CHAPTER 5

Fundamental theorems of differential calculus

In this chapter we will focus on three important theorems, the local inversion theorem, the implicit function theorem and the finite increment theorem. These theorems are fundamental in the study of functions of several variables. Let us begin by defining the notion of diffeomorphism.

5.1 Difféomorphismes

Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^1, such that $f'(a) \neq 0$, then there exists an open interval I, containing a, where f' keeps a constant sign. Thus f is strictly monotone on I and performs a bijection from I to the open interval $J = f(I)$. Moreover the reciprocal function $f^{-1} : J \to I$ is of class C^1 in J and that for all $x \in I$:

$$(f^{-1})'(f(x)) = (f'(x))^{-1} = \frac{1}{f'(x)}.$$

It is this result that we want to generalize in higher dimension, for this we need the following definitions and results. In this section E and F denote two Banach spaces.

Theorem 1 [(**Open application theorem**)] Let $f : E \to F$ be a continuous surjective linear application. Then f is an open application, that is, the direct image of any open subset of E by f is an open subset of F.

Corollary 5.1. *Let f be a linear, continuous and bijective application of E onto F. Then f^{-1} is continuous from F into E.*

Definition 5.1 ((Isomorphism of Banac spaces))**.** *A continuous linear application f from E to F, is an isomorphism from E to F if, and only if, f is bijective. (By the previous corollary, f^{-1} also belongs to $\mathcal{L}(E; F)$). When such an application exists, we say that E and F are isomorphic. The set of isomorphisms from E to F is denoted Isom(E, F). If $E = F$, we simply denote Isom(E) instead of Isom(E, E).*

Properties 5.1. *In finite dimension, any linear application $f : E \to F$ is continuous, so two isomorphic spaces are of the same dimension, and conversely two spaces of the same dimension are isomorphic.*

Example 5.1. *The continuous linear application $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (2x +$*

$y, x-y)$ *is an isomorphism. Indeed*

$$\begin{aligned}(x,y) \in \ker f &\iff f(x,y) = (0,0)\\ &\iff \begin{cases} 2x+y=0 \\ x-y=0 \end{cases} \iff (x,y) = (0,0).\end{aligned}$$

Ainsi $\ker f = \{(0,0)\}$ *et donc* f *est injective. De plus, The rank theorem therefore affirms that* $rg(f) = 2$. *Since* $Im(f) \subseteq \mathbb{R}^2$, *we necessarily have that* $Im(f) = \mathbb{R}^2$, *i.e., that* f *is surjective.*

f is injective and surjective therefore bijective hence the isomorphism.

Example 5.2. *The linear application* $f : \mathbb{R}^2 \to \mathbb{R}^4$ *defined by*

$$f(x,y) = (y, 0, x-7y, x+y).$$

Without any calculation, f *cannot be an isomorphism because the arrival space is of strictly higher dimension than the departure space.*

Theorem 2 Let E and F be two finite-dimensional normed vector spaces and $f : E \to F$ be a linear application and $A = M_{B_E,B_F}(f)$ be the matrix of f with respect to the arbitrary bases B_E and B_F of E and F respectively. Then f is an isomorphism if and only if A is invertible. Moreover, the matrix of the reciprocal f^{-1} is equal to A^{-1}, the inverse of the matrix of A. This can be written

$$M_{B_F,B_E}(f^{-1}) = (M_{B_E,B_F}(f))^{-1}.$$

Example 5.3. *Returning to Exercise and equipping* $\mathbb{R}^2$ *with the canonical basis* $\{e_1, e_2\}$. *The matrix of* f *with respect to the canonical basis is*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}.$$

We have $\det A = -3 \neq 0$ *then* A *is invertible and therefore* f *is an isomorphism.*

Let U and V be two open (non-empty) sets of $\mathbb{R}^n$.

Definition 5.2 ((Diffeomorphism)). *We say that an application* $f : U \to V$ *is a diffeomorphism from* U *to* V *if, and only if*

❶ f *is bijective.*

❷ f *is differentiable in* U.

❸ f^{-1} *is differentiable in* V.

If in addition f *and* f^{-1} *are of class* C^r, *with* $1 \leqslant r \leqslant \infty$, *we say that* f *is a* C^r*-diffeomorphism (or a diffeomorphism of class* C^r*).*

Example 5.4. *Let* $f \in L(\mathbb{R}^n)$ *be bijective. Then* f *is a* C^1*-diffeomorphism from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$.

Indeed

$$\forall a \in \mathbb{R}^n, d_a f = f.$$

Furthermore, we have

$$\forall m \geqslant 2, \forall a \in \mathbb{R}^n, d_a^m f = 0.$$

Which shows that f is a C^∞-diffeomorphism. For example, $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x + 2y, 3x - y)$ is a C^1-diffeomorphism from $\mathbb{R}^2$ onto $\mathbb{R}^2$.

Properties 5.2.

❶ *For any open U of $\mathbb{R}^n$, the identity map Id_U from U to U is a C^1-diffeomorphism from U to U.*

❷ *If $g : U \to V$ is a C^1-diffeomorphism and if $f : V \to W$ is a C^1-diffeomorphism, then $f \circ g$ is a C^1-diffeomorphism from U to W.*

❸ *If $f : U \to V$ is a C^1-diffeomorphism from U to V, then $f^{-1} : V \to U$ is a C^1-diffeomorphism from V to U.*

Proposition 5.1. *If $f : U \to V$ is a diffeomorphism (so necessarily $V = f(U)$), then its differential is at every point of U an isomorphism (from $\mathbb{R}^n$ to itself), that is,*

$$\forall x \in U, d_x f \in \mathit{Isom}(\mathbb{R}^n).$$

The differential of its reciprocal function f^{-1} is related to that of f by the formula:

$$\forall x \in U, d_{f(x)} f^{-1} = (d_x f)^{-1}.$$

In terms of Jacobian matrices, we can write

$$\forall x \in U, \det(J_f(x)) \neq 0 and J_{f^{-1}(f(x))} = (J_f(x))^{-1}.$$

Proof. By differentiating the two equalities

$$f^{-1} \circ f = \mathrm{Id}_U \quad f \circ f^{-1} = \mathrm{Id}_V.$$

Using the differentiation rule for composite functions, we have for all $x \in U$

$$(d_{f(x)} f^{-1}) \circ (d_x f) = d_{f^{-1} of} = d_x(\mathrm{Id}_U) = \mathrm{Id}(d_x f) \circ (d_{f(x)} f^{-1}) = d_x(f \circ f^{-1}) = d_x(\mathrm{Id}_V) = \mathrm{Id},$$

where Id denotes the identity application of $\mathbb{R}^n$. Thus $d_x f$ and $d_{f(x)} f^{-1}$ are isomorphisms réciproques. □

Remark 5.1. *The converse of the proposition 5.1 is false. We can have functions of class C^1 whose differential, at each point of their domain of definition, is an isomorphism, and which are not diffeomorphisms.*

See the following counter-example

Example 5.5. *The function*

$$\begin{aligned} L : \mathbb{R}^2 \setminus \{(0,0)\} &\to \mathbb{R}^2 \\ (x,y) &\mapsto f(x,y) = (x^2 - y^2, 2xy), \end{aligned}$$

is of class C^1 *and*

$$\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} : \det J_f(x,y) = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2) \neq 0.$$

But f *is not injective, because* $\texttt{forall}(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, f(x,y) = f(-x,-y)$. *Therefore* f *cannot be a diffeomorphism of* $\mathbb{R}^2 \setminus \{(0,0)\}$ *on* $\mathbb{R}^2$.

