CHAPTER 3

Differentiable functions

The concept of differentiability is the generalisation to functions of several variables of the concept of derivability of a real variable. We begin by showing how to define and calculate a partial derivative from the concept of derivative of a function of a single variable.

3.1 First-order partial derivatives

Let f be a function defined on a neighbourhood Ω of the point $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ with values in $\mathbb{R}^p$ with $(n,p) \in (\mathbb{N}^*)^2$. Let $f_{i,a} : t \mapsto f(a_1, \dots, t, \dots, a_n)$ be the i^{textth} partial function of f in a. It is assumed that $f_{i,a}$ is derivable in a_i. Then the derivative of $f_{i,a}$ in a_i

$$f'_{i,a}(a_i) = \lim_{h \to 0} \frac{f(a_1, \dots, a_i + h, \dots, a_n) - f(a_1, \dots, a_n)}{h},$$

we will call the i^{th} partial derivative of f at a and which we will note $\dfrac{\partial f}{\partial x_i}(a)$.

Definition 3.1. *Let $i \in \{1, \dots, n\}$. We say that the function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^p$ admits a partial derivative with respect to its i^{th} variable at point a if the ith partial function of f at a is differentiable at point a_i. In this case, $f'_{i,a}(a_i)$ is called the partial derivative of f with respect to its i^{th} variable at a and is denoted by $\dfrac{\partial f}{\partial x_i}(a) \in \mathbb{R}^p$:*

$$\frac{\partial f}{\partial x_i}(a) = f'_{i,a}(a_i) = \lim_{h \to 0} \frac{f(a_1, \dots, a_i + h, \dots, a_n) - f(a_1, \dots, a_n)}{h}$$

Example 3.1. *Let us calculate the partial derivatives of the function f defined by $f(x,y) = \ln(x + y^2)$ at the point $a = (1,0)$.*

- $\dfrac{\partial f}{\partial x}(1,0) = \lim\limits_{h \to 0} \dfrac{f(1+h,0) - f(1,0)}{h} = \lim\limits_{h \to 0} \dfrac{\ln(1+h) - \ln(1)}{h} = \lim\limits_{h \to 0} \dfrac{\ln(1+h)}{h} = 1.$
- $\dfrac{\partial f}{\partial y}(1,0) = \lim\limits_{k \to 0} \dfrac{f(1,k) - f(1,0)}{k} = \lim\limits_{k \to 0} \dfrac{\ln(1+k^2) - \ln(1)}{k} = \lim\limits_{k \to 0} \dfrac{\ln(1+k^2)}{k} = 0.$

Remark 3.1. *The partial derivative of the function*

$$\begin{aligned} f : \Omega \subseteq \mathbb{R}^n &\to \mathbb{R}^p \\ (x_1, \dots, x_n) &\mapsto f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_p(x_1, \dots, x_n)), \end{aligned}$$

with respect to the i^{th} *variable at point* a *is given by*

$$\frac{\partial f}{\partial x_i}(a) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_i}(a) \\ \vdots \\ \dfrac{\partial f_p}{\partial x_i}(a) \end{pmatrix}.$$

Remark 3.2. *In practice, to find a partial derivative of* f*, all you have to do is derive the expression of* f *with respect to the variable under consideration, the others being taken as constant.*

Example 3.2. *Consider the function* $f(x,y,z) = xy^2 + y^3 + xz$*. Then the partial derivatives at any point* $a = (x,y,z)$ *are*

$$\frac{\partial f}{\partial x}(x,y,z) = y^2 + z, \quad \frac{\partial f}{\partial y}(x,y,z) = 2xy + 3y^2, \quad \frac{\partial f}{\partial z}(x,y,z) = x.$$

Example 3.3. *The partial derivatives of the function*

$$\begin{aligned} f:\ \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x,y,z) &\longmapsto f(x,y,z) = (xy^2 + 3x, yz^2), \end{aligned}$$

at the point $a = (x,y,z)$ *are given by*

$$\frac{\partial f}{\partial x}(x,y,z) = \begin{pmatrix} y^2+3 \\ 0 \end{pmatrix}, \quad \frac{\partial f}{\partial y}(x,y,z) = \begin{pmatrix} 2xy \\ xz^2 \end{pmatrix}, \quad \frac{\partial f}{\partial z}(x,y,z) = \begin{pmatrix} 0 \\ 2yz \end{pmatrix}.$$

Remark 3.3. *A function can have partial derivatives at a point without being continuous at that point.*

Example 3.4. *The function defined on* $\mathbb{R}^2$ *by*

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

has partial derivatives at the point $(0,0)$*, because*

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0,$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0.$$

But it is not continuous at the point $(0,0)$ *because*

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{r\to 0} \frac{r^2 \cos\theta \sin\theta}{r^2} = \cos\theta \sin\theta.$$

Remark 3.4. *A function can be continuous at a point without having partial derivatives at that point.*

Example 3.5. *Consider the function defined on* $\mathbb{R}^2$ *by*

$$f(x,y) = |x+y|.$$

It is clear that f *is continuous on* $\mathbb{R}^2$ *as a composite of continuous functions. In particular, it is continuous at the point* $(0,0)$*. Let us now study the existence of partial derivatives in* $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

This limit does not exist and therefore $\frac{\partial f}{\partial x}(0,0)$ *does not exist. Similarly, we can check that* $\frac{\partial f}{\partial y}(0,0)$ *does not exist.*

Properties 3.1. *We have the same derivation rules as for functions of one variable. Let* Ω *be an open of* $\mathbb{R}^n$, $a \in \Omega$ *and* $i \in \{1,\dots,n\}$.

- *Let* $f,g:\ \Omega \subset \mathbb{R}^n \to \mathbb{R}^p$ *two functions admitting an* i^{th} *dérivée partielle en* a*. Alors pour tout (* $\mathbb{R}^2$, *the function* $\alpha f + \beta g$*admits a* i^{th} *partial derivative in* a *and we have,*

$$\frac{\partial(\alpha f+\beta g)}{\partial x_i}(a) = \alpha\frac{\partial f}{\partial x_i}(a) + \beta\frac{\partial g}{\partial x_i}(a).$$

- *Let* $f,g:\ \Omega \subset \mathbb{R}^n \to \mathbb{R}$ *be two functions with an* i^{th} *partial derivative in* a*. Then,* $f\cdot g$ *and* $\frac{f}{g}$ (if $g(a)\neq 0$) *have an* i^{th} *partial derivative at* a *and we have*

  - $$\frac{\partial(f\cdot g)}{\partial x_i}(a) = \frac{\partial f}{\partial x_i}(a)g(a) + f(a)\frac{\partial g}{\partial x_i}(a),$$
  - $$\frac{\partial\left(\frac{f}{g}\right)}{\partial x_i}(a) = \frac{\frac{\partial f}{\partial x_i}(a)g(a) - f(a)\frac{\partial g}{\partial x_i}(a)}{(g(a))^2}.$$

- *Let* $f:\ \Omega \subset \mathbb{R}^n \to \mathbb{R}$ *and* $g:\ f(\Omega) \subset \mathbb{R} \to \mathbb{R}$ *is a function of a real variable derivable at the point* $f(a)$*. Then* $g\circ f$ *has a partial* i^{th} *derivative at* a *and we have*

$$\frac{\partial(g\circ f)}{\partial x_i}(a) = g'(f(a))\frac{\partial f}{\partial x_i}(a).$$

3.1.1 Geometric representation

Let S be the surface with equation $z = f(x,y)$ and the point $M_0(x_0,y_0,z_0)$. The section of the surface S through the plane with equation $y = y_0$ is a curve C_{x_0}. In this plane, C_{x_0} is the graph of $z = f_{1,x_0}(x) = f(x,y_0)$. The first partial function of f in $a = (x_0,y_0)$.

We know that $f'_{1,x_0}(x_0) = \frac{\partial f}{\partial x}(x_0,y_0)$ is the slope of the tangent to the curve C_{x_0} at M_0, as can be seen in the figure opposite.

3.2 Partial derivative functions

Definition 3.2. *Let* $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^p$ *a function defined on a non-empty open* Ω *and let* $i \in \{1, \dots, n\}$. f *has an* i^{th} *partial derivative on* Ω *if and only if* f *has an* i^{th} *partial derivative at each point* a *on* Ω. *In this case, we define the partial derivative function of* f *with respect to* x_i *on* Ω *as follows*

$$\frac{\partial f}{\partial x_i} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^p$$
$$(x_1, \dots, x_n) \mapsto \frac{\partial f}{\partial x_i}(x_1, \dots, x_n).$$

Example 3.6. *Consider the function defined on* $\mathbb{R}^2$ *by*

$$f(x, y) = \ln(x^2 + y^2 + 1).$$

The partial derivatives of f *exist at any point* $(x, y) \in \mathbb{R}^2$ *and we have,*

$$\frac{\partial f}{\partial x}(x, y) = \frac{2x}{x^2 + y^2 + 1} \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = \frac{2y}{x^2 + y^2 + 1}.$$

Example 3.7. *Consider the function defined on* $\mathbb{R}^2$ *by*

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

Partial derivatives in $(0,0)$. *We have already seen in the example 3.4 that* f *admits partial derivatives at the point*$(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = 0.$$

Partial derivatives in $\mathbb{R}^2 \setminus \{(0,0)\}$. f *has on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *a partial derivative with respect to* x *as a quotient of functions with a partial derivative with respect to their variable* x *whose denominator does not cancel on* $\mathbb{R}^2 \setminus \{(0,0)\}$. *Moreover, for any* $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ *we have,*

$$\frac{\partial f}{\partial x}(x,y) = \frac{y(x^2 + y^2) - 2x(xy)}{(x^2 + y^2)^2} = \frac{y^3 - x^2 y}{(x^2 + y^2)^2}.$$

Similarly, for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ *we have*

$$\frac{\partial f}{\partial y}(x,y) = \frac{x^3 - xy^2}{(x^2 + y^2)^2}.$$

In summary, f *admits on* $\mathbb{R}^2$ *partial derivatives with respect to each of its two variables and we have,*

$$\forall (x,y) \in \mathbb{R}^2 : \frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{y^3 - x^2 y}{(x^2 + y^2)^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

$$\forall (x,y) \in \mathbb{R}^2 : \frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x^3 - xy^2}{(x^2 + y^2)^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

3.3 C^1 class functions

Definition 3.3. *Let $\Omega \subseteq \mathbb{R}^n$ be an open and $f : \Omega \to \mathbb{R}^p$. We say that f is of class C^1 (continuously differentiable) on Ω if all the partial derivatives of f exist and are continuous on Ω. We denote $C^1(\Omega, \mathbb{R}^p)$ the set of functions of class C^1 on Ω with values in $\mathbb{R}^p$, or more simply, $C^1(\Omega)$ when $p = 1$.*

Example 3.8. *The function $f : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f(x,y) = xy^2 + 3x.$$

is of class C^1 on $\mathbb{R}^2$ because

$$\frac{\partial f}{\partial x}(x,y) = y^2 + 3 and \frac{\partial f}{\partial y}(x,y) = 2xy,$$

are continuous on $\mathbb{R}^2$.

Theorem 1 Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^p$. If f is of class C^1 on Ω then it is continuous on Ω and we have the inclusion $C^0(\Omega, \mathbb{R}^p) \subset C^1(\Omega, \mathbb{R}^p)$.

Example 3.9. *Consider the function defined on $\mathbb{R}^2$ by*

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

We have already seen in the example **??** *that f is not continuous $(0,0)$ so it cannot be of class C^1 sur $\mathbb{R}^2$.*

Exercise 3.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ the function defined by*

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

1. *Is f continuous on $\mathbb{R}^2$?*
2. *Study the existence of partial derivatives of f on $\mathbb{R}^2$ and determine them.*
3. *Is f of class $C^1(\mathbb{R}^2)$?*

Solution.

