, *that is,*

$$\forall A > 0, \exists B > 0, \forall x \in \mathbb{R}^n, (\|x\| \geqslant B \implies f(x) \geqslant A).$$

*Then* $f$ *has a global minimum.*

**Example 6.12.** *Returning to example 6.10, we can see that*

$$f(x,0) = x^3 - 6x \longrightarrow +\infty, \quad x \to +\infty,$$

$f$ *is neither upper bound nor lower bound. We conclude that* $f$ *does not admit a global extremum.*

### 6.2.2 Extrema liés

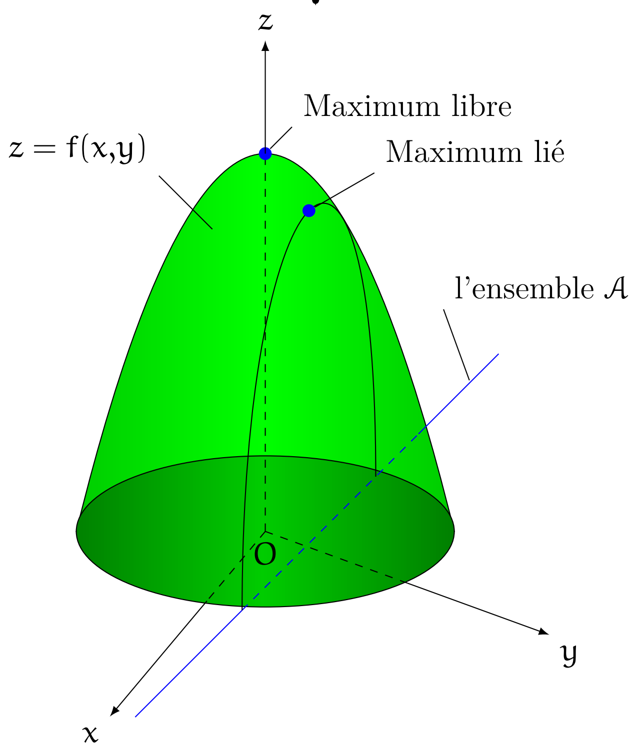
The results obtained in the previous section are very strong, but they cannot be applied to solve all the problems encountered. Indeed, the extrema can belong either to the open set $\Omega = \Omega \setminus \partial\Omega$ or to the boundary $\partial\Omega$. The question of the existence of extrema on the boundary of the domain has led to a discrimination between free and bound extrema. It is emphasized that the boundary can be represented by an equality $g(x) = 0$ where $g = (g_1, \ldots, g_m) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$.

#### 6.2.2.1 Problem position

Let $f$ and $g_1, \ldots, g_m$ be real-valued functions of class $C^1$ on an open $\Omega$ of $\mathbb{R}^n$. We assume,

$$A = \{x \in \Omega : g_j(x) = 0, \ j = 1, \ldots, m\}.$$

**Definition 6.6.** *The function* $f$, *admits at* $a \in A$ *a bound maximum (resp. a bound minimum) under the constraints* $g_1(x) = 0, g_2(x) = 0, \ldots, g_m(x) = 0$, *if at this point it admits a free maximum (resp. a free minimum) in the set* $A$.




## 6.3 Related Extrema

The following example shows that when the constraint takes a relatively simple form, we can reduce ourselves to the search for free extrema on an open $\mathbb{R}^{n-m}$ by substituting the constraint in the expression of $f$.

**Example 6.13.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = 2x - y$. Let's try to find the extrema of $f$ under the constraint $g(x,y) = y - x^2 = 0$. This is equivalent to finding the extrema of the restriction of $f$ to the parabola of equation $f$ at $y = x^2$. Let's put*

$$F(x) = f(x,x^2) = 2x - x^2.$$

*Let us find the extrema of $F$ on $\mathbb{R}$. We have*

$$\forall x \in \mathbb{R}, F'(x) = 2 - 2x.$$

*We have $F`(x) = 0$ if and only if $x = 1$. And since,*

$$F''(1) = -2 < 0,$$

*then $F$ admits a global maximum at $x = 1$. Therefore $f$ admits a global maximum under the constraint $g(x,y) = 0$ at the point $(1,1)$.*

**Remark 6.8.** *The set $A$ consists of the intersection of $m$ hypersurfaces of $\mathbb{R}^n$. Consequently, for $m > 1$, it may contain too few points. This is why, in practice, it is rare to have the number of constraints greater than or equal to the number of variables, so the condition $m < n$ will always be imposed.*

### 6.3.1 Necessary condition and Lagrange's theorem

The following Lagrange theorem is a necessary condition for the existence of extrema with equality constraints.

