CHAPTER 2

Limits and continuity of Functions of several variables

Functions in several variables are very widely used to model and represent phenomena involving several parameters. In this chapter, we are mainly interested in generalising definitions and results known in the case of one variable, such as limit and continuity.

Example 2.1. *The volume function of a rectangular box associates the product xyz with the three sides x, y and z of the box. Although this quantity is defined for all real values of x, y and z, it has physical meaning only for strictly positive values of the variables x, y and z. We say that we have defined a function $(x, y, z) \mapsto xyz$ on the domain $\Omega = \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^*$.*

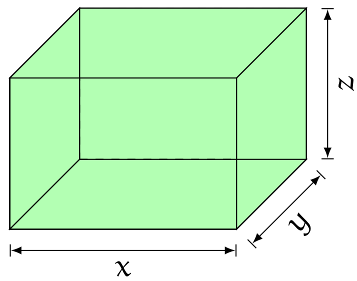


Fig. 2.1 . Rectangular box

Let n and p be two non-zero natural numbers.

Definition 2.1. *Let Ω be a part of $\mathbb{R}^n$. A function of Ω with values in $\mathbb{R}^p$ is a correspondence which associates a single element of Ω with each $(x_1, \ldots, x_n)$ of Ω $\mathbb{R}^p$, noté $f(x_1, \ldots, x_n)$.*

$$f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^p \quad (x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_p(x_1, \ldots, x_n)).$$

The function f is said to be scalar if $p = 1$, with vector values otherwise. Its components $f_1, \ldots, f_p$ are scalar functions.

$$f_i : \Omega \subset \mathbb{R}^n \to \mathbb{R} \quad (x_1, \ldots, x_n) \mapsto f_i(x_1, \ldots, x_n).$$

Example 2.2.

1. *Distance of a point $M(x, y, z) \in \mathbb{R}^3$ from the origin as a function of its coordinates :*

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad (x, y) \mapsto f(x, y) = \sqrt{x^2 + y^2 + z^2}.$$

2. *La fonction de passage des coordonnées polaires en coordonnées cartésiennes*

$$f : \mathbb{R}_+ \times [0, 2\pi[\to \mathbb{R}^2 \quad (r, \theta) \mapsto f(r, \theta) = (r \cos\theta, r \sin\theta).$$

has two component functions, the functions

$$f_1 : \mathbb{R}_+ \times [0, 2\pi[\to \mathbb{R} \quad (r, \theta) \mapsto f_1(r, \theta) = r\cos\theta,$$

$$f_2 : \mathbb{R}_+ \times [0, 2\pi[\to \mathbb{R} \quad (r, \theta) \mapsto f_2(r, \theta) = r\sin\theta.$$

3. *The simplest functions are constant applications that associate the same image* c *with any* x.
$$f : \mathbb{R}^n \to \mathbb{R}^p \quad x = (x_1, \dots, x_n) \mapsto c = (c_1, \dots, c_p).$$

4. *Let* A *be a matrix with* p *rows and* n *columns and real coefficients (we denote by* $M_{p,n}(\mathbb{R})$ *the set of these matrices). We can associate with it the so-called linear application, or endomorphism, which to a vector* $x = (x_1, \dots, x_n)$ *associates a vector* y *by*

$$f : \mathbb{R}^n \to \mathbb{R}^p \quad x \mapsto y = Ax.$$

Linear applications are essential in differential calculus, since most of the major theorems are based essentially on the local approximation of a function by the sum of a constant (the value of the function at the point under study) and a linear application (the differential of the function at this point).

5. *Let* A *be a symmetric square matrix of order* n. *The quadratic form associated with* A *is the scalar function defined by*

$$f : \mathbb{R}^n \to \mathbb{R} \quad x = (x_1, \dots, x_n) \mapsto y = {}^t x A x.$$

Later, we will come across quadratic forms when studying the extrema of a numerical function.

Definition 2.2. *A vector field (resp. scalar) is a function from* $\mathbb{R}^n$ *to* $\mathbb{R}^p$ *with* $p > 1$ *(resp.* $p = 1$*).*

Definition 2.3. *Let* Ω *be a subset of* $\mathbb{R}^n$ *and* f *is a function of* Ω *with values in* $\mathbb{R}^p$.

1. *The set* Ω *is called the domain of definition of* f. *More precisely, the domain of definition of* f *is the largest set* $\Omega \subseteq \mathbb{R}^n$, *such that for each* $x = (x_1, \dots, x_n) \in \Omega$, $f(x)$ *has a meaning (is an element of* $\mathbb{R}^p$*).*

2. *If* $E \subseteq \Omega$, *we call the image of* E *by* f, *the set*

$$f(E) = \{f(x) : x \in E\}.$$

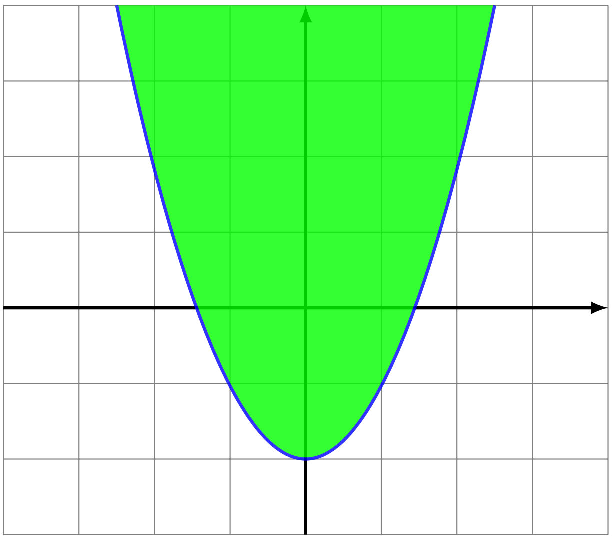
3. *If* $F \subseteq \mathbb{R}^p$, *we call the reciprocal image of* F *by* f, *the set*

$$f^{-1}(F) = \{x \in \Omega : f(x) \in F\}.$$

Example 2.3.