1. *The function f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ as a quotient of continuous functions on $\mathbb{R}^2 \setminus \{(0,0)\}$ whose denominator does not vanish on $\mathbb{R}^2 \setminus \{(0,0)\}$. For all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$*

$$|f(x,y)| = \left|\frac{xy^3}{x^2+y^2}\right| = \frac{|xy||y^2|}{x^2+y^2} \leqslant \frac{|xy|(x^2+y^2)}{x^2+y^2} = |xy|.$$

Since $\lim\limits_{(x,y)\to(0,0)} |xy| = 0$, we deduce that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$. This shows that

f *is continuous in* $(0,0)$. *In summary,* f *is continuous on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *and in* $(0,0)$ *and therefore,* f *is continuous on* $\mathbb{R}^2$.

2. *Partial derivatives in* $\mathbb{R}^2 \setminus \{(0,0)\}$. f *has on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *a partial derivative with respect to* x *as a quotient of functions having a partial derivative with respect to their variable where the denominator does not cancel on* $\mathbb{R}^2 \setminus \{(0,0)\}$. *Moreover, for all* $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ *we have*
$$\frac{\partial f}{\partial x}(x,y) = \frac{y^3(x^2+y^2) - 2x(xy^3)}{(x^2+y^2)^2} = \frac{y^5 - x^2y^3}{(x^2+y^2)^2}.$$
Similarly, for any $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$
$$\frac{\partial f}{\partial y}(x,y) = \frac{xy^2(x^2+y^2) - 2y(xy^3)}{(x^2+y^2)^2} = \frac{xy^4 + 3xy^3}{(x^2+y^2)^2}.$$
Partial derivatives in $(0,0)$. *We have*
$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$$

- $\dfrac{\partial f}{\partial y}(0,0) = \lim_{k \to 0} \dfrac{f(0,k) - f(0,0)}{k} = 0.$

To sum up, f *has partial derivatives on* $\mathbb{R}^2$ *with respect to each of its two variables and we have,*
$$\forall (x,y) \in \mathbb{R}^2 : \frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{y^5 - x^2y^3}{(x^2+y^2)^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$
$$\forall (x,y) \in \mathbb{R}^2 : \frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{xy^4 + 3xy^3y^2}{(x^2+y^2)^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

3. *The partial derivative functions* $\dfrac{\partial f}{\partial x}$ *and* $\dfrac{\partial f}{\partial y}$ *are continuous on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *as quotients of continuous functions on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *whose denominator does not cancel on* $\mathbb{R}^2 \setminus \{(0,0)\}$. *Furthermore, we have*
$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = \lim_{(x,y)\to(0,0)} \frac{y^5 - x^2y^3}{(x^2+y^2)^2} = \lim_{r \to 0} \frac{r^5(\sin^5\theta - \cos^2\theta \sin^3\theta)}{r^4} = 0 = \frac{\partial f}{\partial x}(0,0).$$
$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^4 + 3xy^3y^2}{(x^2+y^2)^2} = \lim_{r \to 0} \frac{r^5(\cos\theta \sin^4\theta + 3\cos^3\theta \sin^2\theta)}{r^4} = 0 = \frac{\partial f}{\partial y}(0$$
This shows that the partial derivatives $\dfrac{\partial f}{\partial x}$ *and* $\dfrac{\partial f}{\partial y}$ *are continuous in* $(0,0)$ *and therefore on* $\mathbb{R}^2$. *The function* f *is therefore of class* $C^1(\mathbb{R}^2)$.

Exercise 3.2. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be the function defined by*
$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

1. *Is* f *continuous on* $\mathbb{R}^2$?

2. *Study the existence of partial derivatives of* f *on* $\mathbb{R}^2$ *and determine them.*

3. *Is* f *of class* $C^1(\mathbb{R}^2)$?

Solution.

1. *The function* f *is continuous on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *as a quotient of continuous functions on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *whose denominator does not vanish on* $\mathbb{R}^2 \setminus \{(0,0)\}$*. Moreover*

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2} = 0.$$

This shows that f *is continuous at* $(0,0)$*. In summary,* f *is continuous on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *and at* $(0,0)$ *and therefore,* f *is continuous on* $\mathbb{R}^2$.

2. ***Partial derivatives in*** $\mathbb{R}^2 \setminus \{(0,0)\}$ **:** f *admits on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *a partial derivative with respect to* x *as a quotient of functions admitting a partial derivative with respect to their variable* x *whose denominator does not vanish on* $\mathbb{R}^2 \setminus \{(0,0)\}$*. Moreover, for all* $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$*, we have,*

$$\frac{\partial f}{\partial x}(x,y) = \frac{3x^2(x^2+y^2) - 2x(x^3-y^3)}{(x^2+y^2)^2} = \frac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2}.$$

Similarly, for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\frac{\partial f}{\partial y}(x,y) = \frac{y^4+3x^2y^2+2xy^3}{(x^2+y^2)^2}.$$

Partial derivatives at $(0,0)$. *We have,*

- $\displaystyle \frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h^3}{h} = 1.$
- $\displaystyle \frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{-k^3}{k} = -1.$

In summary, f *admits on* $\mathbb{R}^2$ *partial derivatives with respect to each of its two variables and we have,*

$$\forall (x,y) \in \mathbb{R}^2 : \frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2} & : (x,y) \neq (0,0) \\ 1 & : (x,y) = (0,0) \end{cases}$$

$$\forall (x,y) \in \mathbb{R}^2 : \frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{y^4+3x^2y^2+2xy^3}{(x^2+y^2)^2} & : (x,y) \neq (0,0) \\ -1 & : (x,y) = (0,0) \end{cases}$$

3. ***Study on*** $\mathbb{R}^2 \setminus \{(0,0)\}$. *The partial derivative functions* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *are continuous on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *as quotients of continuous functions on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *whose denominator does not vanish on* $\mathbb{R}^2 \setminus \{(0,0)\}$*. Then* f *is of class* $C^1(\mathbb{R}^2 \setminus \{(0,0)\})$.

Study at $(0,0)$. *the partial derivatives are continuous at* $(0,0)$ *if and only if*

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial x}(0,0) \quad \text{and} \quad \lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}(x,y) = \frac{\partial f}{\partial y}(0,0).$$

Since $\displaystyle \lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = 1 \neq \frac{\partial f}{\partial x}(0,0)$ *we conclude that* $\frac{\partial f}{\partial x}$ *is not continuous at* $(0,0)$*. Then the function* f *is not of class* $C^1(\mathbb{R}^2)$.

3.4 Higher order partial derivatives

Definition 3.4. *Let $i \in \{1, \dots, n\}$, Ω be a non-empty open set of $\mathbb{R}^n$, $a \in \Omega$ and $f : \Omega \to \mathbb{R}^p$ be a function admitting on Ω a partial derivative $\frac{\partial f}{\partial x_i}$. If the function $\frac{\partial f}{\partial x_i} : \Omega \to \mathbb{R}^p$ itself has a partial derivative with respect to x_j at a, $j \in \{1, \dots, n\}$, will say that is twice differentiable with respect to x_i and x_j at a. We denote the second partial derivative (or of order 2)*

Remark 3.5. *By recurrence, we define the partial derivatives of order k of f at the point a as being the partial derivatives of the partial derivative functions of order $k-1$. The partial derivative of order k of a function of variables $x_1, x_2, \dots, x_n$ is obtained by deriving k_1 times with respect to x_1, ..., k_n times with respect to x_n where $(k_1, \dots, k_n) \in \mathbb{N}^n$ with $k_1 + \dots + k_n = k$ and is denoted by*

$$\frac{\partial^k f}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}}.$$

3.4.1 Functions of the class C^k

Definition 3.5. *Let $\Omega \subset \mathbb{R}^n$ be an open and $f : \Omega \to \mathbb{R}^p$.*

- *We say that f is of class C^k on Ω if all partial derivatives of f up to order k exist and are continuous on Ω. We denote by $C^k(\Omega; \mathbb{R}^p)$ the set of functions of class C^k on Ω with values in $\mathbb{R}^p$ and, more simply, $C^k(\Omega)$ when $p = 1$.*
- *We say that f is of class C^∞ on Ω if f is of class C^k for all $k \in \mathbb{N}$:*

$$C^\infty(\Omega; \mathbb{R}^p) = \bigcap_{k \in \mathbb{N}} C^k(\Omega; \mathbb{R}^p).$$

Example 3.10. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$f(x, y) = x^3 y^2 + \cos x - \sin y.$$

- *The first-order partial derivatives are*

$$\frac{\partial f}{\partial x}(x, y) = 3x^2y^2 - \sin x \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = 2x^3 y - \cos y.$$

- *The partial derivatives of order 2 are*

$$\frac{\partial^2 f}{\partial x^2}(x, y) = 6xy^2 - \cos x, \quad \frac{\partial^2 f}{\partial x \partial y}(x, y) = 6x^2 y,$$

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = 6x^2 y \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2}(x, y) = 2x^3 + \sin y.$$

- *The partial derivatives of order 3 are*

$$\frac{\partial^3 f}{\partial x^3}(x, y) = 6y^2 + \sin x, \quad \frac{\partial^3 f}{\partial y \partial x^2}(x, y) = 12xy = \frac{\partial^3 f}{\partial x^2 \partial y}(x, y).$$

$$\frac{\partial^3 f}{\partial y^3}(x, y) = \cos y, \quad \frac{\partial^3 f}{\partial x^2 \partial y}(x, y) = 6x^2 = \frac{\partial^3 f}{\partial y^2 \partial x}(x, y).$$

Noting that all partial derivatives of f *up to order 3 exist and are continuous on* $\mathbb{R}^2$*, then* $f \in C^3(\mathbb{R}^2)$*. We can even show that* $f \in C^\infty(\mathbb{R}^2)$*.*

Example 3.11. *Let* $f: \mathbb{R}^2 \to \mathbb{R}$ *be the function defined by*

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

Let's calculate $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$, $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$, $\dfrac{\partial^2 f}{\partial x^2}(0,0)$ *and* $\dfrac{\partial^2 f}{\partial y^2}(0,0)$*.*
We have already seen in 3.1 that

$$\forall (x,y) \in \mathbb{R}^2 : \frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{y^5 - x^2y^3}{(x^2+y^2)^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

$$\forall (x,y) \in \mathbb{R}^2 : \frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{xy^4 + 3x^3y^2}{(x^2+y^2)^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0). \end{cases}$$

We have,

- $\dfrac{\partial^2 f}{\partial x^2}(0,0) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)(0,0) = \lim\limits_{h \to 0} \dfrac{\frac{\partial f}{\partial x}(h,0) - \frac{\partial f}{\partial x}(0,0)}{h} = 0.$
- $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)(0,0) = \lim\limits_{k \to 0} \dfrac{\frac{\partial f}{\partial x}(0,k) - \frac{\partial f}{\partial x}(0,0)}{k} = 1.$
- $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)(0,0) = \lim\limits_{h \to 0} \dfrac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = 0.$
- $\dfrac{\partial^2 f}{\partial y^2}(0,0) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)(0,0) = \lim\limits_{k \to 0} \dfrac{\frac{\partial f}{\partial y}(0,k) - \frac{\partial f}{\partial y}(0,0)}{k} = 0.$

Consequently, $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) \neq \dfrac{\partial^2 f}{\partial y \partial x}(0,0)$*.*

The following Schwarz theorem provides a sufficient condition for the reversal of partial derivatives.