**Theorem 6** [**(Lagrange's theorem)**] Let $\Omega$ be a non-empty open set of $\mathbb{R}^n$, $1 \leqslant m < n$ and $f, g_1, \dots, g_m \in C^1(\Omega)$. We assume that,

❶ $f$ admits at $a = (a_1, \dots, a_n)$ a related(bound or linked) extremum under the constraints
$$g_1(x) = 0, \dots, g_m(x) = 0.$$

❷ The vectors $\nabla g_j(a)$, $j = 1, \dots, m$ are linearly independent (in this case the Jacobian matrix $\left(\dfrac{\partial g_j}{\partial x_i}(a)\right)_{\substack{1 \leqslant j \leqslant m \\ 1 \leqslant i \leqslant n}}$ est de rang $m$).

Then there exist $\lambda_1, \dots, \lambda_m \in \mathbb{R}^m$, called Lagrange multipliers, such that

$$\nabla f(a) = -\sum_{j=1}^{m} \lambda_j \nabla g_j(a). \tag{6.2}$$

This means that at point $a$ the gradient $\nabla f(a)$ is a linear combination of the gradients of the constraints $\nabla g_j(a)$, $j = 1, \dots, m$.

### 6.3.2 Geometric interpretation

In the case $n = 2$ and $m = 1$ (two variables and one constraint), condition (6.2) is written as follows
$$\exists \lambda \in \mathbb{R}, \nabla f(a) = -\lambda \nabla g(a).$$
This means that, in the $\mathbb{R}^2$ plane, the level curve $f(x,y) = f(a_1,a_2)$ and the curve $g(x,y) = 0$ have a common tangent at the point $a = (a_1, a_2)$.

**Definition 6.7.** *A point* $a \in A$ *such that*

$$\operatorname{rg}\left(\frac{\partial g_j}{\partial x_i}(a)\right) = m,$$

*is called the regular point of the constraints. Non-regular points of* $A$, *known as singular points, are retained a priori as candidates for related extrema.*

Lagrange's theorem only allows us to find candidate bound extrema of $f$, but does not allow us to determine their nature. The condition (6.2) can be seen as a necessary condition for the existence of free extrema for the function of $n + m$ variables called the Lagrangian function (or the Lagrangian) defined as follows

$$L: \ \Omega \times \mathbb{R}^m \to \mathbb{R}$$
$$(x, \lambda) \mapsto L(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x),$$

où $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$. The result of the previous theorem can be expressed as,

$$\exists \lambda \in \mathbb{R}^m : \nabla_x L(a, \lambda) = 0 \text{ with } \nabla_x L(x, \lambda) = \left(\frac{\partial L}{\partial x_1}(x, \lambda), \dots, \frac{\partial L}{\partial x_n}(x, \lambda)\right).$$

Partial derivatives with respect to the $\lambda_i$ do not appear in this notation. The condition $\nabla_\lambda L(a,\lambda) = 0$ is always verified since $\frac{\partial L}{\partial \lambda_j}(x,\lambda) = g_j(x) = 0$. The solutions $(x,\lambda)$ of the system $\nabla_x L(x,\lambda) = 0$ are called critical points of the Lagrangian. Lagrange's theorem can be reformulated in the following form

**Theorem 7** Let $\Omega$ be a non-empty open set of $\mathbb{R}^n$, $1 \leqslant m < n$ and $f, g_1, \ldots, g_m \in C^1(\Omega)$. If $f$ admits, at the regular point $a$, an extremum linked under the constraints $g_1(x) = 0, \ldots, g_m(x) = 0$, then

$$\exists \lambda \in \mathbb{R}^m : \nabla L(a,\lambda) = 0.$$

### 6.3.3 Second order conditions

The second-order conditions for determining whether it is a maximum or a minimum rely on the computation of the minors of the bordered Hessian matrix introduced in the following definition.