1. *The function*
$$\begin{aligned} f : \ &\mathbb{R}^2 \to \mathbb{R} \\ &(x, y) \mapsto f(x, y) = \sqrt{2 - x^2 + y}, \end{aligned}$$
is defined if, and only if $2 - x^2 + y \geqslant 0$. *Then the domain of definition of* f *is*
$$\Omega = \{(x, y) \in \mathbb{R}^2 : y \geqslant x^2 - 2\},$$

so Ω *is the part above the parabola with equation* $y = x^2 - 2$.

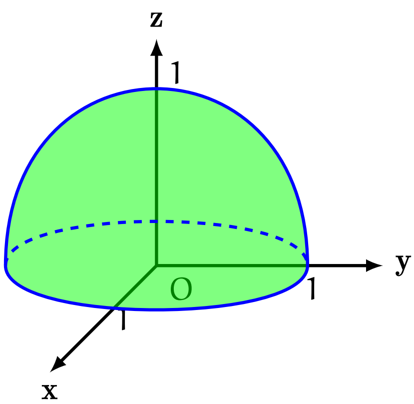


2. *The function*

$$f: \mathbb{R}^3 \to \mathbb{R}$$
$$(x, y, z) \mapsto f(x, y, z) = \sqrt{z \ln(1 - x^2 - y^2 - z^2)},$$

is defined if, and only if $1 - x^2 - y^2 - z^2 > 0$ *and* $z \geqslant 0$. *Then the domain of definition of* f *is*

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : 1 - x^2 - y^2 - z^2 > 0 and z \geqslant 0\}.$$



2.1 Graph of a function

Definition 2.4. *Let* $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^p$ *be a function. The graph of* f *is the set*

$$G_f = \{(x, f(x)) : x \in \Omega\} \subseteq \Omega \times \mathbb{R}^p \subseteq \mathbb{R}^{n+p}.$$

We can see that the graph of a function from $\mathbb{R}^n$ into $\mathbb{R}^p$ is an n-dimensional object in $(n + p)$. The graph of a function is only possible for numerical functions of one or two variables.

Example 2.4. 1. *The graph of a function of a single variable:*

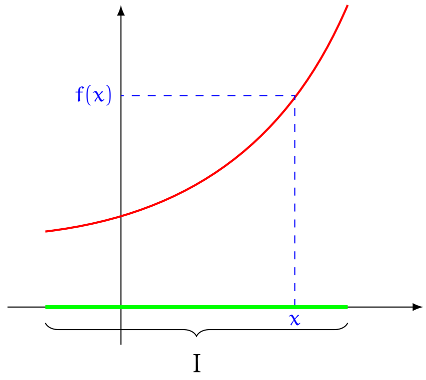
$$f: I \subset \mathbb{R} \to \mathbb{R}$$

$$x \mapsto f(x),$$

is a curve (object of dimension 1) in the plane, defined by

$$G_f = \{(x, f(x)) \in \mathbb{R}^2 : x \in I\},$$

which is represented by the curve in the plane with equation $y = f(x)$.



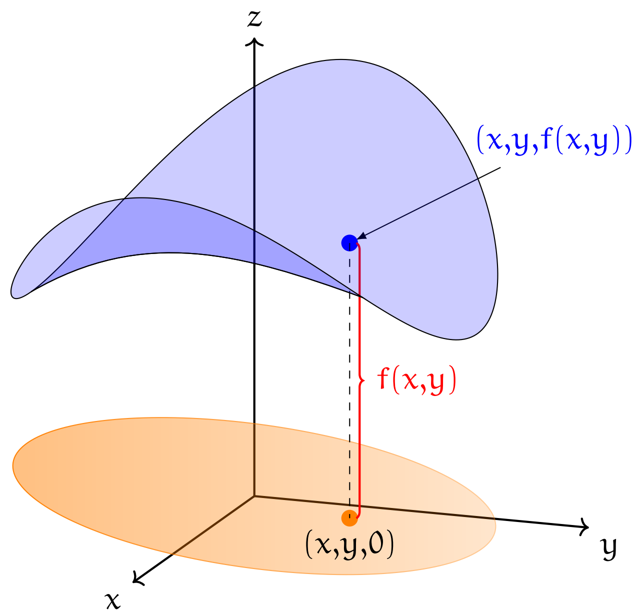
2. *We define the graph of*

$$f : \ \Omega \subset \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto f(x, y),$$

by

$$G_f = \{(x, y, f(x, y)) \in \mathbb{R}^3 : x \in \Omega\},$$

which we represent as a surface with equation $z = f(x, y)$ *in* $\mathbb{R}^3$.



2.2 Level lines

Another way of representing a function of $\mathbb{R}^2$ in $\mathbb{R}$ is to draw its level lines

Definition 2.5. *Let* $k \in \mathbb{R}$ *and* $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ *be a real-valued function. We call the level curve* k *of* f, *the set*

$$C_k = \{x \in \Omega : f(x) = k\}.$$

chapters/chap2/fig/Cas d'une fonction Ãă deux variables.PNG

Fig. 2.2 . Case of a function with two variables

Example 2.5. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x, y) = \sqrt{x^2 + y^2}$.