Theorem 2 (Schwarz's theorem) Let Ω be a non-empty open set $\mathbb{R}^n$, $a \in \Omega$ and $i, j \in \{1, \ldots, n\}$. Let $f: \Omega \to \mathbb{R}^p$ a function such that $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ and $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$ exist on Ω and are continuous at the point a. Then

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a).$$

Remark 3.6. *Le théorème précédant peut s'appliquer dans le cas des dérivées The previous theorem can be applied in the case of partial derivatives of order* $k \geqslant 3$ *as follows, If* $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^p$ *is of class* C^k *on an open* Ω*. Then the order in which any partial derivative of* *order* k *is computed is unimportant. If* σ *is a permutation of* $\{1, 2, \ldots, k\}$ *then*

$$\frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_k}} = \frac{\partial^k f}{\partial x_{i_{\sigma(1)}} \partial x_{i_{\sigma(2)}} \ldots \partial x_{i_{\sigma(k)}}} \quad \text{on } D, \text{ with} 1 \leqslant i_1, \ldots, i_k \leqslant n.$$

Exercise 3.3. *Let* f *be the function defined on* $\mathbb{R}^2$ *by*

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

1. *Show that* f *is continuous on* $\mathbb{R}^2$.
2. *Show that* f *is of class* C^1 *on* $\mathbb{R}^2$ *?*
3. *Calculate* $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$ *and* $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$. *Is the function* f *of class* C^2 *on* $\mathbb{R}^2$*?*

Solution.

1. *The function* f *is continuous on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *as a quotient of continuous functions on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *whose denominator does not cancel on* $\mathbb{R}^2 \setminus \{(0,0)\}$*. Moreover*

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{r^4 \cos\theta \sin\theta(\cos^2\theta - \sin^2\theta)}{r^2} = 0.$$

This shows that f *is continuous at* $(0,0)$*. In summary,* f *is continuous on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *and at* $(0,0)$ *and therefore,* f *is continuous on*$\mathbb{R}^2$.

2. ***Partial derivatives in*** $\mathbb{R}^2 \setminus \{(0,0)\}$. f *has on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *a partial derivative with respect to* x *as a quotient of functions having a partial derivative with respect to their variable* x *whose denominator does not cancel on* $\mathbb{R}^2 \setminus \{(0,0)\}$*. Furthermore, for all* $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ *we have,*

$$\frac{\partial f}{\partial x}(x,y) = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2+y^2)^2}.$$

Similarly, for any $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\frac{\partial f}{\partial y}(x,y) = \frac{x^5 - 4x^3 y^2 - xy^4}{(x^2+y^2)^2},$$

(obtained by exchanging the roles of x *and* y *and changing sign), because*

$$f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2} = -f(y,x).$$

Partial derivatives at $(0,0)$. *we have,*

- $\dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = 0,$
- $\dfrac{\partial f}{\partial y}(0,0) = \lim\limits_{k\to 0} \dfrac{f(0,k) - f(0,0)}{k} = 0.$

In summary, f *admits on* $\mathbb{R}^2$ *partial derivatives with respect to each of its two variables and we have*

$$\forall (x,y)\in\mathbb{R}^2 : \frac{\partial f}{\partial x}(x,y)=\begin{cases}\dfrac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} & : (x,y)\neq(0,0)\\ 0 & : (x,y)=(0,0)\end{cases}$$

$$\forall (x,y)\in\mathbb{R}^2 : \frac{\partial f}{\partial y}(x,y)=\begin{cases}\dfrac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2} & : (x,y)\neq(0,0)\\ 0 & : (x,y)=(0,0)\end{cases}$$

3. ***Study on*** $\mathbb{R}^2\setminus\{(0,0)\}$. *The partial derivative functions* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *are continuous on* $\mathbb{R}^2\setminus\{(0,0)\}$ *as a quotient of continuous functions on* $\mathbb{R}^2\setminus\{(0,0)\}$ *whose denominator does not cancel on* $\mathbb{R}^2\setminus\{(0,0)\}$.

   Study at $(0,0)$. *the partial derivatives are continuous in* $(0,0)$, *in fact*

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{\partial f}{\partial x}(x,y)&=\lim_{(x,y)\to(0,0)}\frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}\\&=\lim_{r\to0}\frac{r^5(\cos^4\theta\sin\theta+4\cos^2\theta\sin^3\theta-\sin^5\theta)}{r^4}=0\\&=\frac{\partial f}{\partial x}(0,0).\end{aligned}$$

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{\partial f}{\partial y}(x,y)&=\lim_{(x,y)\to(0,0)}\frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}\\&=\lim_{r\to0}\frac{r^5(\cos^5\theta-4\cos^3\theta\sin^2\theta-\cos\theta\sin^4\theta)}{r^4}\\&=0\\&=\frac{\partial f}{\partial y}(0,0).\end{aligned}$$

   In summary, $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *are continuous on* $\mathbb{R}^2\setminus\{(0,0)\}$ *and at* $(0,0)$ *and therefore, they are continuous on* $\mathbb{R}^2$. *Then the function* f *is of class* $C^1(\mathbb{R}^2)$.

4. *We have,*

   - $\dfrac{\partial^2 f}{\partial y\partial x}(0,0)=\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)(0,0)=\lim\limits_{k\to0}\dfrac{\frac{\partial f}{\partial x}(0,k)-\frac{\partial f}{\partial x}(0,0)}{k}=-1.$
   - $\dfrac{\partial^2 f}{\partial x\partial y}(0,0)=\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)(0,0)=\lim\limits_{h\to0}\dfrac{\frac{\partial f}{\partial y}(h,0)-\frac{\partial f}{\partial y}(0,0)}{h}=1.$

   Consequently $\frac{\partial^2 f}{\partial x\partial y}(0,0)\neq\frac{\partial^2 f}{\partial y\partial x}(0,0)$. *According to Schwarz's theorem, either* $\frac{\partial^2 f}{\partial x\partial y}$ *is not continuous in* $(0,0)$. *So* f *cannot be of class* $C^2(\mathbb{R}^2)$.

3.5 Jacobian matrix

Definition 3.6. *Let*

$$\begin{aligned} f : \Omega \subset \mathbb{R}^n &\to \mathbb{R}^p \\ (x_1, ..., x_n) &\mapsto f(x_1, ..., x_n) = (f_1(x_1, ..., x_n), f_2(x_1, ..., x_n), ..., f_p(x_1, ..., x_n)), \end{aligned}$$

admitting partial derivatives in $a \in \Omega$*. The matrix with p rows and n columns, denoted by* $J_f(a)$*, defined by*

$$J_f(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \cdots & \frac{\partial f_2}{\partial x_n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1}(a) & \frac{\partial f_p}{\partial x_2}(a) & \cdots & \frac{\partial f_p}{\partial x_n}(a) \end{pmatrix} \in \mathbb{M}_{p,n}(\mathbb{R}).$$

In other words, the Jacobian matrix of f *has as columns the vectors* $\frac{\partial f}{\partial x_i}$.
If $n = p$*, the determinant of the matrix* $J_f(a)$ *is called Jacobian of* f *at* a *and we write,*

$$\mathit{det} J_f(a) = \frac{\partial(f_1, f_2, ..., f_p)}{\partial(x_1, x_2, ..., x_n)}.$$

Example 3.12. *Let* $f : \mathbb{R}^3 \to \mathbb{R}^2$ *a function defined by*

$$f(x,y,z) = \Big(f_1(x,y,z), f_2(x,y,z)\Big) = \Big(xy^2, \sin yz\Big).$$

For all $(x, y, z) \in \mathbb{R}^3$*:*

$$J_f(x, y, z) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x, y, z) & \frac{\partial f_1}{\partial y}(x, y, z) & \frac{\partial f_1}{\partial z}(x, y, z) \\ \frac{\partial f_2}{\partial x}(x, y, z) & \frac{\partial f_2}{\partial y}(x, y, z) & \frac{\partial f_2}{\partial z}(x, y, z) \end{pmatrix} = \begin{pmatrix} y^2 & 2xy & 0 \\ 0 & z\cos yz & y\cos yz \end{pmatrix}.$$

Example 3.13. *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be a function defined by*

$$f(r, \theta) = \Big(f_1(r, \theta), f_2(r, \theta)\Big) = \Big(r\cos\theta, r\sin\theta\Big).$$

Pour tout $(r, \theta) \in \mathbb{R}^2$*:*

$$J_f(r, \theta) = \begin{pmatrix} \frac{\partial f_1}{\partial r}(r, \theta) & \frac{\partial f_1}{\partial \theta}(r, \theta) \\ \frac{\partial f_2}{\partial r}(r, \theta) & \frac{\partial f_2}{\partial \theta}(r, \theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

so the Jacobian of f *is*

$$\mathit{det} J_f(r, \theta) = \frac{\partial(f_1, f_2)}{\partial(r, \theta)} = r.$$

3.6 Classical differential operators

Most real-world phenomena are modeled by differential equations that involve differential operators. The gradient, divergence, and rotational are first-order linear differential operators. This means that they only involve first partial derivatives of the fields, unlike, for example, the Laplacian that involves second-order partial derivatives.

3.6.1 Gradient

Definition 3.7. *Let* $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ *admit partial derivatives in* $a \in \Omega$. *The gradient of f at the point a is defined by*

$$\operatorname{grad} f(a) = \nabla f(a) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(a) \\ \frac{\partial f}{\partial x_2}(a) \\ \vdots \\ \frac{\partial f}{\partial x_n}(a) \end{pmatrix}.$$

Example 3.14. *Let* $f : \mathbb{R}^3 \to \mathbb{R}$ *be the function defined by*

$$f(x, y, z) = x^3 y^2 z.$$

We have for all $(x, y, z) \in \mathbb{R}^3$

$$\nabla f(x, y, z) = \begin{pmatrix} \frac{\partial f}{\partial x}(x, y, z) \\ \frac{\partial f}{\partial y}(x, y, z) \\ \frac{\partial f}{\partial z}(x, y, z) \end{pmatrix} = \begin{pmatrix} 3x^2y^2z \\ 2x^3yz \\ x^3y^2 \end{pmatrix}.$$

Properties 3.2.

- *Let* $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^p$ *two functions have partial derivatives at* a. *Then for all* $(\alpha, \beta) \in \mathbb{R}^2$, *we have,*

$$\nabla(\alpha f + \beta g)(a) = \alpha \nabla f(a) + \beta \nabla g(a).$$

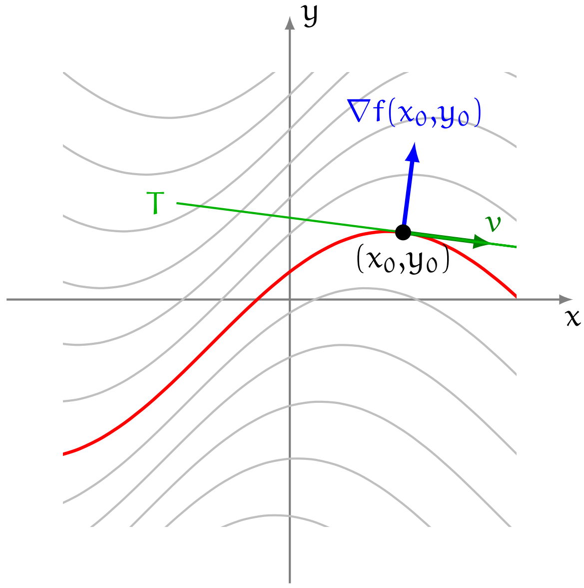
- *Let* $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ *a function has partial derivatives at* a *and* $g : f(\Omega) \subset \mathbb{R} \to \mathbb{R}$ *is a function of a real variable derivable at the point* $f(a)$. *Then we have*

$$\nabla(g \circ f)(a) = g'(f(a)) \nabla f(a).$$

Geometric interpretation of the gradient

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ admit partial derivatives at $a \in \Omega$.

- The gradient $\nabla f(a)$ is perpendicular to the line of level of f passing through the point a.
- The gradient $\nabla f(a)$ indicates the direction of greatest slope, i.e., the direction in which the variations of f grow most strongly in the vicinity of a.