5.2 Local inversion theorem

Theorem 3 [(**Local inversion theorem**)] Let U be an open $\mathbb{R}^n$ and f be an application of U in $\mathbb{R}^n$, of class C^1. Let a be a point in U. If $d_a f$ is an isomorphism of $\mathbb{R}^n$ into itself (i.e., $\det(J_f(a)) \neq 0$). Then there exists an open neighbourhood U_a of a, and an open neighbourhood V_b of $b = f(a)$, such that the restriction of f to U_a (again denoted f) is a C^1-diffeomorphism of U_a onto V_b. Furthermore, at any point $y = f(x)$ of V_b, the differential of the inverse application f^{-1} verifies

$$\forall x \in U_a, \quad d_y f^{-1} = (d_x f)^{-1}.$$

Remark 5.2.

❶ *This theorem remains true if we replace* C^1 *by* C^r, $1 \leqslant r < \infty$.

❷ *The continuity of* $d_y f^{-1}$ *is essential, i.e., the theorem fails if we remove the assumption* $f \in C^1$. *For example, the function* $f : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} \dfrac{x}{2} + x^2 \sin\left(\dfrac{1}{x}\right) & : x \neq 0 \\ 0 & : x = 0, \end{cases}$$

is differentiable and $f'(0) = 1 \neq 0$. *However there is no open interval containing* 0 *on which* f *is invertible. By showing that in any neighborhood of* 0, *the function* f' *defined by*

$$f'(x) = \begin{cases} \dfrac{1}{2} + 2x \sin\left(\dfrac{1}{x}\right) - \cos\left(\dfrac{1}{x}\right) & : x \neq 0 \\ 1 & : x = 0, \end{cases}$$

which is not continuous at 0, *oscillates and takes the values of both signs. Therefore there is no neighborhood of 0 where* f *is monotone.*

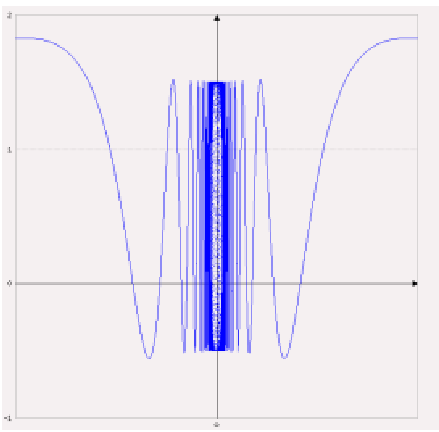


Fig. 5.1 . Curve representative of f'.

By the absurd, suppose that f *is invertible in the neighbourhood of* 0. *Let* $\varepsilon > 0$ *such that* $]-\varepsilon, \varepsilon[$ *is included in this neighbourhood. Since* f *is continuous (because it is derivable), it is strictly monotone. Take* $k \in \mathbb{N}$ *such that* $\frac{1}{2\pi k} < \varepsilon$ *and* $\frac{1}{2\pi k} < \varepsilon$ *then* $f'\left(\frac{1}{2\pi k}\right) = -\frac{1}{2} < 0$ *and* $f'\left(\frac{1}{2\pi k + \frac{1}{4k+1}}\right) > 0$ *and therefore* f' *is not monotonic on* $]-\varepsilon, \varepsilon[$ *which gives the desired contradiction.*

We admit the following theorem.

Theorem 4 [(**Global inversion theorem**)]
Let $f : U \to V$ be a function of class C^1 and injective. For f to be a C^1-diffeomorphism of U onto V, it is necessary and sufficient that

$$\begin{cases} V = f(U) \\ \forall x \in U, \det(J_f(x)) \neq 0. \end{cases}$$

Remark 5.3. *This theorem remains true if we replace* C^1 *by* C^r, $1 \leqslant r < \infty$.

Exercise 5.1. *Let the function* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by*

$$f(x, y) = (x^2 - xy, y - x)$$

and let the set

$$U = \{(x, y) \in \mathbb{R}^2, x - y > 0\}.$$

❶ *Show that* f *is a* C^∞*-diffeomorphism of* U *onto* $f(U)$.

❷ *Find* $f(U)$ *and then* $f^{-1}|_U$*, the reciprocal of the restriction of* f *to* U.

Solution.

❶ *Noting that* U *is an open* $\mathbb{R}^2$ *as the reciprocal image of the open interval* $]0, +\infty[$ *by the continuous function* $g : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $g(x, y) = x - y$.

- *The function* f *is of class* C^∞ *on* $\mathbb{R}^2$ *because each of its two components is of class* C^∞ *as multivariate polynomials.*
- *Pour tout* $(x, y) \in U$

$$\det(J_f(x, y)) = \begin{vmatrix} 2x - y & -x \\ -1 & 1 \end{vmatrix} = x - y > 0.$$

Thus the Jacobian of f *does not cancel at any point of* U.

- *Let us show that* f *is injective. Let* (x, y) *and* (x', y') *be two elements of* U *such that* $f(x, y) = f(x', y')$ *hence*

$$\begin{cases} x(x - y) = x'(x' - y') \\ y - x = y' - x'. \end{cases}$$

Substituting the second equation into the first, we obtain $x = x'$ *and* $y = y'$. *We conclude by the global inversion theorem that* f *is a* C^∞*-diffeomorphism of* U *onto* $f(U)$.

❷ *Let us determine* $f(U)$ *and* $f_{|U}^{-1}$. *Let* $(u,v) \in f(U)$, *since* f *is a bijection from* U *to* $f(U)$, *then there exists a unique* $(x,y) \in U$ *such that* $f(x,y) = (u,v)$, *which means,*

$$\begin{cases} x^2 - xy = u \\ y - x = v. \end{cases}$$

We deduce that $v < 0$ *and that*

$$\begin{cases} x = -\dfrac{u}{v} \\ y = v - \dfrac{u}{v}. \end{cases}$$

On the other hand, it is easy to show that for any $(u,v) \in \mathbb{R}^2$ *with* $v < 0$

$$f\left(-\frac{u}{v}, v - \frac{u}{v}\right) = (u,v).$$

We conclude that $f(U) = \{(u,v) \in \mathbb{R}^2, v < 0\}$ *and*

$$\begin{aligned} f_{|U}^{-1} : f(U) &\longrightarrow U \\ (u,v) &\longmapsto \left(-\frac{u}{v}, v - \frac{u}{v}\right). \end{aligned}$$

5.2.1 Conversion to polar coordinates

The application

$$\begin{aligned} \varphi : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (r,\theta) &\longmapsto (r\cos\theta, r\sin\theta), \end{aligned}$$

a pour jacobien

$$\det\left(J_\varphi(r,\theta)\right) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Note that, by virtue of the local inversion theorem, f is a local C^1-diffeomorphism on $\mathbb{R}^2 \setminus \{(0;0)\}$.