- *If* $k < 0$, *the level line* C_k *is empty (no point has a negative altitude).*
- *If* $k = 0$, *the level line* C_k *reduces to* $\{(0,0)\}$.
- *If* $k > 0$, *the level line* C_k *is the circle of the plane of centre* $(0,0)$ *and radius* k.

We go back C_k *to the altitude* $z = k$*: the level curve is then the horizontal circle of space with center* $(0,0,k)$ *and radius* k. *The graph is then a superposition of circles horizontals of the space with center* $(0,0,k)$ *and radius* k.

2.3 Limites

The notion of limit and continuity of numerical functions of a single variable is trivially generalized for functions of several variables, it suffices to replace the absolute value by norms in the starting space $\mathbb{R}^n$ and in the arrival space $\mathbb{R}^p$. We assume that $\mathbb{R}^n$ and $\mathbb{R}^p$ each have a norm that we denote indifferently $\|\cdot\|$.

Definition 2.6. *Let* $f : \Omega \to \mathbb{R}^p$ *be a function defined on some subset* $\Omega \subset \mathbb{R}^n$ *and* a *is an accumulation point of* Ω *(i.e.,* $a \in \Omega'$*). We say that* f *has a limit of* $l \in \mathbb{R}^p$ *when* x *tends to* a, *and we denote* $\lim\limits_{x\to a} f(x) = l$, *if*

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \Omega : \|x - a\| \leqslant \delta \implies \|f(x) - l\| \leqslant \epsilon.$$

Remark 2.1. *In the previous definition, in practice we prefer to assume that*

$$\exists r > 0, \; B(a,r) \setminus \{a\} \subset \Omega,$$

which means that f *is defined around the point* a.

Properties 2.1.

- *The notion of limit does not depend on the norms used.*
- *Uniqueness: If it exists, the limit is unique.*
- *Boundedness: If* f *admits a limit at* a*, then* f *is bounded in the neighborhood of* a*. This means that there exist* $\delta > 0$ *and* $M > 0$ *such that*

$$\forall x \in \Omega \cap B(a,\delta), \|f(x)\| \leqslant M.$$

Example 2.6. *Let* $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ *be the function defined by* $f(x,y) = \dfrac{3xy^2}{x^2+y^2}$*. Let us show, using the definition of the limit that*

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

∕et $\epsilon > 0$*, since we have* $x^2 + y^2 \geqslant y^2$*, then for all* $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ *we have,*

$$|f(x,y) - 0| = \left|\frac{3xy^2}{x^2+y^2}\right| \leqslant \left|\frac{3xy^2}{y^2}\right| = 3|x| \leqslant 3\sqrt{x^2+y^2} = 3\|(x,y)\|_2 < \epsilon,$$

so, it is sufficient to take $\delta = \dfrac{\epsilon}{3}$*.*

Proposition 2.1. *Let* Ω *be any subset of* $\mathbb{R}^n$*,* $a \in \Omega'$*,* $l = (l_1, \dots, l_p) \in \mathbb{R}^p$ *and*

$$\begin{aligned} f : \Omega &\to \mathbb{R}^p \\ x = (x_1, \dots, x_n) &\mapsto f(x) = (f_1(x), \dots, f_p(x)). \end{aligned}$$

Then,

$$\lim_{x\to a} f(x) = l \iff \lim_{x\to a} f_i(x) = l_i \text{ for all } i = 1, \dots, p.$$

In view of the above proposition, we can say that f has a limit at a if and only if each of its components has a limit at a and in this case

$$\lim_{x\to a} f(x) = \left(\lim_{x\to a} f_1(x), \dots, \lim_{x\to a} f_p(x)\right) = (l_1, \dots, l_p) = l.$$

Example 2.7. *Let* $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^3$ *the function defined by*

$$f(x,y) = \left(\frac{3xy^2}{x^2+y^2}, x^2 + \sin y - 1, \cos xy\right).$$

We have,

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f_1(x,y) &= \lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^2} = 0, \\ \lim_{(x,y)\to(0,0)} f_2(x,y) &= \lim_{(x,y)\to(0,0)} x^2 + \sin y - 1 = -1, \\ \lim_{(x,y)\to(0,0)} f_3(x,y) &= \lim_{(x,y)\to(0,0)} \cos xy = 1. \end{aligned}$$

Then,

$$\lim_{(x,y)\to(0,0)} f(x,y) = (0, -1, 1).$$

The properties of the limit of a function of several variables are the same as those of the limit of a function of a single variable. The framing theorem is used instead for real-valued functions.

Theorem 1 (**Frame theorem**) Let Ω be a non-empty part of $\mathbb{R}^n$, $a \in \Omega'$, and $f, g_1, g_2 : \Omega \to \mathbb{R}$ be three real-valued functions such that

- $\forall x \in \Omega : \; g_1(x) \leqslant f(x) \leqslant g_2(x)$.
- $\lim\limits_{x \to a} g_1(x) = \lim\limits_{x \to a} g_2(x) = l$.

Then $\lim\limits_{x \to a} f(x) = l$.