**Definition 6.8** ((The Bordered Hessian Matrix)). *If the Lagrangian* $L$ *is of class* $C^2$, *we define the bordered Hessian matrix (the Hessian matrix of* $L$*) by*

$$H = \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{\partial g_1}{\partial x_1}(a) & \cdots & \frac{\partial g_1}{\partial x_n}(a) \\ 0 & 0 & \cdots & 0 & \frac{\partial g_2}{\partial x_1}(a) & \cdots & \frac{\partial g_2}{\partial x_n}(a) \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \frac{\partial g_m}{\partial x_1}(a) & \cdots & \frac{\partial g_m}{\partial x_n}(a) \\ \frac{\partial g_1}{\partial x_1}(a) & \frac{\partial g_1}{\partial x_2}(a) & \cdots & \frac{\partial g_1}{\partial x_n}(a) & \frac{\partial^2 L}{\partial x_1^2}(a,\lambda) & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n}(a,\lambda) \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1}(a) & \frac{\partial g_m}{\partial x_2}(a) & \cdots & \frac{\partial g_m}{\partial x_n}(a) & \frac{\partial^2 L}{\partial x_n x_1}(a,\lambda) & \cdots & \frac{\partial^2 L}{\partial x_n^2}(a,\lambda) \end{bmatrix}$$

Let $H_p$ be the primary principal submatrix of order $m+p$ of $H$ $(1 \leqslant p \leqslant n)$, that is, the submatrix obtained by keeping only the first $m+p$ rows and the $m+p$ columns of $H$. The sufficient condition for the existence of an extremum is given in the following result

**Theorem 8** [**(Sufficient conditions)**] Let $\Omega$ be a convex open set of $\mathbb{R}^n$, $1 \leqslant m < n$ and $f, g_1, \ldots, g_m \in C^2(\Omega)$. If $(a,\lambda) \in \mathbb{R}^n \times \mathbb{R}^m$ is a critical point of the Lagrangian $L$, then

❶ If $(-1)^m \det(H_p) > 0$ for $p = m+1, \ldots, n$, then $f$ admits a local minimum bound at $a$.

❷ If $(-1)^p \det(H_p) > 0$ for $p = m+1, \ldots, n$, then $f$ admits a related local maximum at $a$.

### 6.3.4 Case of a function of two variables ($n = 2$ and $m = 1$)

Let $\Omega$ be a non-empty open of $\mathbb{R}^2$ and $f, g \in C^2(\Omega)$. The aim is to find the extrema of $f$ under the constraint $g(x, y) = 0$. We have the Lagrangian

$$L(x, y, \lambda) = f(x, y) + \lambda g(x, y).$$

Let $(a, b) \in \Omega$ such that $g(a, b) = 0$. If $\nabla g(a, b) \neq 0$, then

$$\exists \lambda \in \mathbb{R}, \nabla L(a, b, \lambda) = \nabla f(a, b) + \lambda \nabla g(a, b) = 0.$$

thus $(a, b, \lambda)$ is a critic of the Lagrangian $L$. To determine whether this is a maximum or a minimum, we introduce the Bordered Hessian matrix,

$$H = \begin{bmatrix} \frac{\partial^2 L}{\partial \lambda^2}(a,b,\lambda) & \frac{\partial^2 L}{\partial \lambda \partial x}(a,b,\lambda) & \frac{\partial^2 L}{\partial \lambda \partial y}(a,b,\lambda) \\ \frac{\partial^2 L}{\partial x \partial \lambda}(a,b,\lambda) & \frac{\partial^2 L}{\partial x^2}(a,b,\lambda) & \frac{\partial^2 L}{\partial x \partial y}(a,b,\lambda) \\ \frac{\partial^2 L}{\partial y \partial \lambda}(a,b,\lambda) & \frac{\partial^2 L}{\partial y \partial x}(a,b,\lambda) & \frac{\partial^2 L}{\partial y^2}(a,b,\lambda) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \frac{\partial g}{\partial x}(a,b) & \frac{\partial g}{\partial y}(a,b) \\ \frac{\partial g}{\partial x}(a,b) & \frac{\partial^2 L}{\partial x^2}(a,b,\lambda) & \frac{\partial^2 L}{\partial x \partial y}(a,b,\lambda) \\ \frac{\partial g}{\partial y}(a,b) & \frac{\partial^2 L}{\partial y \partial x}(a,b,\lambda) & \frac{\partial^2 L}{\partial y^2}(a,b,\lambda) \end{bmatrix}$$