3.6.2 Divergence

Definition 3.8. *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ admit partial derivatives at $a \in \Omega$. The divergence of f at the point a is defined by*

$$\operatorname{div} f(a) = \operatorname{tr}(J_f(a)) = \frac{\partial f_1}{\partial x_1}(a) + \frac{\partial f_2}{\partial x_2}(a) + \ldots + \frac{\partial f_n}{\partial x_n}(a) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(a),$$

where $\operatorname{tr}(J_f(a))$ is the trace of the Jacobian matrix of f en a.

Example 3.15. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by*

$$f(x, y) = (f_1(x, y), f_2(x, y)) = \left(-x^2y, -xy^2\right).$$

We have for all $(x, y) \in \mathbb{R}^2$

$$\operatorname{div} f(a) = \frac{\partial f_1}{\partial x}(x, y) + \frac{\partial f_2}{\partial y}(x, y) = 2xy - 2xy = 0.$$

Properties 3.3.

- *Let $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ two functions with partial derivatives at a. Then for all $(\alpha, \beta) \in \mathbb{R}^2$, we have,*

$$\operatorname{div}(\alpha f + \beta g)(a) = \alpha \operatorname{div} f(a) + \beta \operatorname{div} g(a).$$

- *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ two functions with partial derivatives at a. Then we have*

$$\operatorname{div}(fg)(a) = f(a) \operatorname{div} g(a) + \langle \nabla f(a), g(a) \rangle,$$

where $\langle \cdot, \cdot \rangle$ *represents the scalar product on* $\mathbb{R}^n$ *:*

$$\langle x, y \rangle = x \cdot y = (x_1, \dots, x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} x_i y_i.$$

3.6.3 Rotational

Definition 3.9. *Let* $f : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ *admitting partial derivatives at* $a \in \Omega$*. The Rotational of* f *at the point* a *is a vector defined by*

$$\operatorname{rot} f(a) = \begin{pmatrix} \frac{\partial f_3}{\partial x_2}(a) - \frac{\partial f_2}{\partial x_3}(a) \\ \frac{\partial f_1}{\partial x_3}(a) - \frac{\partial f_3}{\partial x_1}(a) \\ \frac{\partial f_2}{\partial x_1}(a) - \frac{\partial f_1}{\partial x_2}(a) \end{pmatrix}.$$

In Anglo-Saxon notation, $\operatorname{rot} f$ *is written as* $\operatorname{curl} f$.

Properties 3.4.

- *Let* $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ *two functions have partial derivatives in* a*. Then for all* $(\alpha, \beta) \in \mathbb{R}^2$*, we have,*

$$\operatorname{rot}(\alpha f + \beta g)(a) = \alpha \operatorname{rot} f(a) + \beta \operatorname{rot} g(a).$$

- *Let* $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ *of class* C^2 *on* Ω*. Then for all* $(x, y, z) \in \mathbb{R}^3$

$$\operatorname{rot}(\operatorname{grad} f(x, y, z)) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

- *Let* $f : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ *of class* C^2 *on* Ω*. Then for all* $(x, y, z) \in \mathbb{R}^3$ *:*

$$\operatorname{div}(\operatorname{rot} f(x, y, z)) = 0.$$

3.6.4 Laplacian

Definition 3.10. *Let* $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ *of class* C^2 *on an open* Ω*. The Laplacian of f is the function*

$$\Delta f : \mathbb{R}^n \to \mathbb{R}$$
$$x \mapsto \Delta f(x) = \operatorname{div}(\operatorname{grad} f(x)) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(x).$$

Remark 3.7. *If*

$$\begin{aligned} f : \Omega \subset \mathbb{R}^n &\to \mathbb{R}^p \\ (x_1, \dots, x_n) &\mapsto f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_p(x_1, \dots, x_n)), \end{aligned}$$

of class C^2 *on an open* Ω. *The Laplacian of f at the point x is given by*

$$\Delta f(x) = \begin{pmatrix} \Delta f_1(x) \\ \vdots \\ \Delta f_p(x) \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{i=1}^{n} \frac{\partial^2 f_1}{\partial x_i^2}(x) \\ \vdots \\ \displaystyle\sum_{i=1}^{n} \frac{\partial^2 f_p}{\partial x_i^2}(x) \end{pmatrix}.$$

3.7 Derivative along a vector

Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^p$ be a function with a i^{th} partial derivative at $a = (a_1, \dots, a_n) \in \Omega$. Then we have

$$\frac{\partial f}{\partial x_i}(a) = \lim_{t \to 0} \frac{f(a_1, \dots, a_i + t, \dots, a_n) - f(a_1, \dots, a_n)}{t}.$$

Noting $(e_i)_{1 \leqslant i \leqslant n}$ the canonical basis of $\mathbb{R}^n$, we obtain

$$\frac{\partial f}{\partial x_i}(a) = \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t}.$$

We then say that we have derived the function f in a according to the vector e_i. The following definition generalizes this notion to any vector of $\mathbb{R}^n$.

Definition 3.11. *Let* f *be a function defined on an open* Ω *of* $\mathbb{R}^n$ *with values in* $\mathbb{R}^p$, $a = (a_1, \dots, a_n)$ *a point of* Ω *and* $v = (v_1, \dots, v_n)$ *a non-zero vector of* $\mathbb{R}^n$. f *is derivable in* a *along the vector* v *if and only if the function of one real variable* $g : t \mapsto f(a + tv)$ *is derivable in* $t = 0$. *In this case,* $g'(0)$ *is called the derivative of* f *in* a *along the vector* v *and is noted* $\frac{\partial f}{\partial v}(a)$ *:*

$$\frac{\partial f}{\partial v}(a) = \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t} = \lim_{t \to 0} \frac{f(a_1 + tv_1, \dots, a_n + tv_n) - f(a_1, \dots, a_n)}{t}.$$

Example 3.16. *The directional derivative of the*

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R}^2, \\ (x, y, z) &\mapsto (f_1(x, y, z), f_2(x, y, z)) = (xy^2 + 3x, yz), \end{aligned}$$

at the point $a = (0, 1, 0)$ *in the direction* $v = (0, 1, -1)$ *is given by*

$$\frac{\partial f}{\partial v}(0, 1, 0) = \lim_{t \to 0} \frac{f(0, 1 + t, -t) - f(0, 1, 0)}{t} = (0; 1).$$

Remark 3.8. *A function can be derivable along any vector at a point without being continuous at that point.*

Example 3.17. *Consider the function defined on* $\mathbb{R}^2$ *by*

$$f(x,y) = \begin{cases} \dfrac{y^2}{x} & : x \neq 0 \\ 0 & : \textit{sinon} \end{cases}$$

Alors, pour $v = (\alpha, \beta) \in \mathbb{R}^2$*, at* a

$$\lim_{t\to 0} \frac{f(a+tv) - f(a)}{t} = \lim_{t\to 0} \frac{f(\alpha t, \beta t) - f(0,0)}{t} = \begin{cases} \dfrac{\beta^2}{\alpha} & : \alpha \neq 0 \\ 0 & : \textit{sinon} \end{cases}$$

so f *has a derivative along* v *at the point* $(0,0)$*. But* f *is not continuous at* $(0,0)$*, in fact*

- *If we take the path* $\gamma_1(t) = (t,t)$*, then* $\lim\limits_{t\to 0} f \circ \gamma_1(t) = \lim\limits_{t\to 0} 0 = 0$.
- *If we take the path* $\gamma_2(t) = (t^2, t)$*, then* $\lim\limits_{t\to 0} f \circ \gamma_2(t) = 1$.

Remark 3.9. *If* $(e_i)_{1\leqslant i\leqslant n}$ *the canonical basis of* $\mathbb{R}^n$*, then*

$$\frac{\partial f}{\partial x_i}(a) = \frac{\partial f}{\partial e_i}(a).$$

It follows that if f *is derivable along any vector in* a*, then* f *has partial derivatives with respect to each of its variables in* a*. Let us now give an example where the converse is false.*

Example 3.18. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be the function defined by*

$$f(x,y) = \begin{cases} 1 & : x \neq 0 \textit{ and } y \neq 0 \\ 0 & : x = 0 \textit{ or } y = 0 \end{cases}$$

We have,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0,$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0.$$

So f *has partial derivatives at* $(0,0)$.
Let $v = (1,1)$*. For all* $t \neq 0$, $\dfrac{f(t,t) - f(0,0)}{t} = \dfrac{1}{t}$ *is an expression which has no limit when* t *tends towards* 0*. So* f *is not derivable in* $(0,0)$ *along the vector* $v = (1,1)$*. Thus,* f *has partial derivatives with respect to each of its two variables in* $(0,0)$ *but is not derivable along any vector at* $(0,0)$.

Proposition 3.1. *Let* $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ *be a function derivable in* $a \in \Omega$ *along the vector* $v = (v_1, \ldots, v_n)$*. Then,*

$$\frac{\partial f}{\partial v}(a) = \langle \nabla f(a), v \rangle = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(a) v_i.$$

3.8 Homogeneous functions

Definition 3.12 (**cône**). *A part* A *of* $\mathbb{R}^n$ *is a cone if and only if* A *is stable for multiplication by any strictly positive real. This is written as*

$$\forall x \in A, \forall t > 0 : tx \in A.$$

Fig. 3.1 . Cone in the plane $\mathbb{R}^2$.

Example 3.19. *The set* $A = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$ *is a cone of* $\mathbb{R}^2$.

Definition 3.13 (**Homogeneous function**). *Let* $k \in \mathbb{R}$ *and* A *be a cone of* $\mathbb{R}^n$. *A function* $f : A \to \mathbb{R}^p$ *is said to be homogeneous of degree* k *if it verifies,*

$$\forall x \in A, \forall t > 0 : f(tx) = t^k f(x).$$

Example 3.20.

- *The function defined on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *by* $f(x,y) = \arctan\left(\frac{xy}{x^2+y^2}\right)$ *is homogeneous of degree* 0. *on* $A = \{(x,y) \in \mathbb{R}^2 :\ xy \geqslant 0\}$ *is homogeneous of degree* 1.

Properties 3.5. *If a homogeneous function is of degree* k, *its partial derivatives, if they exist, are homogeneous of degree* $k-1$.

Example 3.21. *The function defined on* $A = \{(x,y) \in \mathbb{R}^2 : xy > 0\}$ *by* $f(x,y) = \sqrt{xy}$ *is homogeneous of degree 1 and of class* $C^1(A)$. *We have the function* $\frac{\partial f}{\partial x}(x,y) = \frac{y}{2\sqrt{xy}}$ *is homogeneous of degree* 0.

Theorem 3 (**Euler's theorem**) Let $f : A \to \mathbb{R}^p$ be a homogeneous function of degree k on a cone A of $\mathbb{R}^n$, admitting partial derivatives with respect to all variables. Then,

$$\forall x = (x_1, x_2, \dots, x_n) \in A : \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x) = kf(x).$$

Example 3.22. *Using the example 3.21, we have* $\frac{\partial f}{\partial x}(x,y) = \frac{y}{2\sqrt{xy}}$ *et* $\frac{\partial f}{\partial y}(x,y) = \frac{x}{2\sqrt{xy}}$. *For all* $(x,y) \in A$*:*

$$x \cdot \frac{\partial f}{\partial x}(x,y) + y \cdot \frac{\partial f}{\partial y}(x,y) = \frac{xy}{2\sqrt{xy}} + \frac{xy}{2\sqrt{xy}} = \frac{xy}{\sqrt{xy}} = \sqrt{xy} = f(x,y).$$

Exercise 3.4. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be a homogeneous function of degree* 3 *such that* $f(1,0) = 2$.

1. *Find* $f(2,0)$.
2. *Determine* f *knowing that* $\forall (x,y) \in \mathbb{R}^2 : \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y)$.

Solution.