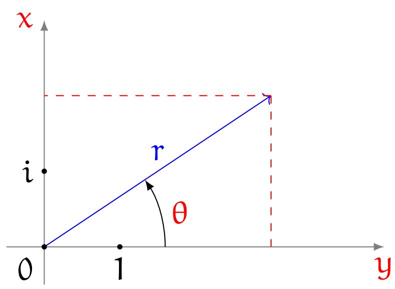


Fig. 5.2 . Polar coordinates

Moreover, we must have $r = \sqrt{x^2+y^2}$ and the point $\left(\frac{y}{r}, \frac{x}{r}\right)$ is in the unit circle minus the point $(-1,0)$. Therefore there exists a unique $\theta \in]-\pi,\pi[$ such that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Thus φ is bijective and it is obvious that it is of class C^1, so according to the global inversion theorem, f is a C^1-diffeomorphism of $U = \mathbb{R}_+^* \times]-\pi,\pi[$ (on $V = f(U) = \mathbb{R}^2 \setminus \{0\}$). and its reciprocal application is defined by

$$\varphi^{-1}(x,y) = \left(\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right)\right).$$

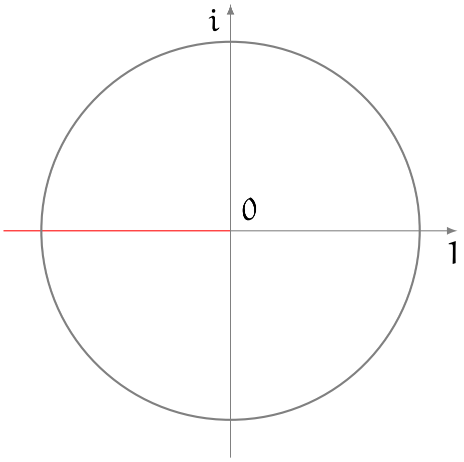


Fig. 5.3 . Unit circle centered at point $(-1,0)$.

To calculate the partial derivatives of r and θ, we use the matrix inversion of the Jacobian. Indeed

$$J_{\varphi^{-1}}(r,\theta) = \begin{pmatrix} \frac{\partial x}{\partial r} \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial r} \frac{\partial y}{\partial \theta} \end{pmatrix} = (J_\varphi(r,\theta))^{-1} = \frac{1}{r}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

This gives us

$$\begin{pmatrix} \frac{\partial r}{\partial x} \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} \frac{x}{x^2+y^2} \end{pmatrix}.$$

Now consider an application $f : (x,y) \to f(x,y)$ from $V = \mathbb{R}^2 \setminus (\mathbb{R}_- \times \{0\})$ into $\mathbb{R}$. To use the coordinates, we must replace the old coordinates (x,y) by the new coordinates (r,θ), and therefore consider the function g from $U = \mathbb{R}_+^* \times]-\pi,\pi[$ in $\mathbb{R}$ which to (r,θ) associates,

$$g(r,\theta) = f(\varphi(r,\theta)) = f(x(r,\theta), y(r,\theta)).$$

The derivation formula for compound functions gives,

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = \frac{\partial g}{\partial r}(r,\theta)\frac{\partial r}{\partial x} + \frac{\partial g}{\partial \theta}(r,\theta)\frac{\partial \theta}{\partial x}, \\ \frac{\partial f}{\partial y}(x,y) = \frac{\partial g}{\partial r}(r,\theta)\frac{\partial r}{\partial y} + \frac{\partial g}{\partial \theta}(r,\theta)\frac{\partial \theta}{\partial y}. \end{cases}$$

Donc,

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = \cos\theta\frac{\partial g}{\partial r}(r,\theta) - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}(r,\theta), \\ \frac{\partial f}{\partial y}(x,y) = \sin\theta\frac{\partial g}{\partial r}(r,\theta) + \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}(r,\theta). \end{cases} \tag{5.1}$$

Solving a partial differential equation

Consider in $U = \{(x,y) \in \mathbb{R} : x > 0\}$ the following partial differential equation

$$(E) : x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \sqrt{x^2+y^2}.$$

Let $V = \mathbb{R}_+^* \times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and consider the application $\varphi : V \to U$ defined by

$$\varphi(r,\theta) = (r\cos\theta, r\sin\theta).$$

Let f be a function of class C^1 solution of (E) on U. Consider the function g defined on V by

$$g(r,\theta) = f(\varphi(r,\theta)) = f(x(r,\theta), y(r,\theta)).$$

Rewrite the relations 5.1 in equation (E) to obtain

$$r\frac{\partial g}{\partial r}(r,\theta) = r,$$

ainsi

$$(E') : \frac{\partial g}{\partial r}(r,\theta) = 1,$$

We deduce that g is a solution of an equation of the type (E'), so $g(r,\theta) = r + \psi(\theta)$ where ψ is a function of class C^1 on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. We conclude that f is of the form,

$$f(x,y) = \sqrt{x^2+y^2} + \psi\left(\arctan\left(\frac{y}{x}\right)\right).$$

5.2.2 Transition to spherical coordinates

The application

$$\begin{aligned} \varphi : \mathbb{R}^3 &\to \mathbb{R}^3 \\ (r,\theta,\varphi) &\mapsto (\rho\sin\theta\cos\varphi, \rho\sin\theta\sin\varphi, \rho\cos\theta), \end{aligned}$$

has for Jacobian

$$\det(J_\varphi(\rho,\theta,\varphi)) = \begin{vmatrix} \sin\theta\cos\varphi & \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\theta & -\rho\sin\theta & 0 \end{vmatrix} = \rho^2\sin\theta$$

is a C^1-diffeomorphism de $U = \mathbb{R}^* \times]0,\pi[\times]-\pi,\pi[$ sur $f(U) = \mathbb{R}^3 \setminus \{(x,y,z) \in \mathbb{R}^3 : x \leqslant 0, y = 0\}$.