In this case, the sufficient condition for the existence of an extremum can be reduced to the following statement

❶ If $\det(H) < 0$, then $f$ has a local minimum bound in $(a, b)$.

❷ If $\det(H) > 0$, then $f$ admits a local maximum bound in $(a, b)$.

❸ If $\det(H) = 0$, we cannot conclude anything.

**Example 6.14.** *We are searching for the extrema of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x, y) = x + y,$$

*under the constraint $x^2 + y^2 = 2$. Let's start by studying the level curves of $f$ and their intersections with the graph of the constraint*

❶ $f(x, y) = k \iff y = k - x$, *the level curves of $f$ are parallel lines with slope $-1$.*

❷ $x^2 + y^2 = 2$ *is the equation of the circle with centre $(0, 0)$ and radius $\sqrt{2}$.*

❸ *the level curves of $f$ which are tangent to the constraint are the straight lines $y = -x - 2$ and $y = -x + 2$ at the points $(1, 1)$ and $(-1, -1)$ respectively.*

❹ *By examining the direction of growth of $f$ along the constraint, we deduce that $f$ admits, under the constraint $g(x, y) = x^2 + y^2 - 2 = 0$, a maximum at $(1, 1)$ and a minimum at $(-1, -1)$.*

We apply Lagrange's method to find the results obtained graphically. The Lagrangian in this case is defined by

$$\begin{aligned} L : \mathbb{R}^2 \times \mathbb{R} &\to \mathbb{R} \\ (x, y, \lambda) &\mapsto L(x, y, \lambda) = f(x, y) + \lambda g(x, y) = x + y + \lambda \left(x^2 + y^2 - 2\right), \end{aligned}$$

We are looking for the critical points of $L$ :

$$\nabla L(x, y, \lambda) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} 1 + 2\lambda x = 0 \\ 1 + 2\lambda y = 0 \\ x^2 + y^2 - 2 = 0 \end{cases}$$

$$\iff (x, y, \lambda) \in \left\{ \left(1,1,-\frac{1}{2}\right), \left(-1,-1,\frac{1}{2}\right) \right\}$$

To determine whether it is a maximum or a minimum, we consider the bordered Hessian matrix

$$H = H(x, y, \lambda) = \begin{bmatrix} 0 & \frac{\partial g}{\partial x}(a, b) & \frac{\partial g}{\partial y}(a, b) \\ \frac{\partial g}{\partial x}(a, b) & \frac{\partial^2 L}{\partial x^2}(a, b, \lambda) & \frac{\partial^2 L}{\partial x \partial y}(a, b, \lambda) \\ \frac{\partial g}{\partial y}(a, b) & \frac{\partial^2 L}{\partial y \partial x}(a, b, \lambda) & \frac{\partial^2 L}{\partial y^2}(a, b, \lambda) \end{bmatrix} = \begin{bmatrix} 0 & 2x & 2y \\ 2x & 2\lambda & 0 \\ 2y & 0 & 2\lambda \end{bmatrix}$$

We have,

$$\det\left(H\left(-1,-1,\frac{1}{2}\right)\right) = \begin{vmatrix} 0 & -2 & -2 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{vmatrix} = -8 < 0.$$

So $f$ admits, under the constraint $g(x, y) = x^2 + y^2 - 2 = 0$ a global minimum at $(-1,-1)$.

For the second critical point, we have

$$\det\left(H\left(1,1,-\frac{1}{2}\right)\right) = \begin{vmatrix} 0 & 2 & 2 \\ 2 & -1 & 0 \\ 2 & 0 & -1 \end{vmatrix} = 8 > 0.$$

So $f$ admits, under the constraint $g(x, y) = x^2 + y^2 - 2 = 0$ a global maximum at $(1,1)$.