1. *The function* f *being homogeneous of degree* 3, *then,*

$$f(2,0) = f(2(1,0)) = 2^3 f(1,0) = 2^4 = 16.$$

2. *Determine* f *knowing that*

$$\forall (x,y) \in \mathbb{R}^2 : \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y). \tag{*}$$

According to Euler's theorem, we have for all $(x,y) \in \mathbb{R}^2$:

$$x \cdot \frac{\partial f}{\partial x}(x,y) + y \cdot \frac{\partial f}{\partial y}(x,y) = f(x,y).$$

Using (*) for all $(x,y) \in \mathbb{R}^2$:

$$(x+y) \cdot \frac{\partial f}{\partial x}(x,y) = f(x,y).$$

If we first consider the variable y *as a parameter and set* $f(x,y) = \varphi_y(x)$. *then* varphi_y *satisfies the following differential equation*

$$(x+y)\varphi_y'(x) = 3\varphi_y(x),$$

so,

$$\frac{\varphi_y'(x)}{\varphi_y(x)} = \frac{3}{x+y}.$$

Integrating with respect to x, *we obtain*

$$\ln(|\varphi_y(x)|) = \ln(|(x+y)^3|) + \psi_1(y),$$

then,

$$\varphi_y(x) = f(x,y) = \psi_2(y)(x+y)^3,$$

where ψ_1 *and* ψ_2 *are two continuous functions from* $\mathbb{R}$ *into* $\mathbb{R}$. *Condition (*) gives,*

$$\forall (x,y) \in \mathbb{R}^2, \quad \psi_2'(y)(x+y)^3 + 3\psi_2(y)(x+y)^2 = 3\psi_2(y)(x+y)^2.$$

This implies that $\forall y \in \mathbb{R}, \quad \psi_2'(y) = 0$, *and since* ψ_2 *is continuous on* $\mathbb{R}$, *then* $\forall y \in \mathbb{R}, \quad \psi_2(y) = c$ *where* c *is a real constant. On the other hand we have*

$$f(1,0) = 2 = c(1+0)^3 = c.$$

Finally, we conclude that $f(x,y) = 2(x+y)^3$.

3.9 Differentiable functions

Let $f: \ I \subset \mathbb{R} \to \mathbb{R}$ a function defined on an open interval I and $a \in I$, we know that if f is derivable in a then,

$$f(a+h) = f(a) + f'(a) \cdot h + o(h),$$

where $\lim_{h\to 0} \frac{o(h)}{h} = 0$. The linear application

$$\begin{aligned} L : \mathbb{R} &\to \mathbb{R} \\ h &\mapsto L(h) = f'(a)\cdot h, \end{aligned}$$

is a linear approximation of the difference $f(a+h) = f(a)$ when h tends to 0. Since we are interested in the differentiability of functions from $\mathbb{R}^n$ into $\mathbb{R}^p$, we need to define linear applications from $\mathbb{R}^n$ into $\mathbb{R}^p$. We say that a function $L : \mathbb{R}^n \to \mathbb{R}^p$ is linear if for all $x, y \in \mathbb{R}^n$ and for all $\alpha, \beta \in \mathbb{R}$:

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y).$$

If we consider

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{pmatrix} \in \mathcal{M}_{p,n}(\mathbb{R}),$$

the matrix of the linear application L relative to the canonical bases $\mathcal{B}_n$ and $\mathcal{B}_p$ of $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively. Then for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$:

$$L(x) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pn}x_n \end{pmatrix}.$$

In particular, if $L : \mathbb{R}^n \to \mathbb{R}$ is linear, then there exist $a_1, a_2, \ldots, a_n \in \mathbb{R}$ for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$:

$$L(x) = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

Definition 3.14. *Let f be a function defined in the neighbourhood of $a \in \mathbb{R}^n$, with values in $\mathbb{R}^p$. We say that f is differentiable at a if there exists a linear application*

$$\begin{aligned} L : \mathbb{R}^n &\to \mathbb{R}^p \\ h = (h_1, h_2, \ldots, h_n) &\mapsto L(h), \end{aligned}$$

Suth that,

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - L(h)}{\|h\|} = 0.$$

It is the same to say that f is differentiable at a if and only if there exists a linear application $L : \mathbb{R}^n \to \mathbb{R}^p$ such that

$$\forall h \in \mathbb{R}^n, \quad f(a+h) = f(a) + L(h) + \|h\| \cdot \epsilon(h),$$

where ϵ is a function defined on a neighbourhood of $0_{\mathbb{R}^n}$ in $\mathbb{R}^p$ and satisfying $\lim_{h\to 0} \epsilon(h) = 0_{\mathbb{R}^p}$. Equivalently (just put $x = a + h$), if there is a linear application $L : \mathbb{R}^n \to \mathbb{R}^p$: such that,

$$f(x) = f(a) + L(x-a) + \|x-a\|\epsilon(x-a),$$

where ϵ is a function defined on a neighbourhood of a in $\mathbb{R}^p$ and satisfying $\lim_{x\to a} \epsilon(x-a) = 0_{\mathbb{R}^p}$.

Theorem 4 The application L, if it exists, is unique and is called the differential of f at point a. It is denoted $d_a f$.

$$\begin{aligned} d_a f : \mathbb{R}^n &\to \mathbb{R}^p \\ h = (h_1, h_2, \dots, h_n) &\mapsto d_a f(h). \end{aligned}$$

Proof. We assume that there exist two linear applications L_1 and L_2 from $\mathbb{R}^n$ to $\mathbb{R}^p$ verifying

$$\forall h \in \mathbb{R}^n, f(a+h) = f(a) + L_1(h) + \|h\|\epsilon_1(h) = f(a) + L_2(h) + \|h\|\epsilon_2(h),$$

with $\lim\limits_{h\to 0} \epsilon_1(h) = \lim\limits_{h\to 0} \epsilon_2(h) = 0$. Then,

$$\lim_{h\to 0} \frac{(L_1 - L_2)(h)}{\|h\|} = 0.$$

We set $h = tv$ with $v \in \mathbb{R}^n \setminus \{0\}$ and t is a strictly positive real

$$0 = \lim_{h\to 0} \frac{(L_1 - L_2)(h)}{\|h\|} = \lim_{t\to 0} \frac{(L_1 - L_2)(tv)}{\|tv\|} = \frac{(L_1 - L_2)(v)}{\|v\|}.$$

We deduce that for any non-zero vector v, $L_1(v) = L_2(v)$ which is true even for $v = 0$ and therefore $L_1 = L_2$. $\square$

Example 3.23. *If $f : \mathbb{R}^n \to \mathbb{R}^p$ is a linear application, then it is differentiable in all $a \in \mathbb{R}^n$ and its differential is*

$$\begin{aligned} d_a f : \mathbb{R}^n &\to \mathbb{R}^p \\ h = (h_1, h_2, \dots, h_n) &\mapsto d_a f(h) = f(h). \end{aligned}$$

Indeed, for all $a \in \mathbb{R}^n$. If f is linear, then

$$\forall h \in \mathbb{R}^n, f(a+h) = f(a) + f(h) = f(a) + d_a f(h) + \|h\|\epsilon(h),$$

with $d_a f(h) = f(h)$ and $\epsilon(h) = 0$ for all $h \in \mathbb{R}^n$.
In particular if $f : \mathbb{R}^2 \to \mathbb{R}$ is a linear application defined by $f(x,y) = 2x - 3y$, then f is differentiable at all $a = (a_1, a_2) \in \mathbb{R}^2$ and its differential is

$$\begin{aligned} d_a f : \mathbb{R}^2 &\to \mathbb{R} \\ h = (h_1, h_2) &\mapsto d_a f(h_1, h_2) = f(h_1, h_2) = 2h_1 - 3h_2. \end{aligned}$$

Remark 3.10. *Let f be a function defined in the neighborhood of a point $a \in \mathbb{R}^n$ with values in $\mathbb{R}^p$.*

$$f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_p(x_1, \dots, x_n)).$$

The function f is differentiable at a if and only if each component $f_i, i \in \{1, \dots, p\}$, is differentiable at a. In this case

$$d_a f = \begin{pmatrix} d_a f_1 \\ \vdots \\ d_a f_p \end{pmatrix}.$$

Example 3.24. *Let the application*

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x,y,z) &\longrightarrow f(x,y,z) = (f_1(x,y,z), f_2(x,y,z)) = (x+3y-2z, 2y+z). \end{aligned}$$

It is clear that f *is linear, then* f *is differentiable in all* $a \in \mathbb{R}^3$ *and we have*

$$\begin{aligned} d_a f(h_1,h_2,h_3) &= \begin{pmatrix} d_a f_1 (h_1,h_2,h_3) \\ d_a f_2 (h_1,h_2,h_3) \end{pmatrix} \\ &= \begin{pmatrix} h_1 + 3h_2 - 2h_3 \\ 2h_2 + h_3 \end{pmatrix}. \end{aligned}$$

Example 3.25. *Let* f *be an application of a non-empty open set* Ω *from* $\mathbb{R}^n$ *to* $\mathbb{R}^p$*. If* f *is constant on* $\Omega\,(\forall x \in \mathbb{R}^n, f(x) = C \in \mathbb{R}^p)$*, then for all* $a \in \Omega, f$ *is differentiable at a and* $d_a f = 0$*. Indeed*

$$\forall h \in \mathbb{R}^n, f(a+h) = C = f(a) + d_a f(h) + \|h\|\varepsilon(h),$$

with $d_a f(h) = 0$ *and* $\varepsilon(h) = 0$ *for all* $h \in \mathbb{R}^n$*.*

Example 3.26. *Let* $\mathbb{R}^n$ *have the scalar product* $\langle\cdot,\cdot\rangle$*. Let* $\|\mathsf{cdot}\|_2$ *be the associated Euclidean norm. Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a function defined by* $f(x) = \|x\|_2^2 = x_1^2 + x_2^2 + \ldots + x_n^2$*. Let* $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$*. For all* $h \in \mathbb{R}^n$ *:*

$$\|a+h\|_2^2 = \langle a+h, a+h\rangle = \|a\|_2^2 + 2\langle a,h\rangle + \|h\|_2^2.$$

So,

$$f(a+h) = f(a) + 2\langle a,h\rangle + \|h\|_2\varepsilon(h).$$

avec $\varepsilon(h) = \|h\|_2$ *which tends to* 0 *when* h *tends to* 0*. Therefore* f *is differentiable at* a *and*

$$\forall h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n,\ d_a f(h) = \langle a,h\rangle = \sum_{i=1}^{n} a_i h_i$$

Theorem 5 Let f be a function defined in the neighborhood of $a \in \mathbb{R}^n$, with values in $\mathbb{R}^p$. If f is differentiable at a, then f is continuous at a (and the converse of this implication is false).

Proof. Since f is differentiable at a, then

$$\forall h \in \mathbb{R}^n, f(a+h) = f(a) + d_a f(h) + \|h\|\varepsilon(h),$$

with $\lim_{h\to 0} \varepsilon(h) = 0$. Since $\mathbb{R}^n$ is finite-dimensional, we know that the linear application $d_a f$ is continuous on $\mathbb{R}^n$. So, $\lim_{h\to 0} d_a f(h) = d_a f(0) = 0$. We deduce that

$$\lim_{h\to 0} f(a+h) = \lim_{h\to 0} ((a) + d_a f(h) + \|h\|\varepsilon(h) = f(a).$$

Which shows that f is continuous in a. □

3.9.1 Link with the derivative along a vector

Theorem 6 Let f be a function defined on an open set Ω of $\mathbb{R}^n$ with values in $\mathbb{R}^p$ and $a \in \Omega$. If f is differentiable at a, then f is differentiable along any vector at a and in this case, for any $v \in \mathbb{R}^n \backslash \{0\}$:

$$\frac{\partial f}{\partial v}(a) = d_a f(v)$$

Proof. Since f is differentiable at a, then,

$$\forall h \in \mathbb{R}^n, f(a+h) = f(a) + d_a f(h) + \|h\|\varepsilon(h),$$

with $\lim\limits_{h \to 0} \varepsilon(h) = 0$. Let $v \in \mathbb{R}^n \backslash \{0\}$ and $t \in \mathbb{R}$.

$$f(a+tv) = f(a) + t\ d_a f(v) + |t|\|v\|\varepsilon(tv),$$

avec $\lim\limits_{t \to 0} \varepsilon(tv) = 0$. Donc

$$\lim_{t \to 0} \frac{f(a+tv) - f(a)}{t} = d_a f(v) + \lim_{t \to 0} \frac{|t|\|v\|\varepsilon(tv)}{t} = d_a f(v).$$

This shows that f is differentiable along the vector v and that $\dfrac{\partial f}{\partial v}(a) = d_a f(v)$. □

Remark 3.11. *The converse of the previous theorem is false.*

Example 3.27. *We have already seen in the example 3.17 that the function f defined on $\mathbb{R}^2$ by*

$$f(x,y) = \begin{cases} \dfrac{y^2}{x} & : x \neq 0 \\ 0 & : \textit{sinon} \end{cases}$$

admits a derivative along any vector at the point $(0,0)$, but f is not continuous at $(0,0)$. So by Theorem 3.9, f cannot be differentiable at $(0,0)$.