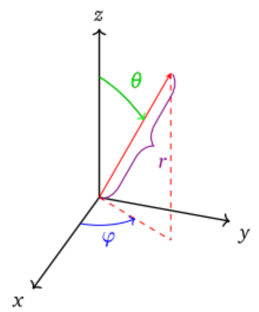


Fig. 5.4 . Spherical coordinates

5.3 Implicit Function Theorem

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x,y) = x^2+y^2-1$. Consider the curve defined by the equation $f(x,y) = 0$ is a circle of radius 1 centered at $(0,0)$. This circle is not globally the graph of a function. However, for $a \in]-1,1[$, there exists $\varepsilon > 0$ such that for $x \in]a-\varepsilon, a+\varepsilon[$, the curve $f(x,y) = 0$ coincides with the graph of $\varphi(x) = y = \sqrt{1-x^2}$. This is related to the fact that the tangent to the curve at $(a, b = \sqrt{1-a^2})$ is not vertical, that is to say that $\frac{\partial f}{\partial y}(a,b) = 2b \neq 0$. On the contrary, at $(1,0)$ where the tangent to the curve is vertical, the curve is never the graph of a function of the type $y = \varphi(x)$.

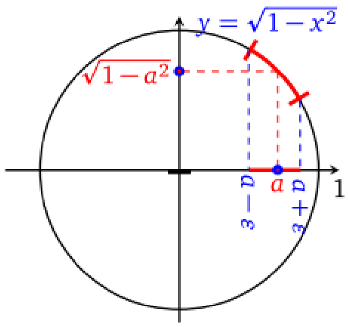


Fig. 5.5 . In the neighbourhood of $(a, \sqrt{1-a^2})$ with $a \neq 1$.

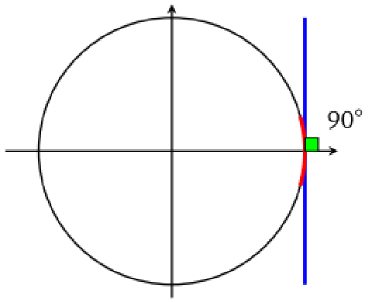


Fig. 5.6 . In the neighbourhood of $(1, 0)$.

This is a special case of the implicit function theorem which allows us to determine the points at which a curve defined by an equation of the type $f(x,y)=0$ is locally the graph of a function $y=\varphi(x)$ where the regularity of $varphi$ is linked to that of f.

Theorem 5 [Implicit Function Theorem (case $\Omega \subset \mathbb{R}^2 \to \mathbb{R}$.)] Let Ω be an open set of $\mathbb{R}^2$ and $f:\ \Omega \to \mathbb{R}$ be a function of class C^1 on Ω. Assume that there exists $(a,b) \in \Omega$ such that $f(a,b)=0$ and $\dfrac{\partial f}{\partial y}(a,b) \neq 0$. Then there exists an open interval I containing a, an open interval J containing b (with $I \times J \subset \Omega$) and a unique function $\varphi:\ I \to \mathbb{R}$ class C^1 on I such that

$$\forall (x,y) \in I \times J, f(x,y)=0 \iff y=\varphi(x).$$

In addition, the derivative of φ is given by

$$\forall x \in I, \varphi'(x) = -\frac{\frac{\partial f}{\partial x}(x,\varphi(x))}{\frac{\partial f}{\partial y}(x,\varphi(x))}. \tag{5.2}$$

Proof. This is an immediate consequence of the local inversion theorem. Let $f:\Omega \subset \mathbb{R}^2 \to \mathbb{R}$ a function C^1 such that $f(a,b)=0$ and $frac\partial f\partial y(a,b) \neq 0$. Consider the function $g:\ \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$g(x,y)=(x,f(x,y)).$$

The Jacobian of g at the point (a,b) is

$$\det(J_g(a,b)) = \begin{vmatrix} 1 & 0 \\ \dfrac{\partial f}{\partial x}(a,b) & \dfrac{\partial f}{\partial y}(a,b) \end{vmatrix} = \frac{\partial f}{\partial y}(a,b) \neq 0,$$

so by the local inversion theorem, g is a local C^1-diffeomorphism on a neighborhood U of (a,b) in Ω. Thus g is a bijection of U onto its image and the inverse application, let us call it h, is of class C^1 on the open $g(U)$. Let us write $h(s,t)=(h_1(s,t),h_2(s,t))$ the components of h. Since h is the inverse of g we have, for all $(s,t) \in g(U)$ (using the definition of g),

$$(s,t)=g(h_1(s,t),h_2(s,t))=(h_1(s,t),f(h_1(s,t),h_2(s,t))).$$

So we have

$$\begin{cases} h_1(s,t) = s \\ f(s, h_2(s,t)) = t. \end{cases}$$

The points (x, y) of U for which $f(x, y) = 0$ are the points whose image by g is of the form $(x, 0)$. These are therefore the points $h(x, 0)$ for $(x, 0) \in g(U)$, or again, according to the form of the application h, the points $(x, h_2(x, 0))$ for $(x, 0) \in g(U)$. Now $g(U)$ is an open set containing $(a, 0)$. There therefore exists $\alpha > 0$ such that, for $x \in I =]a - \alpha, a + \alpha[$, $(x, y) \in U$, the equation $f(x, y) = 0$ is equivalent to $y = h_2(x, 0)$. It is then sufficient to set $\varphi(x) = h_2(x, 0)$ to conclude.

For the form (5.2), we derive the equation $f(x, \varphi(x)) = 0$ with respect to x using the chain derivation rule

$$\frac{\partial f}{\partial x}(x, \varphi(x)) + \frac{\partial f}{\partial y}(x, \varphi(x))\varphi'(x) = 0.$$

hence (5.2). □

Remark 5.4. *In theorem 5.3, if f is of class C^r, $r \geqslant 2$, φ is also.*

If we derive the equation $\dfrac{\partial f}{\partial x}(x, \varphi(x)) + \dfrac{\partial f}{\partial y}(x, \varphi(x))\varphi'(x) = 0$ with respect to x using the chain rule, we obtain

$$\frac{\partial^2 f}{\partial x^2}(x, \varphi(x)) + \left[2\frac{\partial^2 f}{\partial x \partial y}(x, \varphi(x)) + \frac{\partial^2 f}{\partial y^2}(x, \varphi(x))\varphi'(x)\right]\varphi'(x) + \frac{\partial f}{\partial y}(x, \varphi(x))\varphi''(x) = 0$$

hence,

$$\begin{aligned}
\varphi''(x) &= \frac{\frac{\partial^2 f}{\partial x^2}(x, \varphi(x)) + \left[2\frac{\partial^2 f}{\partial x \partial y}(x, \varphi(x)) + \frac{\partial^2 f}{\partial y^2}(x, \varphi(x))\varphi'(x)\right]\varphi'(x)}{-\frac{\partial f}{\partial y}(x, \varphi(x))}. \\
&= \frac{\frac{\partial^2 f}{\partial x^2}(x, \varphi(x)) - \left[2\frac{\partial^2 f}{\partial x \partial y}(x, \varphi(x)) + \frac{\partial^2 f}{\partial y^2}(x, \varphi(x))\varphi'(x)\right]\frac{\partial f}{\partial x}(x, \varphi(x))}{\left(\frac{\partial f}{\partial y}(x, \varphi(x))\right)^2}. \\
&= \frac{\frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial f}{\partial x} - \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)^2}{\left(\frac{\partial f}{\partial y}\right)^3}.
\end{aligned}$$