Example 2.8. *Let* $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ *the function defined by* $f(x,y) = \dfrac{x^4 y^2}{x^4 + y^2}$. *Let us show, using the framing theorem that* $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. *Since* $x^2 + y^2 \geqslant y^2$, *then for all* $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ *we have*

$$0 \leqslant f(x,y) \leqslant \frac{x^4 y^2}{y^2} = x^4 \xrightarrow[(x,y)\to(0,0)]{} 0.$$

The following theorem states that the study of the limits of vector-valued functions can be reduced to the study of real-valued functions.

Theorem 2 Let Ω be a nonempty part of $\mathbb{R}^n$, $a \in \Omega'$, $f : \Omega \to \mathbb{R}^p$, $l \in \mathbb{R}^p$ and $g : \Omega \to \mathbb{R}$ such that

- $\forall x \in \Omega : \|f(x) - l\| \leqslant g(x)$.
- $\lim\limits_{x \to a} g(x) = 0$.

Alors $\lim\limits_{x \to a} f(x) = l$.

2.3.1 Sequential characterization of the limit

The sequential characterization of the limit also extends to the case of a function with several variables.

Theorem 3 Let Ω be a nonempty part of $\mathbb{R}^n$, $a \in \Omega'$, $f : \Omega \to \mathbb{R}^p$, $l \in \mathbb{R}^p$. The following assertions are equivalent.

1. f admits l as limit at a.
2. For any sequence $(x_k)_k$ of elements of Ω that converges to a, we have

$$\lim_{k \to \infty} f(x_k) = l.$$

This characterization is often used to show that a function does not admit a limit, for example by showing two sequences $(x_k)_k$ and $(y_k)_k$ converging to a such that $(f(x_k))_k$ and $(f(y_k))_k$ do not have the same limit.

Example 2.9. *Let* $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$*the function defined by* $f(x,y) = \dfrac{xy}{x^2+y^2}$. *Let us show that* $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ *does not exist. Consider the two sequences of elements of* $\mathbb{R}^2 \setminus \{(0,0)\}$ *defined by*

$$x_k = \left(\frac{1}{k}, \frac{1}{k}\right) \quad \text{and} \quad y_k = \left(\frac{1}{k}, \frac{2}{k}\right).$$

It is clear that

$$x_k = \left(\frac{1}{k}, \frac{1}{k}\right) \xrightarrow[k\to\infty]{} (0,0) \quad \text{and} \quad y_k = \left(\frac{1}{k}, \frac{2}{k}\right) \xrightarrow[k\to\infty]{} (0,0).$$

While,

$$f(x_k) = \frac{\frac{1}{k}\cdot\frac{1}{k}}{\left(\frac{1}{k}\right)^2 + \left(\frac{1}{k}\right)^2} = \frac{1}{2} \xrightarrow[k\to\infty]{} \frac{1}{2} \quad \text{and} \quad f(y_k) = \frac{\frac{1}{k}\cdot\frac{2}{k}}{\left(\frac{1}{k}\right)^2 + \left(\frac{2}{k}\right)^2} = \frac{2}{5} \xrightarrow[k\to\infty]{} \frac{2}{5}.$$

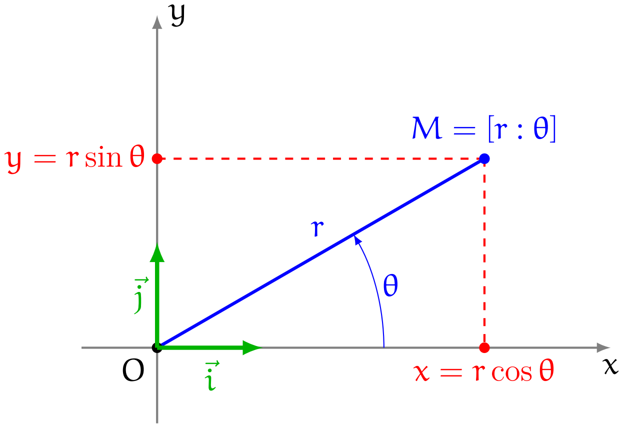
2.3.2 Change of variables

It is often useful to switch to spherical coordinates to reduce the calculation of the limit of a function of several variables to that of the limit of a function of a single variable. For a point x of $\mathbb{R}^2$, with Cartesian coordinates $(x_1, \ldots, x_n)$, we define the spherical coordinates $(r, \theta_1, \ldots, \theta_{n-1})$ by

$$\begin{aligned}
r &= \sqrt{x_1^2 + \ldots + x_n^2},\\
x_1 &= r\cos\theta_1,\\
x_2 &= r\sin\theta_1\cos\theta_2,\\
&\vdots\\
x_{n-1} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\cos\theta_{n-1},\\
x_n &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\sin\theta_{n-1}.
\end{aligned}$$

Polar coordinates constitute the special case $n = 2$. We define the (bijective) application of change of variable in polar coordinates by

$$\begin{aligned}
\varphi : \mathbb{R}_+^* \times [0, 2\pi[&\longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}\\
(r,\theta) &\longmapsto (r\cos\theta, r\sin\theta).
\end{aligned}$$



When considering applications $f : \ \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, it is sometimes easier to calculate the limit by going through polar coordinates.