3.9.2 Link with partial derivatives

By combining the theorems 3.9.1 and the remark 3.9, we have the following result

Corollary 3.1. *Let f be a function defined on an open set Ω of $\mathbb{R}^n$ with values in $\mathbb{R}^p$ and $a \in \Omega$. If f is differentiable at a, then f admits partial derivatives with respect to each of its variables at a and moreover,*

$$\forall i \in \{1, \dots, n\}, \frac{\partial f}{\partial x_i}(a) = d_a f(e_i).$$

With this result, we are able to express the differential of a function f at a point a as a function of the partial derivatives of f at a.

Theorem 7 Let f be a function defined on an open set Ω of $\mathbb{R}^n$ with values in $\mathbb{R}^p$ and $a \in \Omega$. If f is differentiable at a, then

$$\forall h = (h_1, h_2, \dots, h_n) \in \mathbb{R}^n,\ d_a f(h) = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a).$$

Proof. Let $(e_i)_{1 \leqslant i \leqslant n}$ the canonical basis of $\mathbb{R}^n$. Since $d_a f \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$, then,

$$\forall h \in \mathbb{R}^n,\ d_a f(h) = d_a f\left(\sum_{i=1}^{n} h_i e_i\right) = \sum_{i=1}^{n} h_i\ d_a f(e_i) = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a).$$

□

Corollary 3.2. *Let f be a function defined on an open set Ω of $\mathbb{R}^n$ with values in $\mathbb{R}$ and $a \in \Omega$. If f is differentiable at a, then*

$$\forall h \in \mathbb{R}^n,\ d_a f(h) = \langle \nabla f(a), h \rangle.$$

Remark 3.12. *For a function from $\mathbb{R}$ to $\mathbb{R}^p$, the above theorem allows us to make the link between the notions of derivability and differentiability. More precisely, a function $f : \mathbb{R} \to \mathbb{R}^p$ is derivable at a if and only if it is differentiable at a, and we then have*

$$\forall h \in \mathbb{R}, d_a f(h) = h f'(a).$$

Proposition 3.2. *Let f be a function defined on an open Ω of $\mathbb{R}^n$ with values in $\mathbb{R}^p$ and $a \in \Omega$. f is differentiable in a if and only if f admits partial derivatives with respect to each of its variables in a and*

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a)}{\|h\|} = 0.$$

The reciprocal of the implication of 3.1 is false in general. In fact, there are functions which have partial derivatives without being differentiable, as the following example proves.

Example 3.28. *The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0), \end{cases}$$

has partial derivatives at the point $(0,0)$.

- $\dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = 1.$
- $\dfrac{\partial f}{\partial y}(0,0) = \lim\limits_{k \to 0} \dfrac{f(0,k) - f(0,0)}{k} = -1.$

We have,

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(h_1,h_2)-f(0,0)-h_1\frac{\partial f}{\partial x}(0,0)-h_2\frac{\partial f}{\partial y}(0,0)}{\sqrt{h_1^2+h_2^2}} = \lim_{(h_1,h_2)\to(0,0)} \frac{\frac{h_1^3-h_2^3}{h_1^2+h_2^2}-h_1+h_2}{\sqrt{h_1^2+h_2^2}}$$
$$= \lim_{(h_1,h_2)\to(0,0)} \frac{h_1^3-h_2^3-h_1^3+h_1h_2^2}{(h_1^2+h_2^2)\sqrt{h_1^2+h_2^2}}$$
$$= \lim_{r\to 0} \frac{h_2h_1^2-h_1h_2^2}{(h_1^2+h_2^2)\sqrt{h_1^2+h_2^2}}$$
$$= \lim_{r\to 0} \frac{h_2h_1^2-h_1h_2^2}{r^3\sqrt{1}} = \cos^2\theta\sin\theta-\cos\theta\sin^2\theta$$

So f is not differentiable at the point $(0,0)$.

3.9.3 Classical differential notation

Let $f:\ \Omega\subset\mathbb{R}^n\to\mathbb{R}^p$ be a function defined on a neighbourhood of $a\in\Omega$. We have seen that if f is differentiable in a, then for any $h=(h_1,h_2,\ldots,h_n)\in\mathbb{R}^n$ we have,

$$d_af(h)=h_1\frac{\partial f}{\partial x_1}(a)+h_2\frac{\partial f}{\partial x_2}(a)+\cdots+h_n\frac{\partial f}{\partial x_n}(a). \tag{3.1}$$

For each $i\in\{1,\ldots,n\}$, we define the linear application

$$dx_i:\mathbb{R}^n\to\mathbb{R}^p$$
$$h=(h_1,h_2,\ldots,h_n)\mapsto dx_i(h)=h_i.$$

Substituting into (3.1), we obtain

$$d_af(h)=\frac{\partial f}{\partial x_1}(a)dx_1(h)+\frac{\partial f}{\partial x_2}(a)dx_2(h)+\cdots+\frac{\partial f}{\partial x_n}(a)dx_n(h).$$

This is written briefly

$$d_af=\frac{\partial f}{\partial x_1}(a)dx_1+\frac{\partial f}{\partial x_2}(a)dx_2+\cdots+\frac{\partial f}{\partial x_n}(a)dx_n.$$

Example 3.29. *Let* $f:\Omega\subset\mathbb{R}^2\to\mathbb{R}$ *be a polynomial function defined by*

$$f(x,y)=x^2+xy.$$

f is differentiable at $a=(2,-1)$ and we have

$$d_{(2,-1)}f=\frac{\partial f}{\partial x}(2,-1)\,dx+\frac{\partial f}{\partial y}(2,-1)\,dy=3\,dx+2\,dy.$$

Geometrical interpretation of differential

Let $f:\Omega\subset\mathbb{R}^n\to\mathbb{R}^p$ be a function defined on a neighborhood of $a=(a_1,\ldots,a_n)\in\Omega$. We have seen that if f is differentiable at a, therefore,

$$f(x)=f(a)+d_af(x-a)+\|x-a\|\epsilon(x-a),\quad \text{with}\lim_{x\to a}\epsilon(x-a)=0.$$

Then, in the neighborhood of a, the function f is well approximated by the affine function

$$x \mapsto f(a) + d_a f(x-a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i).$$

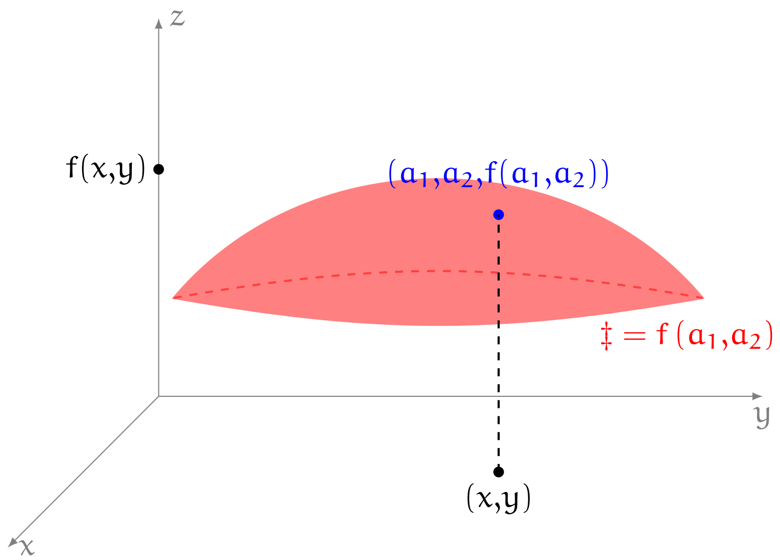
Geometrically this means that in $\mathbb{R}^{n+1}$, in the neighborhood of the point $(a, f(a))$, the graph of the function f differs little from the hyperplane of $\mathbb{R}^{n+1}$ whose equation

$$x_{n+1} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i).$$

This hyperplane is called the tangent hyperplane to the graph of f at the point $(a, f(a))$.

In dimension 2 the graph of f is a surface S of $\mathbb{R}^3$ with equation $z = f(x, y)$ and the tangent plane at $(a_1, a_2, f(a_1, a_2))$ has the equation

$$z = f(a_1, a_2) + \frac{\partial f}{\partial x}(a_1, a_2)(x - a_1) + \frac{\partial f}{\partial y}(a_1, a_2)(y - a_2).$$



Example 3.30. *Let* $f: \ \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ *be a polynomial function defined by*

$$f(x, y) = x^2 + xy^2.$$

f *is differentiable at* $a = (1, -1)$*. Moreover, the equation of the tangent plane to* S *at* a *is*

$$\begin{aligned} z &= f(1,-1) + \frac{\partial f}{\partial x}(1,-1)(x-1) + \frac{\partial f}{\partial y}(1,-1)(y+1) \\ &= 2 + 3(x-1) - 2(y+1) \\ &= 3x - 2y - 3. \end{aligned}$$

3.9.4 Differential of a function on an open number

Definition 3.15. *Let* f *be a function defined on an open set* Ω *of* $\mathbb{R}^n$ *with values in* $\mathbb{R}^p$*. We say that* f *is differentiable on* Ω *if it is differentiable at every point* $x \in \Omega$*. In this case, we call the differential of* f *the function*

$$\begin{aligned} df : \Omega &\to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p) \\ x &\mapsto d_x f, \end{aligned}$$

where $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ *is the set of linear applications (therefore continuous because we are in finite dimension) of* $\mathbb{R}^n$ *with values in* $\mathbb{R}^p$.

Example 3.31. *The function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$f(x,y) = x^2 + y^3,$$

is differentiable on $\mathbb{R}^2$*. Indeed, let*$(x_0, y_0) \in \mathbb{R}^2$:

$$\begin{aligned} &\lim_{(h_1,h_2)\to(0,0)} \frac{f(x_0+h_1, y_0+h_2) - f(x_0,y_0) - h_1 \frac{\partial f}{\partial x}(x_0,y_0) - h_2 \frac{\partial f}{\partial y}(x_0,y_0)}{\sqrt{h_1^2+h_2^2}} \\ &= \lim_{(h_1,h_2)\to(0,0)} \frac{(x_0+h_1)^2 + (y_0+h_2)^3 - x_0^2 - y_0^3 - 2h_1x_0 - 3h_2y_0^2}{\sqrt{h_1^2+h_2^2}} \\ &= \lim_{r\to 0} \frac{h_1^2 + 3h_2^2 y_0 + h_3}{\sqrt{h_1^2+h_2^2}} \\ &= \lim_{r\to 0} \frac{r^2(\cos^2\theta + 3y_0 \sin^2\theta + r\sin^3\theta)}{r} = 0. \end{aligned}$$

This is sufficient to conclude that f *is differentiable over* $\mathbb{R}^2$ *and for all* $(x,y) \in \mathbb{R}^2$:

$$d_{(x,y)} f(h_1,h_2) = 2xh_1 - 3y^2 h_2.$$

Proposition 3.3. *Soient* f *une fonction définie sur un ouvert* Ω *de* $\mathbb{R}^n$ *à valeurs dans* $\mathbb{R}^p$. f *est de classe* C^1 *sur* Ω *si et seulement si* f *est différentiable sur* Ω *et de plus l'application* $df : \Omega \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ *est continue sur* Ω.