Exercise 5.2. *Verify that the equation $\cos(xy) + y^3 - 2xy - y + x = 1$ defines y as a function of x on a neighborhood of $(0, 1)$ and show that this function admits a limited development to any order in the neighborhood of $x = 0$, then calculate this limited development to order 2.*

Solution. *Let's put $f(x, y) = \cos(xy) + y^3 - 2xy - y + x - 1$. We have,*

$$\frac{\partial f}{\partial y}(x, y) = -x\sin(xy) + 3y^2 - 2x - 1 \quad \Rightarrow \quad \frac{\partial f}{\partial y}(0, 1) = 2 \neq 0.$$

since in addition $f(0, 1) = 0$, the implicit function theorem applies, and there exist two open *intervals* I *and* J, *with* $0 \in I$ *and* $1 \in J$, *and a function* varphi : $I \to J$ *of class* C^1 *such that*

$$\forall (x,y) \in I \times J, f(x,y) = 0 \iff y = \varphi(x).$$

Moreover, since f *is of class* C^∞, φ *is also of class* C^∞ *on* I. *In particular, it admits limited developments to any order in* 0, *that of order* 2 *being written as*

$$\varphi(x) = \varphi(0) + \varphi'(0)x + \frac{\varphi''(0)}{2}x^2 + o(x^2).$$

We already know that varphi$(0) = 1$. *We need to calculate the second derivatives of* f *in* $(0,1)$

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= -y\sin(xy) - 2y + 1 && \Rightarrow && \frac{\partial f}{\partial x}(0,1) = -1,\\
\frac{\partial^2 f}{\partial x^2}(x,y) &= -y^2\cos(xy) && \Rightarrow && \frac{\partial^2 f}{\partial x^2}(0,1) = 0,\\
\frac{\partial^2 f}{\partial x \partial y}(x,y) &= -\sin(xy) - xy\cos(xy) - 2 && \Rightarrow && \frac{\partial^2 f}{\partial x \partial y}(0,1) = -2,\\
\frac{\partial^2 f}{\partial y^2}(x,y) &= -x^2\cos(xy) + 6y && \Rightarrow && \frac{\partial^2 f}{\partial y^2}(0,1) = 6.
\end{aligned}$$

To calculate $\varphi'(0)$ *we use the relation (5.2)*

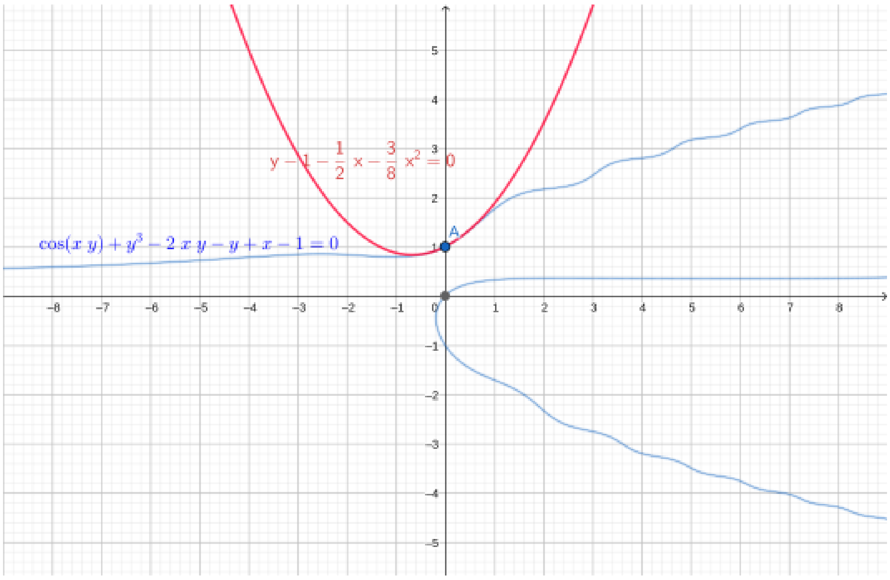
$$\varphi'(0) = \frac{\frac{\partial f}{\partial x}(0,1)}{\frac{\partial f}{\partial y}(0,1)} = \frac{-1}{2}.$$

and

$$\begin{aligned}
\varphi''(0) &= -\frac{\partial^2 f}{\partial x^2}(0,1) + \left[2\frac{\partial^2 f}{\partial x \partial y}(0,1) + \frac{\partial^2 f}{\partial y^2}(0,1)\right]\varphi'(0)^2\\
&= -1 + \left(2 \times (-2) + 6\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right)^2\\
&= -1 + (2 \times (-2) + 6\left(\frac{1}{2}\right))\left(\frac{1}{4}\right)\\
&= \frac{3}{4}
\end{aligned}$$

Thus, the limited expansion of φ *in* 0 *to order* 2 *is*

$$\varphi(x) = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + o(x^2).$$



L'étude est similaire pour les hypersurfaces dans $\mathbb{R}^{n+1}$, où on va pouvoir exprimer une variable en fonction des autres si la dérivée partielle correspondante n'est pas nulle.

Theorem 6 (**Théorème des fonctions implicites (cas $\Omega \subset \mathbb{R}^{n+1} \to \mathbb{R}$)**)
Let Ω an open $\mathbb{R}^{n+1}$ and

$$\begin{aligned} f : \Omega &\to \mathbb{R} \\ (x,y) = (x_1, \dots, x_n, y) &\mapsto f(x_1, \dots, x_n, y), \end{aligned}$$

a function of class C^r on Ω with $r \geqslant 1$. It is assumed that at $(a_1, \dots, a_n, b) \in \Omega$:

- $f(a_1, \dots, a_n, b) = 0.$
- $\dfrac{\partial f}{\partial y}(a_1, \dots, a_n, b) \neq 0.$

Then there exist neighbourhoods U of a and V of b with $U \times V \subset \Omega$ and a unique function $\varphi : U \to V$ of class C^1 on U such that

$$\forall x \in U, y = \varphi(x_1, \dots, x_n).$$

Moreover, the first partial derivatives of the varphi function at the point $x = (x_1, \dots, x_n) \in U$ are given by the expression

$$\frac{\partial \varphi}{\partial x_i}(x) = -\frac{\frac{\partial f}{\partial x_i}(x, \varphi(x))}{\frac{\partial f}{\partial y}(x, \varphi(x))} \qquad 1 \leqslant i \leqslant n. \tag{5.3}$$

Exercise 5.3. *Show that the relation $x + y + z + \sin(xyz) = 0$ defines z as a function of x and y in the neighborhood of the point $(0,0,0)$. Then calculate $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in the neighborhood of this point.*