Proposition 2.2. *Let* f *be a function defined in the neighborhood of* $(a,b) \in \mathbb{R}^2$, *except perhaps in* (a,b). *If*

$$\lim_{r\to 0}(a + r\cos\theta, b + r\sin\theta) = l \in \mathbb{R},$$

exists independently of θ, *then*

$$\lim_{(x,y)\to(a,b)} f(x,y) = l.$$

Example 2.10. *Let* $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ *the function defined by* $f(x,y) = \dfrac{xy}{x^2+y^2}$.

$$f(r\cos\theta, r\sin\theta) = \frac{r^2\cos\theta\sin\theta}{r^2} = r\cos\theta\sin\theta.$$

Since $|\cos\theta\sin\theta| \leqslant 1$ then $r\cos\theta\sin\theta \to 0$ when $r \to 0$. Which implies $f(r\cos\theta, r\sin\theta) \to 0$ when $r \to 0$. The limit exists (independently of θ), So

$$\lim_{(x,y)\to(a,b)} f(x,y) = 0.$$

Remark 2.2. *If the limit* $\lim\limits_{r\to 0} f(a + r\cos\theta, b + r\sin\theta)$ *depends on* θ,, *then* f *has no limit at point* (a,b).

Example 2.11. *Let* $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ *the function defined by* $f(x,y) = \dfrac{2xy}{x^2+y^2}$.

$$\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = \lim_{r\to 0} \frac{2r\cos\theta\sin\theta}{r^2} = \lim_{r\to 0} \frac{2\cos\theta\sin\theta}{r^2} = \sin 2\theta.$$

Since this limit depends on θ, *then* $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ *does not exist.*

2.4 Continuity

Definition 2.7 (Continuous function). *Let* f *be a function defined on a subset* $\Omega \subset \mathbb{R}^n$ *with values in* $\mathbb{R}^p$. *We say that* f *is continuous at* $a \in \Omega$ *if and only if*

$$\lim_{x\to a} f(x) = f(a).$$

In other words

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \Omega : \|x - a\| \leqslant \delta \implies \|f(x) - f(a)\| \leqslant \epsilon.$$

We say that f *is discontinuous at* a *if and only if* f *is not continuous at* a..

Definition 2.8 (Continuity on a set). *Let* Ω *be a non-empty subset of* $\mathbb{R}^n$ *and* $f : \Omega \to \mathbb{R}^p$. *We say that* f *is continuous on* Ω *if it is continuous at every point of* Ω. *The set of continuous functions from* Ω *to* $\mathbb{R}^p$ *will be denoted by* $C(\Omega; \mathbb{R}^p)$ *or* $C^0(\Omega; \mathbb{R}^p)$. *If* $p = 1$, *we will simply denote by* $C(\Omega)$ *or* $C^0(\Omega)$.

Example 2.12. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *a function defined by*

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0). \end{cases}.$$

We have f *is continuous at the point* $(0,0)$ *because*

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} \\
&= \lim_{r\to 0} \frac{r^3(\cos^3\theta+\sin^3\theta)}{r^2} \\
&= \lim_{r\to 0} r(\cos^3\theta+\sin^3\theta) \\
&= 0 = f(0,0).
\end{aligned}$$

Moreover, f *is continuous at every point* $(x,y)\neq 0$ *as the quotient of two continuous functions. Thus,* f *is continuous on* $\mathbb{R}^2$.

Example 2.13. *Let* $f:\ \mathbb{R}^2\to\mathbb{R}$ *the function defined by*

$$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2} & :(x,y)\neq(0,0)\\ 0 & :(x,y)=(0,0)\end{cases}$$

We have

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \\
&= \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta.
\end{aligned}$$

Since this limit depends on θ, *then* $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ *does not exist. Thus,* f *is not continuous at the point* $(0,0)$.

Example 2.14 (Continuity of the projections). *Let* $i\in\{1,\ldots,n\}$, *the* i*-th projection defined as*

$$\begin{aligned}
p_i &: \mathbb{R}^n \to \mathbb{R} \\
x = (x_1,\ldots,x_n) &\mapsto p_i(x) = x_i,
\end{aligned}$$

is continuous on $\mathbb{R}^n$. *In fact, let* $a\in\mathbb{R}^n$ *and let* $\epsilon>0$

$$|p_i(x)-p_i(a)| = |x_i-a_i| \leqslant \|x-a\|_\infty \leqslant \epsilon.$$

We see that we can take $\delta=\epsilon$.

As in the case of functions of a single variable, continuity can be characterized using sequences.

Theorem 4 (Sequential characterization of continuity) Let $f:\ \Omega\subset\mathbb{R}^n\to\mathbb{R}^p$ and $a\in\Omega$. Then, f is continuous at a if and only if, for any sequence $(x_n)_{n\in\mathbb{N}}$ of elements of Ω converging to a, we have,

$$f(x_n)\to f(a).$$

In practice, this theorem is mainly used to prove that a function is discontinuous.

Example 2.15 **(Indicator of $\mathbb{Q}$).** *Let $f: \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} 1 & : x \in \mathbb{Q} \\ 0 & : x \notin \mathbb{Q} \end{cases}$$

The function f is discontinuous at all points $a \in \mathbb{R}$. Indeed, let $a \in \mathbb{R}$. Since $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$, we can find a sequence $(u_n) \subset \mathbb{Q}$ and a sequence $(v_n) \subset \mathbb{R} \setminus \mathbb{Q}$ such that $u_n \to a$ and $v_n \to a$. But, for each n, we have $f(u_n) = 1$ and $f(v_n) = 0$.
The sequences $(f(u_n))$ and $(f(v_n))$ converge to different limits while $f(a)$ is unique.