Remark 3.13. *En particulier, la proposition précédente garantit, en particulier, que si est une fonction* f *de classe* C^1 *sur un voisinage de* a*, alors* f *est différentiable en* a *(et la réciproque de cette implication est fausse).*

Example 3.32. *Consider the function* $f :\ \mathbb{R}^3 \to \mathbb{R}$ *defined by* $f(x,y,z) = x^2y^3 + x\cos z$ *and* $a = (1,1,\frac{\pi}{2})$*. The function is clearly of class* C^1 *on* $\mathbb{R}^3$*, so it is differentiable at the point* a *and its differential at* a *is*

$$d_a f(h_1,h_2,h_3) = h_1 \frac{\partial f}{\partial x}(1,1,\frac{\pi}{2}) + h_2 \frac{\partial f}{\partial y}(1,1,\frac{\pi}{2}) + h_3 \frac{\partial f}{\partial z}(1,1,\frac{\pi}{2}) = 2h_1 + 3h_2 - h_3.$$

Example 3.33. *The partial derivatives of the function*

$$\begin{aligned} f : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x,y,z) &\longmapsto (xy^2 + 3x, yz^2) \end{aligned}$$

at the point (x,y,z) *are given by*

$$\frac{\partial f}{\partial x}(x,y,z) = \begin{pmatrix} y^2+3 \\ 0 \end{pmatrix}, \quad \frac{\partial f}{\partial y}(x,y,z) = \begin{pmatrix} 2xy \\ z^2 \end{pmatrix}, \quad \frac{\partial f}{\partial z}(x,y,z) = \begin{pmatrix} 0 \\ 2yz \end{pmatrix}$$

The function is clearly of class C^1 *on* $\mathbb{R}^3$*, so it is differentiable on* $\mathbb{R}^3$ *and in particular its differential at* $a = (1,1,1)$ *is given for all* $h = (h_1,h_2,h_3) \in \mathbb{R}^3$ *by*

$$\begin{aligned} d_a f(h_1,h_2,h_3) &= h_1 \frac{\partial f}{\partial x}(1,1,1) + h_2 \frac{\partial f}{\partial y}(1,1,1) + h_3 \frac{\partial f}{\partial z}(1,1,1) \\ &= h_1 \begin{pmatrix} 4 \\ 0 \end{pmatrix} + h_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + h_3 \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 4h_1 + 2h_2 \\ h_2 + 2h_3 \end{pmatrix} \end{aligned}$$

Example 3.34. *Let* f *be the function defined on* $\mathbb{R}^2$ *by*

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

Let us show that f *is not of class* C^1 *on* $\mathbb{R}^2$ *but it is differentiable on* $\mathbb{R}^2$*. We have*

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) = \lim_{r\to 0} r^2 \sin\left(\frac{1}{r}\right) = 0.$$

Thus, f *is continuous in* $(0;0)$ *and therefore on* $\mathbb{R}^2$*.*
The function f *admits on* $\mathbb{R}^2$ *partial derivatives with respect to each of its variables defined by*

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 2x\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) - \dfrac{x}{\sqrt{x^2+y^2}}\cos\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} 2y\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) - \dfrac{y}{\sqrt{x^2+y^2}}\cos\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

We have, for all $x \in \mathbb{R}^*$

$$\frac{\partial f}{\partial x}(x,0) = 2x\sin\left(\frac{1}{|x|}\right) - \frac{x}{|x|}\cos\left(\frac{1}{|x|}\right).$$

If we pass to the limit when x *tends to* 0*, we obtain that the first term tends to zero, while the second term has no limit at* 0*. We conclude that the partial derivative of* f *with respect to* x *is not continuous at* $(0,0)$*. In the same way, we show that the partial derivative of* f *with respect to* y *is not continuous at* $(0,0)$*. with respect to* y *is not continuous at* $(0,0)$*. Although* f *is not* *of class* C^1 *on* $\mathbb{R}^2$, *it is differentiable on* $\mathbb{R}^2$ *because*

$$\begin{aligned}\lim_{(h_1,h_2)\to(0,0)} \frac{f(h_1,h_2)-f(0,0)-h_1\frac{\partial f}{\partial x}(0,0)-h_2\frac{\partial f}{\partial y}(0,0)}{\sqrt{h_1^2+h_2^2}} &= \lim_{(h_1,h_2)\to(0,0)} \frac{(h_1^2+h_2^2)\sin\left(\frac{1}{\sqrt{h_1^2+h_2^2}}\right)}{\sqrt{h_1^2+h_2^2}}\\ &= \lim_{(h_1,h_2)\to(0,0)} \sqrt{h_1^2+h_2^2}\sin\left(\frac{1}{\sqrt{h_1^2+h_2^2}}\right)\\ &= \lim_{r\to 0} r\sin\left(\frac{1}{r}\right) = 0.\end{aligned}$$

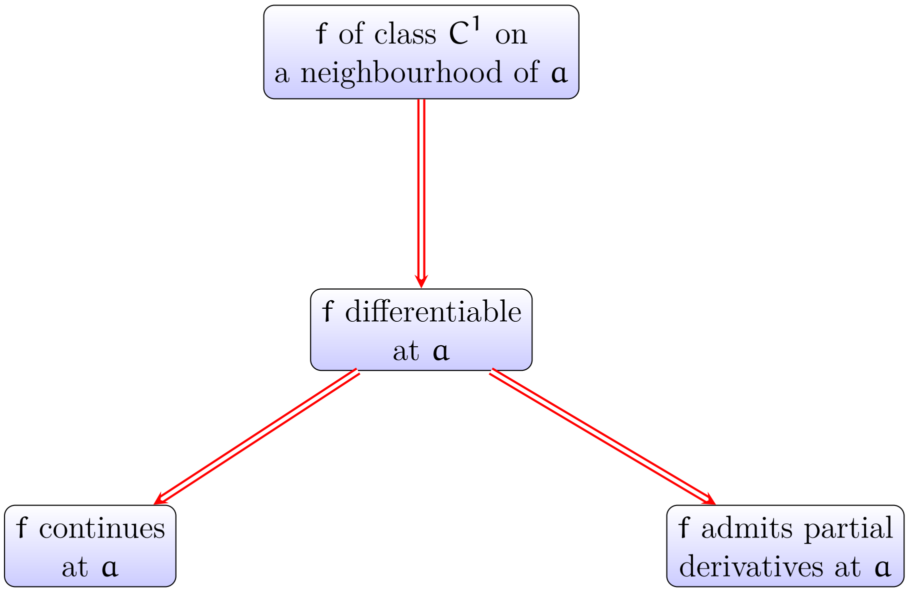


Fig. 3.2 . A diagram showing the relationship between the previous concepts.

3.10 Matrix representation of the differential . Jacobian matrix

Definition 3.16. *Let* Ω *be a non-empty open* $\mathbb{R}^n$ *and*

$$\begin{aligned} f:\ D &\longrightarrow \mathbb{R}^p\\ (x_1,\dots,x_n) &\longmapsto (f_1(x_1,\dots,x_n), f_2(x_1,\dots,x_n),\dots,f_p(x_1,\dots,x_n)),\end{aligned}$$

a function differentiable in $a\in\Omega$. *The matrix of the linear application* $d_a f$ *relative to the canonical bases* B_n *and* B_p *of* $\mathbb{R}^n$ *and* $\mathbb{R}^p$ *respectively is the Jacobian matrix of* f *in* a.

$$M_{B_n,B_p}(d_a f) = J_f(a).$$

It is the same thing to say that, for any $h\in\mathbb{R}^n$:

$$d_a f(h) = J_f(a)h.$$

We denote by $(e_1,\dots,e_n)$ the canonical basis of $\mathbb{R}^n$. The j-th column of $M_{B_n,B_p}(d_a f)$ is the column vector whose components are the coordinates of $d_a f(e_j)$ in the canonical basis of $\mathbb{R}^p$. We have

$$d_a f(e_j) = \frac{\partial f}{\partial x_j}(a).$$

Then,

$$M_{B_n,B_p}(d_a f) = \begin{pmatrix} d_a f(e_1) & d_a f(e_2) & \cdots & d_a f(e_n) \\ \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \cdots & \frac{\partial f_2}{\partial x_n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1}(a) & \frac{\partial f_p}{\partial x_2}(a) & \cdots & \frac{\partial f_p}{\partial x_n}(a) \end{pmatrix} \in M_{p,n}(\mathbb{R}).$$

- If $n = p = 1$, the matrix representing $d_a f$ is the derived number $f'(a) \in M_{1,1}(\mathbb{R}) = \mathbb{R}$.
- If $p = 1$, the matrix representing $d_a f$ is elements of $M_{1,n}(\mathbb{R})$ is a row matrix. Thus, the matrix representation of $d_a f$ is

$$J_f(a) = \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \dots, \frac{\partial f}{\partial x_n}(a)\right) = \nabla f(a).$$

3.11 Operations on differentiable functions

The algebraic operations on differentiable functions concerning sums, products, quotients and multiplication by a scalar are the same as those seen previously in the case of functions derivable from a real variable.

Theorem 8 (Linéarité) Let $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^p$ be two functions differentiable at $a \in \Omega$ and $\lambda, \mu \in \mathbb{R}$. Then the function $\lambda f + \mu g$ is differentiable at a and we have

$$d_a(\lambda f + \mu g) = \lambda d_a f + \mu d_a g.$$

In terms of Jacobians

$$J_{\lambda f + \mu g}(a) = \lambda J_f(a) + \mu J_g(a).$$

Proof. For all $h \in \mathbb{R}^n$, we have,

$$f(a+h) - f(a) = d_a f(h) + \|h\|\epsilon_1(h) \quad \text{and} \quad g(a+h) - g(a) = d_a g(h) + \|h\|\epsilon_2(h),$$

with $\lim_{h\to 0} \epsilon_1(h) = \lim_{h \to 0} \epsilon_2(h) = 0$. So

$$\begin{aligned}(\lambda f + \mu g)(a+h) - (\lambda f + \mu g)(a) &= \lambda(f(a+h) - f(a)) + \mu(g(a+h) - g(a)) \\ &= \lambda d_a f(h) + \mu d_a g(h) + \|h\|(\lambda \epsilon_1(h) + \mu \epsilon_2(h)).\end{aligned}$$

Since $\lambda d_a f + \mu d_a g$ is a linear application of $\mathbb{R}^n$ in $\mathbb{R}^p$, then $\lambda f + \mu g$ is differentiable at a and $d_a(\lambda f + \mu g) = \lambda d_a f + \mu d_a g$. □

Example 3.35. *The function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$\varphi(x, y) = 2x - 3y + 4,$$

is differentiable at any point $a = (x, y) \in \mathbb{R}^2$ *because* $\varphi = f + g$ *with* f *is a linear application*

$f(x, y) = 2x - 3y$ *and* g *is a constant application* $g(x, y) = 4$*. Then, for all*

$$h = (h_1, h_2) \in \mathbb{R}^2, \quad d_a f(h_1, h_2) = 2h_1 - 3h_2 \quad \textit{and} \quad d_a g(h_1, h_2) = 0. \quad \textit{Thus,}$$

$$d_a \varphi(h_1, h_2) = d_a f(h_1, h_2) + d_a g(h_1, h_2) = 2h_1 - 3h_2 + 0 = 2h_1 - 3h_2.$$

Theorem 9 Let $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be two differentiable functions at $a \in \Omega$. Then,

- The product $f \times g$ is differentiable at a and

$$d_a(f \times g) = g(a) d_a f + f(a) d_a g.$$

- if $f(a) \neq 0$, the inverse $\frac{1}{f}$ is differentiable at a and

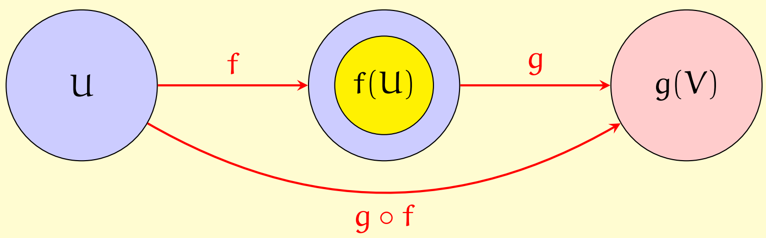
$$d_a \left(\frac{1}{f} \right) = -\frac{d_a f}{(f(a))^2}.$$

- if $g(a) \neq 0$, the quotient $\frac{f}{g}$ is differentiable at a and

$$d_a \left(\frac{f}{g} \right) = \frac{g(a) d_a f - f(a) d_a g}{(f(a))^2}.$$

The chain rule (or rule of differentiation of composite functions) is a central calculus rule in the differentiation of functions of several variables.