Solution. *Let f be the function defined on $\mathbb{R}^3$ by $f(x,y,z) = 2x - y + z + \sin(xyz)$. The function*

f *is of class* C^∞, *it verifies* $f(0,0,0) = 0$ *and*

$$\frac{\partial f}{\partial z}(x,y,z) = 1 + xy\cos(xyz) \implies \frac{\partial f}{\partial z}(0,0,0) = 1 \neq 0.$$

By the implicit function theorem, there therefore exists a neighborhood U *of* $(0,0)$, *an open interval* I *containing 0 and a function* $\varphi : U \to I$ *of class* C^∞ *such that*

$$\forall (x,y,z) \in U \times I, f(x,y,z) = 0 \iff z = \varphi(x,y).$$

Moreover,

$$\frac{\partial \varphi}{\partial x}(0,0) = -\frac{\frac{\partial f}{\partial x}(0,0,0)}{\frac{\partial f}{\partial z}(0,0,0)} = -2,$$

$$\frac{\partial \varphi}{\partial y}(0,0) = -\frac{\frac{\partial f}{\partial y}(0,0,0)}{\frac{\partial f}{\partial z}(0,0,0)} = 1.$$

Theorem 7 [(**Implicit functions theorem (general case)**)]
Let Ω be an open set of $\mathbb{R}^n \times \mathbb{R}^p$ and

$$\begin{aligned} f : \Omega &\to \mathbb{R}^p \\ (x,y) = (x_1, \dots, x_n, y_1, \dots, y_p) &\mapsto f(x,y) = (f_1(x,y), \dots, f_p(x,y)), \end{aligned}$$

a function of class C^1 on Ω (where f_i, $i = 1, \dots, p$ the components of f are defined from Ω to values in $\mathbb{R}$). We assume that at $(a,b) \in \Omega$ ($a \in \mathbb{R}^n$ and $b \in \mathbb{R}^p$) $f(a,b) = 0$ and that the matrix defined by the coefficients $\left(\frac{\partial f_i}{\partial y_j}(a,b)\right)_{1 \leqslant i,j \leqslant p}$ is invertible (in other words the determinant of this matrix is non-zero). Then there exist neighborhoods U of a and V of b with $U \times V \subset \Omega$ and a unique function $\varphi : U \to V$ of class C^1 on U such that

$$\forall x \in U, y = \varphi(x).$$

Moreover, the Jacobian matrix of φ in $x = (x_1, \dots, x_n)$ is given by

$$J_\varphi(x) = -\begin{pmatrix} \frac{\partial f_1}{\partial y_1}(x, \varphi(x)) & \cdots & \frac{\partial f_1}{\partial y_p}(x, \varphi(x)) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial y_1}(x, \varphi(x)) & \cdots & \frac{\partial f_p}{\partial y_p}(x, \varphi(x)) \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x, \varphi(x)) & \cdots & \frac{\partial f_1}{\partial x_n}(x, \varphi(x)) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1}(x, \varphi(x)) & \cdots & \frac{\partial f_p}{\partial x_n}(x, \varphi(x)) \end{pmatrix}.$$

Exercise 5.4 ((Case of a linear system)). *Let* f *be a linear application of* $\mathbb{R}^n \times \mathbb{R}^p$ *in* $\mathbb{R}^p$. *The Jacobian matrix of* f *is none other than the matrix of* f, *since the differential of* f *is none other than* f. *The system*

$$f(x,y) = 0,$$

is written

$$(S) : \begin{cases} f_1(x_1, \dots, x_n, y_1, \dots, y_p) = 0 \\ f_2(x_1, \dots, x_n, y_1, \dots, y_p) = 0 \\ \vdots \\ f_p(x_1, \dots, x_n, y_1, \dots, y_p) = 0 \end{cases}$$

or again

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n + b_{11}y_1 + \ldots + b_{1p}y_p = 0 \\ a_{21}x_1 + \ldots + a_{2n}x_n + b_{21}y_1 + \ldots + b_{2p}y_p = 0 \\ \vdots \\ a_{p1}x_1 + \ldots + a_{pn}x_n + b_{p1}y_1 + \ldots + b_{pp}y_p = 0 \end{cases}$$

We note

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(x,y) & \ldots & \frac{\partial f_1}{\partial y_p}(x,y) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial y_1}(x,y) & \ldots & \frac{\partial f_p}{\partial y_p}(x,y) \end{pmatrix} = \begin{pmatrix} b_{11} & \ldots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{p1} & \ldots & b_{pp} \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x,y) & \ldots & \frac{\partial f_1}{\partial x_n}(x,y) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1}(x,y) & \ldots & \frac{\partial f_p}{\partial x_n}(x,y) \end{pmatrix} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{p1} & \ldots & a_{pn} \end{pmatrix}$$

If we assume that det $A \neq 0$, *this system* (S) *has a unique solution (in the neighbourhood of zero),*

$$y = \varphi(x) = J_\varphi(0) \cdot x = A^{-1}B \cdot x,$$

in other words

$$\begin{cases} y_1 = c_{11}x_1 + \ldots + c_{1n}x_n \\ y_2 = c_{21}x_1 + \ldots + c_{2n}x_n \\ \vdots \\ y_p = c_{p1}x_1 + \ldots + c_{pn}x_n \end{cases}$$

where $C = (c_{ij})_{1 \leqslant i \leqslant p}^{1 \leqslant j \leqslant n} = A^{-1}B$. *This is a well-known linear algebra result. Since the linear system* (S) *is of rank* p, *the principal unknowns* $y_1, \ldots, y_p$ *can be calculated as functions of the others.*

Exercise 5.5. *We consider the system of three equations with four unknowns*

$$(S): \quad \begin{cases} x + y + z + t = 0 \\ x^2 + y^2 + z^2 + t^2 = 2 \\ x^3 + y^3 + z^3 + t^3 = 0 \end{cases}$$

whose solutions are obtained in $\mathbb{R}^4$.

❶ *Verify that* $(0, -1, 1, 0)$ *is a solution.*

❷ *Give a precise statement of the fact that, in the neighborhood of this point, the solutions are of the form* $(x(t), y(t), z(t), t)$.

❸ *What is then the derivative at 0 of the application*

$$\begin{aligned} \varphi : \mathbb{R} &\longrightarrow \mathbb{R}^3 \\ t &\longmapsto (x(t), y(t), z(t)) \end{aligned}$$

Solution. *Let's put*

$$\begin{aligned} f : \mathbb{R}^4 &\longrightarrow \mathbb{R}^3 \\ (x,y,z,t) &\longmapsto \Big(\underbrace{x+y+z+t}_{f_1(x,y,z,t)}, \underbrace{x^2+y^2+z^2+t^2-2}_{f_2(x,y,z,t)}, \underbrace{x^3+y^3+z^3+t^3}_{f_3(x,y,z,t)}\Big). \end{aligned}$$

The function f *is of class* C^∞.

❶ *We have* $f(0,-1,1,0) = (0,0,0)$, *so* $(0,-1,1,0)$ *is a solution of* (S).

❷ *Let's calculate the matrix* A.