Proposition 2.3. *Let Ω be any subset of $\mathbb{R}^n$, $a \in \Omega$ and*

$$\begin{aligned} f:\ \Omega &\to \mathbb{R}^p \\ x = (x_1, \dots, x_n) &\mapsto f(x) = (f_1(x), \dots, f_p(x)). \end{aligned}$$

The function f is continuous at the point a, if and only if, its coordinate functions $f_1, \dots, f_p$ are continuous at a.

Remark 2.3. *The usual functions are continuous on their definition set. In particular, the exponential, logarithmic and trigonometric functions.*

Example 2.16. *The function $f: (x, y) \mapsto (\cos x, \ln y)$ is defined and continues on $\mathbb{R} \times]0, +\infty[$.*

2.5 Operations on continuous functions

Continuous functions of several variables have the same basic properties as continuous functions of a single variable.

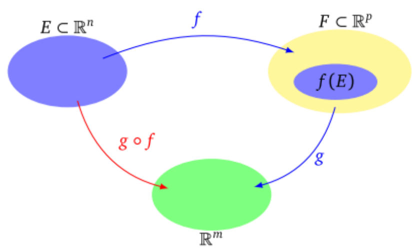
Theorem 5 **[Algebraic properties of continuous functions]**
Let Ω be a non-empty subset of $\mathbb{R}^n$, $a \in \Omega$, $f, g : \Omega \to \mathbb{R}^p$ et $\lambda : \Omega \to \mathbb{R}$ three continuous functions on Ω. Then,

- $f + g$ and λf are continuous on Ω.
- If $\forall x \in \Omega, \lambda(x) \neq 0$, then $\dfrac{1}{\lambda}$ is continuous on Ω.

Theorem 6 **(Composition)**
Let $f: E \subset \mathbb{R}^n \to \mathbb{R}^p$ be continuous on E and $g: F \subset \mathbb{R}^p \to \mathbb{R}^m$ be continuous on F such that $f(E) \subset F$, then the composite

$$\begin{aligned} g \circ f : E &\to \mathbb{R}^m, \\ x &\mapsto g \circ f(x) = g(f(x)), \end{aligned}$$

is continuous on E.



Corollary 2.1 (**Continuity along a path**). *If* $f : \mathbb{R}^n \to \mathbb{R}^p$ *is a continuous function then for any continuous function* $\gamma : I \subset \mathbb{R} \to \mathbb{R}^n$*, the compound* $f \circ \gamma : I \to \mathbb{R}^p$ *is continuous. The application* γ *is called a path (or parametric curve) of* $\mathbb{R}^n$ *and* $f \circ \gamma : t \mapsto f(\gamma(t))$ *is called the restriction of* f *along* γ*.*

Remark 2.4. *Corollary 2.1 is often used in its contrapositive version. To prove that a function* f *is not continuous at* a*, it suffices to find a path* $\gamma : I \to \mathbb{R}^n$ *passing through* a*, that is,*

$$\exists t_0 \in I, \gamma(t_0) = a,$$

such that the restriction of f *along* γ *is discontinuous at* t_0*.*

Example 2.17. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *the function defined by*

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & : x \neq -y \\ 0 & : x = -y \end{cases}$$

If we take the path $\gamma(t) = (t, t^4 - t)$ *which passes through* $(0,0)$ *because* $\gamma(0) = (0,0)$*. Then the restriction of* f *along* γ*:*

$$f \circ \gamma(t) = f(t, t^4 - t) = \begin{cases} t^2 - \frac{1}{t} & : t \neq 0 \\ 0 & : t = 0 \end{cases}$$

is discontinuous at $t_0 = 0$*. Therefore* f *is not continuous at the point* $(0,0)$*.*

In view of Corollary 2.1, we obtain,

Corollary 2.2 (**Limit along a path**). *Let* $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^p$ *a function defined in the neighbourhood of* $a \in \mathbb{R}^n$*, except perhaps in* a*. If* f *has a limit* l *in* a*, the restriction of* f *to any continuous curve passing through* a *has the same limit* l*.*

Remark 2.5. *The uniqueness of the limit implies that, however you tend towards* a*, the limit value remains the same.*

Remark 2.6. *To prove that a function* f *does not have a limit in* a*, it is sufficient to explain a restriction of* f *to a continuous curve passing through* a *which does not have a limit in* a*, or two restrictions of* f *to two continuous curves passing through* a *have different limits.*

Example 2.18. *Let* $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ *the function defined by* $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$*.*

- *If we take the path* $\gamma_1(t) = (t, 0)$*, the restriction of* f *along* γ_1 *is*

$$f \circ \gamma_1(t) = f(t,0) = \frac{t^2}{t^2} = 1 \quad \longrightarrow_{t\to 0} 1.$$

- *If we take the path* $\gamma_2(t) = (0, t)$*, the restriction of* f *along* γ_2 *is*

$$f \circ \gamma_2(t) = f(0,t) = \frac{-t^2}{t^2} = -1 \quad \longrightarrow_{t\to 0} -1.$$

Since the restrictions of f *to two continuous curves passing through* $(0,0)$ *give two different limits, we conclude that* $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ *does not exist.*

Example 2.19. *Let* $f : \Omega = \{(x,y) \in \mathbb{R}^2 : y \neq -x\} \to \mathbb{R}$ *the function defined by*

$$f(x,y) = \frac{x^2 y}{x+y}.$$

If we take the path $\gamma(t) = (t, t^4 - t)$*, then the restriction of* f *along* γ*:*

$$f \circ \gamma(t) = f(t, t^4 - t) = t^2 - \frac{1}{t}.$$

We have $\lim\limits_{t\to 0} t^2 - \frac{1}{t} = -\infty$ *and* $\lim\limits_{t\to 0} t^2 - \frac{1}{t} = +\infty,$*, hence* $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ *does not exist.*