### Case of a function of three variables and a single constraint ( $n = 3$ and $m = 1$ )

Let $\Omega$ be a non-empty open set of $\mathbb{R}^3$ and $f, g \in \mathcal{C}^2(\Omega)$. The goal is to find the extrema of $f$ under the constraint $g(x, y, z) = 0$. We have the Lagrangian

$$L(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z).$$

Let $(a, b, c) \in \Omega$ be such that $g(a, b, c) = 0$. If $\nabla g(a, b, c) \neq 0$, then

$$\exists \lambda \in \mathbb{R}, \nabla L(a, b, c, \lambda) = \nabla f(a, b, c) + \lambda \nabla g(a, b, c) = 0.$$

To determine the nature of the critical point $(a,b,c,\lambda)$, consider the bordered Hessian matrix

$$H = \begin{bmatrix} \frac{\partial^2 L}{\partial \lambda^2}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial \partial x}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial \lambda \partial y}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial \lambda \partial z}(a,b,c,\lambda) \\ \frac{\partial^2 L}{\partial x \partial \lambda}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial x^2}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial x \partial y}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial x \partial z}(a,b,c,\lambda) \\ \frac{\partial^2 L}{\partial y \partial \lambda}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial y \partial x}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial y^2}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial y \partial z}(a,b,c,\lambda) \\ \frac{\partial^2 L}{\partial z \partial \lambda}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial z \partial x}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial z \partial y}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial z^2}(a,b,c,\lambda) \end{bmatrix}$$

$H_2$ the primary principal sub-matrix of order 3

$$= \begin{bmatrix} 0 & \frac{\partial g}{\partial x}(a,b,c) & \frac{\partial g}{\partial y}(a,b,c) & \frac{\partial g}{\partial z}(a,b,c) \\ \frac{\partial g}{\partial x}(a,b,c) & \frac{\partial^2 L}{\partial x^2}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial x \partial y}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial x \partial z}(a,b,c,\lambda) \\ \frac{\partial g}{\partial y}(a,b,c) & \frac{\partial^2 L}{\partial y \partial x}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial y^2}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial y \partial z}(a,b,c,\lambda) \\ \frac{\partial g}{\partial z}(a,b,c) & \frac{\partial^2 L}{\partial z \partial x}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial z \partial y}(a,b,c,\lambda) & \frac{\partial^2 L}{\partial z^2}(a,b,c,\lambda) \end{bmatrix}$$

In this case, the sufficient condition for the existence of an extremum can be reduced to the following statement

❶ If $\det(H) < 0$ and $\det(H_2) < 0$, then $f$ admits a local minimum related at $(a,b,c)$.

❷ If $\det(H) < 0$ and $\det(H_2) > 0$, then $f$ admits a local maximum bound at $(a,b,c)$.

**Example 6.15.** *Among the parallelepipeds of surface $S = 24$, which ones have a maximum volume ? Let $x, y, z$ be the lengths of the sides of the parallelepiped, its surface is $S = 2(xy + yz + xz)$ and its volume is $xyz$. So it's about maximization of function*

$$\begin{aligned} f : \left(\mathbb{R}_+^*\right)^3 &\to \mathbb{R} \\ (x,y,z) &\mapsto f(x,y,z) = xyz, \end{aligned}$$

*sous la contrainte* $g(x,y,z) = 2(xy + yz + xz) - 24 = 0$.

$$\begin{aligned} L : \left(\mathbb{R}_+^*\right)^3 \times \mathbb{R} &\to \mathbb{R} \\ (x,y,z,\lambda) &\mapsto L(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z) = xyz + \lambda(2xy + 2yz + 2xz - 24). \end{aligned}$$