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x}(0,-1,1,0) & \frac{\partial f_1}{\partial y}(0,-1,1,0) & \frac{\partial f_1}{\partial z}(0,-1,1,0) \\ \frac{\partial f_2}{\partial x}(0,-1,1,0) & \frac{\partial f_2}{\partial y}(0,-1,1,0) & \frac{\partial f_2}{\partial z}(0,-1,1,0) \\ \frac{\partial f_3}{\partial x}(0,-1,1,0) & \frac{\partial f_3}{\partial y}(0,-1,1,0) & \frac{\partial f_3}{\partial z}(0,-1,1,0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 3 & 3 \end{pmatrix}$$

We have $\det A = -12 \neq 0$ *so the implicit function theorem applies. There exists a neighbourhood* U *of* $(0,-1,1)$ *in* $\mathbb{R}^3$, *an open interval* I *containing 0 and a function* $\varphi : I \to U$ *of class* C^∞ *such that,*

$$\forall (x,y,z,t) \in U \times I,\ f(x,y,z,t) = 0 \iff (x,y,z) = \varphi(t).$$

❸ *Let's calculate the derivative of* φ *at* 0. *By virtue of the relation (5.3), we have,*

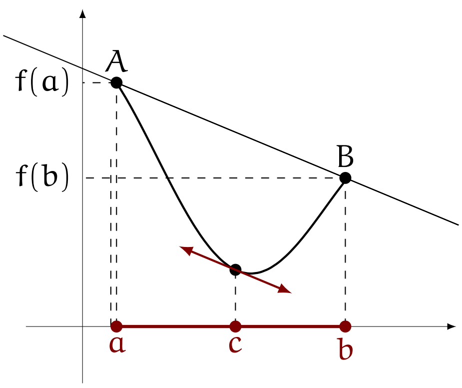
$$\varphi'(0) = -A^{-1} \begin{pmatrix} \frac{\partial f_1}{\partial t}(0,-1,1,0) \\ \frac{\partial f_2}{\partial t}(0,-1,1,0) \\ \frac{\partial f_3}{\partial t}(0,-1,1,0) \end{pmatrix} = -\begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

5.4 The theorem of finite increments

For a real function with real values, recall the result known as the mean theorem Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ (a and b real such that $a < b$) a function continuous on the closed interval $[a,b]$ and derivable on the open interval $]a,b[$. Then there exists a real c in $]a,b[$ satisfying

$$f(b) - f(a) = f'(c)(b-a).$$

Graphically, this means that there is a point c where the tangent to the curve representing f at the point $(c, f(c))$ is parallel to the chord passing through $(a, f(a))$ and $(b, f(b))$.



We begin by recalling the definition of a segment and a convex set.

Definition 5.3 ((Segment)). *Let* a *and* b *be two points of* $\mathbb{R}^n$. *The closed segment (resp. open segment) of extremities* a *and* b, *denoted* $[a,b]$ *(resp.* $]a,b[$*), is the part of* $\mathbb{R}^n$ *defined by*

$$[a;b]\,(\textit{resp. }]a;b[) = \{a + (1-\lambda)b,\ 0 \leqslant \lambda \leqslant 1\,(\textit{resp. } 0 < \lambda < 1)\}.$$

Definition 5.4 (Convex set). *A set* C *is said to be convex when, for all* x *and* y *of* C*, the closed segment* $[a,b]$ *is any integer contained in* C*, which means,*

$$\forall x, y \in C,\ \forall \lambda \in [0,1],\ \lambda x + (1-\lambda) y \in C.$$

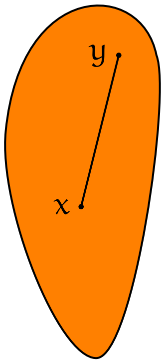


Fig. 5.7 . Ensemble convexe

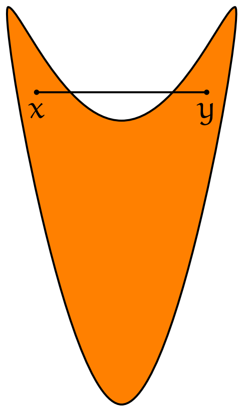


Fig. 5.8 . Non-convex set

Example 5.6.

- *Any ball (open or closed) of* $\mathbb{R}^n$ *is convex.*
- *Open or closed half-spaces of* $\mathbb{R}^n$ *are convex.*

The theorem of finite increments can be easily generalized to the case of a real function of several real variables.

Theorem 8 [(**Equality of finite increments**)] Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a function defined on an open set Ω of $\mathbb{R}^n$. Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be two points of Ω such that the segment $[a,b]$ is contained in Ω. If f is continuous on $[a,b]$, and if f is differentiable on $]a,b[$, then there exists a point c of $]a,b[$ such that

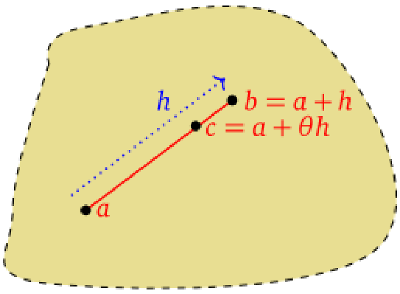
$$f(b) - f(a) = \langle \nabla f(c), b - a \rangle = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(c)(b_i - a_i). \tag{5.4}$$

For a reformulation, we note $b - a = h = (h_1, \dots, h_n) \in \mathbb{R}^n$, then,

$$\exists \theta \in]0,1[: c = a + \theta h.$$

Thus, the equality (5.5) becomes,

$$f(a+h) - f(a) = \langle \nabla f(a+\theta h), h \rangle = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a_i + \theta h_i).$$



Proof. Consider the real function of the real variable

$$\varphi : [0,1] \to \mathbb{R}$$
$$t \mapsto \varphi(t) = f(a + th).$$

which is continuous on $[0,1]$ and differentiable on $]0,1[$ (as composed of two differentiable applications), so according to the theorem of finite increments in one variable, there exists $\theta \in]0,1[$ such that

$$\varphi(1) - \varphi(0) = \varphi'(\theta).$$

Or

$$f(a+h) - f(a) = \frac{\partial f}{\partial h}(a+\theta h).$$

□

The equality of finite increments cannot be extended to the case of vector-valued applications. Consider for example the function f of $\mathbb{R}$ with values in $\mathbb{R}^2$ defined by $f(t) = (\cos t, \sin t)$. It is verified $f(2\pi) - f(0) = (0,0)$ but $f'(t) = (-\sin t, \cos t)$ does not vanish on the interval $]0, 2\pi[$. On the other hand, we have the following results, called inequalities of finite increments.

Theorem 9 [(**Inequality of finite increments**)] Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a function defined on an open Ω of $\mathbb{R}^n$. Let a and b be two points of Ω such that the segment $[a,b]$ is contained in Ω. If f is continuous on $[a,b]$, and if f is differentiable on $]a,b[$, then there exists
a point c of $]a,b[$ such that

$$\|f(b) - f(a)\|_{\mathbb{R}} \leqslant \|J_f(c)\|_{\mathcal{M}_{\mathbb{R}(\mathbb{R})}} \|b-a\|_{\mathbb{R}}. \tag{5.5}$$

where $J_f(c)$ is the Jacobian matrix of f at point c.