Remark 2.7. *To prove that a function* f *admits* l *as a limit at* a*, we need to consider the general case. If we just know that the restriction to any line through* a *admits the same limit, we cannot conclude that the limit of* f *exists.*

Example 2.20. *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *la function defined by*

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2 - 2xy + 3y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

Let us show that the restriction of f *to any line through the origin has limit* 0 *at* $(0,0)$ *but that* f *has no limit at* $(0,0)$*.*
The restriction of f *to the lines* $x = 0$ *and* $y = 0$ *is the zero function. Furthermore, the restriction of* f *to the line* $y = mx$*, with* $m = 0$*, gives*

$$f(x, mx) = \frac{mx}{x^2 - 2mx + 3m^2} \longrightarrow_{x\to 0} 0.$$

But restricting f *to the parabola* $y = x^2$ *gives*

$$f(x, x^2) = \frac{x^4}{x^4 - 2x^4 + 3x^4} = \frac{1}{2} \longrightarrow_{x\to 0} \frac{1}{2}.$$

hence $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ *does not exist.*

2.6 Fundamental examples

Let's give some examples of continuous functions.

2.6.1 Polynomials

Definition 2.9. *A polynomial of* n *variables of degree* d *is a function* $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ *of the form*

$$p(x) = \sum_{\mathbb{N}^n,\, k_1+\cdots+k_n \leqslant d} a_k x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n},$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$*.*

Example 2.21. *In dimension $n = 2$, a polynomial of degree $d = 2$ is written in the form,*

$$p(x,y) = a_0 + a_1x + a_2y + a_3xy + a_4x^2 + a_5y^2,$$

with $a_0, \dots, a_5 \in \mathbb{R}$.

Proposition 2.4. *Polynomial functions are continuous on $\mathbb{R}^n$.*

Example 2.22. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) = \begin{cases} \dfrac{2x^2 + y^2 - 1}{x^2} & : x^2 + y^2 > 1, \\ 1 & : \textit{sinon}. \end{cases}$$

Let's show that f is continuous on $\mathbb{R}^2$.
Let's put $U_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$ et $U_2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. U_1 and U_2 are two open sets of $\mathbb{R}^2$ on which f has a polynomial expression. Therefore f is continuous on U_1 and on U_2. It therefore remains to verify the continuity of f at a point (a, b) such that $a^2 + b^2 = 1$.

- *If $(x,y) \to (a, b)$ with $(x, y) \in U_1$, then,*

$$f(x, y) \to \frac{2a^2 + b^2 - 1}{a^2} = \frac{a^2}{a^2} = 1 = f(a, b)$$

- *If $(x,y) \to (a, b)$ with $(x, y) \in U_2$, then,*

$$f(x, y) \to 1 = f(a, b)$$

We have,

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b),$$

and the function f is continuous in (a, b).

2.6.2 Lipschitzian functions

Definition 2.10. *Let Ω be a non-empty part of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^p$. We say that f is lipschitzian on Ω if there exists a constant $L > 0$ such that*

$$\forall x, y \in \Omega, \|f(x) - f(y)\| \leqslant L\|x - y\|.$$

The constant L is called the Lipschitz constant, and we also say that f is L-Lipschitzian.

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^n$. Any lipschitzian function on Ω is continuous on Ω.*

Proof. Let $a \in \Omega$ and $\epsilon > 0$. We set $\delta = \dfrac{\epsilon}{L}$, then for all $x \in \Omega$ with $\|x - a\| \leqslant \delta$ we have,

$$\|f(x) - f(a)\| \leqslant \delta L = \epsilon.$$

$\square$

Example 2.23. *If $\mathcal{N}$ is a norm on $\mathbb{R}^n$, then $\mathcal{N}$ is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$. Indeed, for all $x, y \in \mathbb{R}^n$*

$$\mathcal{N}(x) = \mathcal{N}(x - y + y) \leqslant \mathcal{N}(x - y) + \mathcal{N}(y),$$

hence,

$$\mathcal{N}(x) - \mathcal{N}(y) \leqslant \mathcal{N}(x - y).$$

By replacing x by y and y by x, we obtain

$$\mathcal{N}(y) - \mathcal{N}(x) \leqslant \mathcal{N}(x - y),$$

and we conclude that for all $x, y \in \mathbb{R}^n$

$$|\mathcal{N}(x) - \mathcal{N}(y)| \leqslant \mathcal{N}(x - y).$$

This implies that $\mathcal{N}$ is a $1-$ lipschitzian function and therefore continuous.

Corollary 2.3. *Any linear application $f : \mathbb{R}^n \to \mathbb{R}^p$ is continuous. More precisely, there exists a constant $M > 0$ such that*

$$\forall x \in \mathbb{R}^n, \|f(x)\| \leqslant M\|x\|.$$

In particular, any linear application is Lipschitzian.

2.6.3 Partial functions

Definition 2.11. *Let f be a function of an open Ω of $\mathbb{R}^n$ in $\mathbb{R}^p$. Let $i \in \{1, \ldots, n\}$ and $a \in \Omega$. Let $\Omega_{i,a} = \{t \in \mathbb{R} : (a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n) \in \Omega\}$ (is an open of $\mathbb{R}$). The function of a single real variable*

$$\begin{aligned} f_{i,a} : \Omega_{i,a} \subset \mathbb{R} &\to \mathbb{R}^p \\ t &\mapsto f_{i,a}(t) = f(a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n), \end{aligned}$$

is called i-th partial function of f in a.