*We are looking for the critical points of* $L$ :

$$\nabla L(x,y,z,\lambda) = \begin{pmatrix} \frac{\partial L}{\partial x}(x,y,z,\lambda) \\ \frac{\partial L}{\partial y}(x,y,z,\lambda) \\ \frac{\partial L}{\partial z}(x,y,z,\lambda) \\ \frac{\partial L}{\partial \lambda}(x,y,z,\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{cases} yz + \lambda(2y + 2z) = 0 \\ xz + \lambda(2x + 2z) = 0 \\ xy + \lambda(2x + 2y) = 0 \\ 2xy + 2yz + 2xz - 24 = 0. \end{cases}$$

*The only solution is the point* $(x,y,z,\lambda) = \left(2{,}2{,}2, -\frac{1}{2}\right)$. *To determine the nature of this* *critical point, let's calculate the bordered Hessian matrix* $H_2$

$$H = H(x,y,z,\lambda) = \begin{bmatrix} 0 & 2y+2z & 2x+2z & 2x+2y \\ 2y+2z & 0 & z+2\lambda & y+2\lambda \\ 2x+2z & z+2\lambda & 0 & x+2\lambda \\ 2x+2y & y+2\lambda & x+2\lambda & 0 \end{bmatrix}$$

*Since*

$$\det\left(H\left(2,2,2,\frac{-1}{2}\right)\right) = \begin{vmatrix} 0 & 8 & 8 & 8 \\ 8 & 0 & 1 & 1 \\ 8 & 1 & 0 & 1 \\ 8 & 1 & 1 & 0 \end{vmatrix} = -192 < 0,$$

*and*

$$\det\left(H_2\left(2,2,2,\frac{-1}{2}\right)\right) = \begin{vmatrix} 0 & 8 & 8 \\ 8 & 0 & 1 \\ 8 & 1 & 0 \end{vmatrix} = 128 > 0.$$

*We conclude that* $f$ *has a bound maximum at* $(2,2,2)$, *i.e., under the constraint* $S = 24$, *the parallelepiped of maximum volume is the cube of side* $2$.

### Case of a function of three variables and two constraints ( $n = 3$ and $m = 2$ )

Let $\Omega$ be a non-empty open of $\mathbb{R}^3$ and $f, g, h \in \mathcal{C}^2(\Omega)$. The aim is to find the extrema of $f$ under the constraints $g(x,y,z) = 0$ and $h(x,y,z) = 0$. We have the Lagrangian

$$L(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z) + \mu h(x,y,y) = 0.$$

Let $(a,b,c) \in \Omega$ suth that $g(a,b,c) = 0$ and $h(a,b,c) = 0$. If

$$\text{rg}\begin{bmatrix} \frac{\partial f}{\partial x}(a,b,c) & \frac{\partial f}{\partial y}(a,b,c) & \frac{\partial f}{\partial z}(a,b,c) \\ \frac{\partial h}{\partial x}(a,b,c) & \frac{\partial h}{\partial y}(a,b,c) & \frac{\partial h}{\partial z}(a,b,c) \end{bmatrix} = 2$$

then,

$$\exists(\lambda,\mu) \in \mathbb{R}^2, \nabla L(a,b,c,\lambda) = \nabla f(a,b,c) + \lambda \nabla g(a,b,c) + \mu \nabla h(a,b,c) = 0.$$

To determine the nature of the critical point $(a,b,c,\lambda,\mu)$, consider the bordered Hessian matrix