Proof. We set $\varphi(t) = f(a+t(b-a))$. We have $\varphi(1) = f(b)$, $\varphi(0) = f(a)$, and φ is continuous on $[0,1]$ (as composed of two continuous applications). Moreover, φ is differentiable at all $t \in]0,1[$ (as composed of two differentiable applications), and $\varphi'(t) = df_{a+t(b-a)}(b-a) = J_f(a+t(b-a))(b-a)$. Applying the inequality of finite increments in one variable to φ, we obtain

$$\|f(b) - f(a)\|_{\mathbb{R}} \leqslant \sup_{t\in[0;1]} \|\varphi'(t)\| \leqslant \sup_{c\in[0;1]} \|J_f(c)\|_{\mathbb{R}} \|b-a\|_{\mathbb{R}}.$$

□

Exercise 5.6. *Let* $f(x,y) = \sin\left(\dfrac{x}{y+1}\right)$. *We will use the inequality of finite increments to upper bound* $f(x,y)$ *on* $\Omega = [-1,1] \times \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$. *We have* f *is of class* $\mathcal{C}^1$ *on the convex* Ω. *Let* $a = (0,0)$ *and* $b = (x,y) \in \Omega$ *(note that* $[a,b] \subset \Omega$*). Let us calculate the partial derivatives,*

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{y+1}\cos\left(\frac{x}{y+1}\right), \quad \frac{\partial f}{\partial y}(x,y) = \frac{-x}{(y+1)^2}\cos\left(\frac{x}{y+1}\right).$$

We upper the cosine by 1, *then for all* $(x,y) \in \Omega$:

$$\left|\frac{\partial f}{\partial x}(x,y)\right| \leqslant \frac{1}{-1+1} = 2, \quad \left|\frac{\partial f}{\partial y}(x,y)\right| \leqslant \frac{1}{\left(-\frac{1}{2}+1\right)^2} = 4.$$

Ainsi pour tout $(x,y) \in \Omega$:

$$\|J_f(x,y)\| = \sqrt{\left|\frac{\partial f}{\partial x}(x,y)\right|^2 + \left|\frac{\partial f}{\partial y}(x,y)\right|^2} \leqslant \sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5}.$$

By applying the inequality of finite increments 5.5 to obtain for all $(x,y) \in \Omega$:

$$|f(x,y)| \leqslant 2\sqrt{5}\sqrt{x^2+y^2}.$$

Corollaire 5.2 [**(Lipschitzian functions)**] Let Ω be a convex open set of $\mathbb{R}^n$, $f:\ \Omega \to \mathbb{R}^p$ be a differentiable function in Ω and $M \in \mathbb{R}$ such that

$$\forall x \in \Omega, \|J_f(x)\| \leqslant M.$$

Then f is M-Lipschitzian, hat is to say

$$\forall x, y \in \Omega, \|f(x) - f(y)\| \leqslant M\|x-y\|.$$

Example 5.7. *We consider the function* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$f(x,y) = \left(\frac{\sin(x+y)}{2}, \frac{\cos(x-y)}{2}\right).$$

The Jacobian matrix of f *at the point* $(x,y) \in \mathbb{R}^2$ *is*

$$J_f(x,y) = \begin{pmatrix} \frac{\cos(x+y)}{2} & \frac{\cos(x+y)}{2} \\ -\frac{\sin(x-y)}{2} & \frac{\sin(x-y)}{2} \end{pmatrix},$$

Then we have

$$J_f(x,y)\begin{pmatrix} h \\ k \end{pmatrix} = \left(\frac{\cos(x+y)}{2}(h+k) \quad \frac{\sin(x-y)}{2}(-h+k)\right),$$

which gives when $\mathbb{R}^2$ *is equipped with the norm* $\|\cdot\|_2$

$$\begin{aligned} \left\| J_f(x,y)\begin{pmatrix} h \\ k \end{pmatrix} \right\|_2 &= \sqrt{\frac{\cos^2(x+y)}{4}(h+k)^2 + \frac{\sin^2(x-y)}{4}(-h+k)^2} \\ &\leqslant \frac{1}{2}\sqrt{(h+k)^2 + (-h+k)^2} \\ &\leqslant \frac{\sqrt{2}}{2}\sqrt{h^2+k^2} \end{aligned}$$

Thus,

$$\left\| J_f(x,y)\begin{pmatrix} h \\ k \end{pmatrix} \right\|_2 \leqslant \frac{\sqrt{2}}{2}\|(h,k)\|_2.$$

from which we finally have for all $(x,y) \in \mathbb{R}^2$,

$$\|J_f(x,y)\| \leqslant \frac{\sqrt{2}}{2}.$$

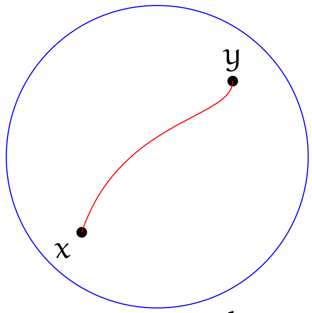
Hence we conclude that f *is contracting (i.e., M-lipschitzian with* $0 \leqslant M < 1$*). Note that the choice of norm is important because, for example,* $\|J_f(\frac{\pi}{4}, -\frac{\pi}{4})\| = 1$ *if we choose the norm* $\|\mathrm{cdot}|_1$ *or the norm* $\|\cdot\|_\infty$.

Corollary 5.2. *Let Ω be a connected open of $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}$ a differentiable function in Ω. For f to be constant on Ω, it is necessary and sufficient that*

$$\forall x \in \Omega,\ df_x = 0.$$

Proof.

If f is constant on Ω, it is clear that then its differential is zero on Ω. Conversely, suppose that the differential of f is zero on Ω. Thus, all the partial derivatives of f are zero on Ω.



If $(x, y) \in \Omega^2$, Ω being connected, there exists a path $\gamma : [0, 1] \to \mathbb{R}^n$ of class C^1 such that

$$\gamma(0) = x,\ \gamma(1) = y \text{ and } \gamma([0,1]) \subset \Omega.$$

We define the function $g : [0,1] \to \mathbb{R}$ by $g(t) = f \circ \varphi(t)$ which is derivable on $[0,1]$ and we have,

$$\forall t \in [0,1],\ g'(t) = \sum_{k=1}^{n} \gamma_k'(t) \frac{\partial f}{\partial x_k}(\gamma(t)) = 0,$$

where $\gamma_1, \ldots, \gamma_n$ indicate the components of γ. So g is constant on $[0,1]$. Consequently,

$$f(x) = g(0) = g(1) = f(y),$$

which shows that f is constant on Ω. □