Theorem 10 (**Differentiation of composite functions**) Let f be a function of a non-empty open $U \subset \mathbb{R}^n$ to $\mathbb{R}^p$ and g an application of an open $V \subset \mathbb{R}^p$ to $\mathbb{R}^m$ such that $f(U) \subset V$.



Let $a \in U$. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and moreover

$$d(g \circ f)_a = d_{f(a)} g \circ d_a f.$$

En termes de Jacobiennes,

$$J_{g \circ f}(a) = J_g(f(a)) \times J_f(a). \tag{3.2}$$

It is very useful to deconstruct the formula (3.2) in order to know how to apply it. Let

$$\begin{aligned} f : U \subset \mathbb{R}^n &\longrightarrow \mathbb{R}^p \\ (x_1, \ldots, x_n) &\longmapsto f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_p(x_1, \ldots, x_n)), \end{aligned}$$

and

$$\begin{aligned} g : V \subset \mathbb{R}^p &\longrightarrow \mathbb{R}^m \\ (y_1, ..., y_p) &\longmapsto g(y_1, ..., y_p) = (g_1(y_1, ..., y_p), ..., g_m(y_1, ..., y_p)). \end{aligned}$$

Since $J_{g\circ f}(a) = J_g(f(a)) \times J_f(a)$, we obtain

$$\begin{pmatrix} \frac{\partial g_1}{\partial y_1}(f(a)) & \cdots & \frac{\partial g_1}{\partial y_p}(f(a)) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial y_1}(f(a)) & \cdots & \frac{\partial g_m}{\partial y_p}(f(a)) \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1}(a) & \cdots & \frac{\partial f_p}{\partial x_n}(a) \end{pmatrix} \begin{pmatrix} \frac{\partial (g\circ f)_1}{\partial x_1}(a) & \cdots & \frac{\partial (g\circ f)_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial (g\circ f)_m}{\partial x_1}(a) & \cdots & \frac{\partial (g\circ f)_m}{\partial x_n}(a) \end{pmatrix}.$$

So,

$$\frac{\partial (g\circ f)_1}{\partial x_1}(a) = \frac{\partial g_1}{\partial y_1}(f(a))\frac{\partial f_1}{\partial x_1}(a) + \cdots + \frac{\partial g_1}{\partial y_p}(f(a))\frac{\partial f_p}{\partial x_1}(a) = \sum_{k=1}^{p} \frac{\partial g_1}{\partial y_k}(f(a))\frac{\partial f_k}{\partial x_1}(a)$$

$$\vdots$$

$$\frac{\partial (g\circ f)_m}{\partial x_n}(a) = \frac{\partial g_m}{\partial y_1}(f(a))\frac{\partial f_1}{\partial x_n}(a) + \cdots + \frac{\partial g_m}{\partial y_p}(f(a))\frac{\partial f_p}{\partial x_n}(a) = \sum_{k=1}^{p} \frac{\partial g_m}{\partial y_k}(f(a))\frac{\partial f_k}{\partial x_n}(a).$$

Thus, to calculate the partial derivatives of compositions of functions or after a change of variable, we can use the following version of the chain rule. This is the writing in components of the matrix formula (3.2).

Corollary 3.3 (Chain Rule).

$$\forall (i,j) \in \{1, ..., m\} \times \{1, ..., n\}, \quad \frac{\partial (g\circ f)_i}{\partial x_j}(a) = \sum_{k=1}^{p} \frac{\partial g_i}{\partial y_k}(f(a))\frac{\partial f_k}{\partial x_j}(a).$$

Exercise 3.5. *We consider the functions* f *and* g *defined by*

$$\begin{aligned} f :\ \mathbb{R}^2 &\to \mathbb{R}^3 \\ (x, y) &\mapsto (f_1(x,y), f_2(x,y), f_3(x,y)) = (x^2y, xy, xy^3) \end{aligned}$$

and

$$\begin{aligned} g : \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto (g_1(x,y,z), g_2(x,y,z)) = (x+y+z, xyz). \end{aligned}$$

1. *Calculate*
   (a) *the Jacobian matrix of* f *at* $a = (x, y) \in \mathbb{R}^2$.
   (b) *the Jacobian matrix of* g *at* $f(a)$.
   (c) *the Jacobian matrix of* $g \circ f$ *at* a.
2. *Find the Jacobian matrix of* $g\circ f$ *at* a *using the explicit expression of the function* $g \circ f$.

Solution. *It is obvious that* $f \in C^1(\mathbb{R}^2)$ *and* $g \in C^1(\mathbb{R}^3)$.

(a) The Jacobian matrix of f at $a = (x, y) \in \mathbb{R}^2$.

$$J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \\ \frac{\partial f_3}{\partial x}(x,y) & \frac{\partial f_3}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} 2xy & x^2 \\ y & x \\ y^3 & 3xy^2 \end{pmatrix}.$$

(b) The Jacobian matrix of g in $f(a)$. The Jacobian matrix of g at $b = (x, y, z) \in \mathbb{R}^3$

$$J_g(x,y,z) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y,z) & \frac{\partial g_1}{\partial y}(x,y,z) & \frac{\partial g_1}{\partial z}(x,y,z) \\ \frac{\partial g_2}{\partial x}(x,y,z) & \frac{\partial g_2}{\partial y}(x,y,z) & \frac{\partial g_2}{\partial z}(x,y,z) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ yz & xz & xy \end{pmatrix}.$$

So, the Jacobian matrix of g *at* $f(a)$ *is*

$$J_g(f(a)) = J_g\left(x^2y, xy, xy^3\right) = \begin{pmatrix} 1 & 1 & 1 \\ x^2y^4 & x^3y^4 & x^3y^2 \end{pmatrix}.$$

(c) The Jacobian matrix of $g \circ f$ at a.

$$\begin{aligned} J_{g\circ f}(x,y) &= J_g\left(x^2y, xy, xy^3\right) \times J_f(x,y) \\ &= \begin{pmatrix} 1 & 1 & 1 \\ x^2y^4 & x^3y^4 & x^3y^2 \end{pmatrix} \begin{pmatrix} 2xy & x^2 \\ y & x \\ y^3 & 3xy^2 \end{pmatrix} \\ &= \begin{pmatrix} 2xy + y + y^3 & x^2 + x + 3xy^2 \\ 4x^3y^5 & 5x^4y^4 \end{pmatrix}. \end{aligned}$$

Let us find the Jacobian matrix of $g \circ f$ *at* a *using the explicit expression of the function* $g \circ f$:

$$\begin{aligned} g \circ f : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x,y) &\mapsto g \circ f(x,y) = g(f(x,y)) = g\left(x^2y, xy, xy^3\right) = \left(x^2y + xy + xy^3, x^4y^5\right) \end{aligned}$$

It is easy to verify that

$$J_{g\circ f}(x,y) = \begin{pmatrix} 2xy + y + y^3 & x^2 + x + 3xy^2 \\ 4x^3y^5 & 5x^4y^4 \end{pmatrix}.$$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "\varphi"] \arrow[rr, bend left, "f \circ \varphi"] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R}
\end{tikzcd}$$

$$(r,\theta) \longmapsto \begin{aligned}\varphi(r,\theta) &= (x(r,\theta), y(r,\theta)) \\ &= (r\cos\theta, r\sin\theta)\end{aligned} \longmapsto \begin{aligned} g(r,\theta) &= f(x,y) = f\circ\varphi(r,\theta) \\ &= f(r\cos\theta, r\sin\theta)\end{aligned} \longmapsto y$$

Example 3.36. *The Jacobian matrix of* φ *at a point* $(r,\theta) \in \mathbb{R}^2$ *is*

$$J_\varphi(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

This can be expressed as follows,

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$
$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

The chain derivation formula then gives

$$\frac{\partial g}{\partial r}(r,\theta) = \frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\frac{\partial x}{\partial r}(r,\theta) + \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\frac{\partial y}{\partial r}(r,\theta)$$
$$\frac{\partial g}{\partial \theta}(r,\theta) = \frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\frac{\partial x}{\partial \theta}(r,\theta) + \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\frac{\partial y}{\partial \theta}(r,\theta),$$

or in more abridged form

$$\frac{\partial g}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}$$
$$\frac{\partial g}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}$$

Example 3.37 (Solving a partial differential equation of order 2). *Let* $c \neq 0$. *Find the solutions of class* C^2 *of the following partial differential equation*

$$(E) : c^2\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2}.$$

using the change of variable $u = x + ct$ *and* $v = x - ct$. *Consider the application* $\varphi(x,t) = (x+ct, x-ct) = (u,v)$. *Let us note for* $g : \mathbb{R}^2 \to \mathbb{R}$, $f = g \circ \varphi$. *On a ainsi :*

$$\forall (x,t) = g(x+ct, x-ct),$$

or again

$$(x,t) \xrightarrow{\varphi} (x+ct, x-ct) \xrightarrow{g} g(x+ct, x-ct) = f(x,t).$$

Moreover, g *is of class class* C^2 *on* $\mathbb{R}^2$ *if and only if* f *is of class class* C^2 *on* $\mathbb{R}^2$, *then by* *applying the chain derivation rule, it comes*

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial g_u}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial g}{\partial u} + \frac{\partial g}{\partial v},\\ \frac{\partial f}{\partial t} &= \frac{\partial g}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial t} = c\frac{\partial g}{\partial u} - c\frac{\partial g}{\partial v}.\end{aligned}$$

Then, still according to the chain rule and using Schwarz's theorem, we obtain,

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial^2 g}{\partial u^2} + 2\frac{\partial^2 g}{\partial u \partial v} + \frac{\partial^2 g}{\partial v^2},\\ \frac{\partial^2 f}{\partial t^2} &= c^2\frac{\partial^2 g}{\partial u^2} - 2c^2\frac{\partial^2 g}{\partial u \partial v} + c^2\frac{\partial^2 g}{\partial v^2}.\end{aligned}$$

Thus f *is a solution on* $\mathbb{R}^2$ *of the equation* (E) *if and only if* g *is a solution on* $\mathbb{R}^2$ *of*

$$(E') : \ \frac{\partial^2 g}{\partial u \partial v} = 0.$$

So, the general solution of (E') *on* $\mathbb{R}^2$ *is*

$$g(u, v) = A(u) + B(v),$$

where $A, B : \mathbb{R} \to \mathbb{R}$ *are of class* $\mathcal{C}^2$. *Finally, the general solution of equation (E) is*

$$f(x, t) = A(x + ct) + B(x - ct),$$

with A *and* B *are two numerical functions of class* $\mathcal{C}^2$ *on* $\mathbb{R}$.