$$H = \begin{bmatrix} \frac{\partial^2 L}{\partial \lambda^2}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial \lambda \partial \mu}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial \lambda \partial x}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial \lambda \partial y}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial \lambda \partial z}(a,b,c,\lambda,\mu) \\ \frac{\partial^2 L}{\partial \mu \partial \lambda}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial \mu^2}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial \mu \partial x}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial \mu \partial y}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial \mu \partial z}(a,b,c,\lambda,\mu) \\ \frac{\partial^2 L}{\partial x \partial \lambda}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial x \partial \mu}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial x^2}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial x \partial y}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial x \partial z}(a,b,c,\lambda,\mu) \\ \frac{\partial^2 L}{\partial y \partial \lambda}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial y \partial \mu}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial y \partial x}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial y^2}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial y \partial z}(a,b,c,\lambda,\mu) \\ \frac{\partial^2 L}{\partial z \partial \lambda}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial z \partial \mu}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial z \partial x}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial z \partial y}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial z^2}(a,b,c,\lambda,\mu) \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0 & \frac{\partial g}{\partial x}(a,b,c,\lambda,\mu) & \frac{\partial g}{\partial y}(a,b,c,\lambda,\mu) & \frac{\partial g}{\partial z}(a,b,c,\lambda,\mu) \\ 0 & 0 & \frac{\partial h}{\partial x}(a,b,c,\lambda,\mu) & \frac{\partial h}{\partial y}(a,b,c,\lambda,\mu) & \frac{\partial h}{\partial z}(a,b,c,\lambda,\mu) \\ \frac{\partial g}{\partial x}(a,b,c,\lambda,\mu) & \frac{\partial h}{\partial x}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial x^2}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial x\partial y}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial x\partial z}(a,b,c,\lambda,\mu) \\ \frac{\partial g}{\partial y}(a,b,c,\lambda,\mu) & \frac{\partial h}{\partial y}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial y\partial x}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial y^2}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial y\partial z}(a,b,c,\lambda,\mu) \\ \frac{\partial g}{\partial z}(a,b,c,\lambda,\mu) & \frac{\partial h}{\partial z}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial z\partial x}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial z\partial y}(a,b,c,\lambda,\mu) & \frac{\partial^2 L}{\partial z^2}(a,b,c,\lambda,\mu) \end{bmatrix}$$

In this case, the sufficient condition for the existence of an extremum is reduced to the following statement

❶ If $\det(H) > 0$, then $f$ admits a local minimum bound at $(a,b,c)$.

❷ If $\det(H) < 0$, then $f$ has a local maximum bound at $(a,b,c)$.

**Example 6.16.** *Let us find the extrema of the function* $f(x,y,z) = 3x - y - 3z$ *under the constraints* $x + y - z = 0$ *and* $x^2 + 2z^2 = 6$. *Let us set* $g(x,y,z) = x + y - z$ *and* $h(x,y,z) = x^2 + 2z^2 - 6$. *In this case, the Lagrangian is given by*

$$L(x,y,z,\lambda,\mu) = 3x - y - 3z + \lambda(x + y - z) + \mu(x^2 + 2z^2 - 6).$$

*We are searching for the critical points of* $L$ :

$$\nabla L(x,y,z,\lambda,\mu) = \begin{pmatrix} \frac{\partial L}{\partial x}(x,y,z,\lambda,\mu) \\ \frac{\partial L}{\partial y}(x,y,z,\lambda,\mu) \\ \frac{\partial L}{\partial z}(x,y,z,\lambda,\mu) \\ \frac{\partial L}{\partial \lambda}(x,y,z,\lambda,\mu) \\ \frac{\partial L}{\partial \mu}(x,y,z,\lambda,\mu) \end{pmatrix} = \begin{pmatrix} 3 + \lambda + 2\mu x \\ -1 + \lambda \\ -3 - \lambda + 4\mu z \\ x + y - z \\ x^2 + 2z^2 - 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

*After solving the system, we find,*

$$(x,y,z,\lambda,\mu) \in \{(-2,3,1,1,1), (2,-3,-1,1,-1)\}.$$

*The bordered Hessian matrix is*

$$H = H(x,y,z,\lambda,\mu) = \begin{bmatrix} 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2x & 0 & 4z \\ 1 & 2x & 2\mu & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 4z & 0 & 0 & 4\mu \end{bmatrix}$$

*We have,*

$$\det(H(-2,3,1,1,1)) = \begin{vmatrix} 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & -4 & 0 & 4 \\ 1 & -4 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & -4 & 0 & 4 \\ 1 & -4 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & 4 \end{vmatrix} = 96 > 0,$$

*so* $f$ *has a local minimum bound at* $(-2,3,1)$. *For the second critical point, we have*

$$\det(H(2,-3,-1,1,-1)) = \begin{vmatrix} 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 4 & 0 & -4 \\ 1 & 4 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & -4 & 0 & 0 & -4 \end{vmatrix} = -96 < 0,$$

*So* $f$ *has a bound local maximum at* $(2,-3,-1)$.