Remark 2.8. *If f has real values, the graph of the i-th partial function of f in a is obtained by intersecting the graph of f with all the hyperplanes $x_j = a_j$ $(j \neq i)$.*

Example 2.24. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = 10 - 4x^2 - y^2$. The partial functions associated with the point $a = (1,2)$ are*

$$\begin{aligned} f_{1,a} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f_{1,a}(x) = f(x,2) = 6 - 4x^2, \end{aligned} \qquad \begin{aligned} f_{2,a} : \mathbb{R} &\to \mathbb{R} \\ y &\mapsto f_{2,a}(y) = f(1,y) = 6 - y^2. \end{aligned}$$

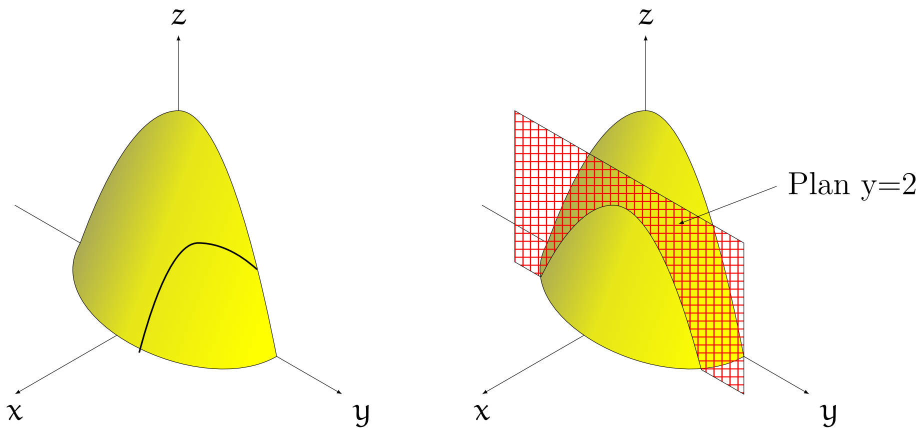


Fig. 2.3 . The graph of the function $f_{1,a}$

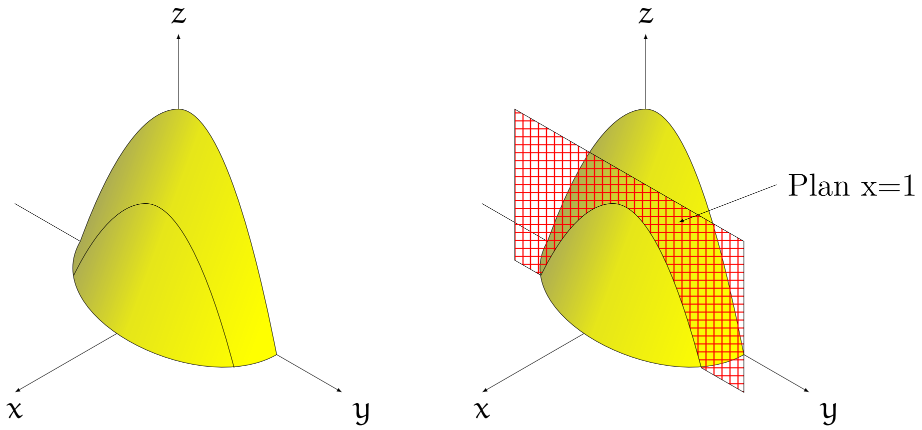


Fig. 2.4 . The graph of the function $f_{2,a}$

Proposition 2.6. *If $f:\ \Omega \subset \mathbb{R}^n \to \mathbb{R}^p$ is continuous on Ω, then the partial function $f_{i,a}$ is continuous on $\Omega_{i,a}$.*

In general, the converse is false as shown in the following example

Example 2.25. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ the function defined by*

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0). \end{cases}$$

The two partial functions of f at $a = (0,0)$ are the functions defined on $\mathbb{R}$ by

- $\forall x \in \mathbb{R},\ f_{1,a}(x) = f(x,0) = 0.$
- $\forall y \in \mathbb{R},\ f_{2,a}(y) = f(0,y) = 0.$

These two partial functions are zero and therefore continuous on $\mathbb{R}$ and in particular at 0. However, we have already shown that f is discontinuous at $(0,0)$ (see Example 2.13).

2.6.4 Extension by continuity

Definition 2.12. *Let Ω be a non-empty part of $\mathbb{R}^n$, a an accumulation point of Ω which does not belong to Ω and $f: \Omega \to \mathbb{R}^p$. If f admits It is said that f is prolongable (extension) by continuity at the point a if it admits a limit l in a. In this case, the prolonged function $\tilde{f}: \Omega \cup \{a\} \to \mathbb{R}^p$ defined by*

$$\tilde{f}(x) = \begin{cases} f(x) & : x \in \Omega \\ l & : x = a \end{cases}$$

is continuous in a.

Example 2.26. *Consider the function $f: \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ defined by*

$$f(x,y) = xy \ln(x^2 + y^2).$$

We have,

$$\forall (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}, |f(x,y)| \leqslant \frac{1}{2}(x^2 + y^2) \ln(x^2 + y^2).$$

Or

$$\lim_{(x,y)\to(0,0)} \frac{1}{2}(x^2 + y^2) \ln(x^2 + y^2) = \lim_{r\to 0} \frac{1}{2} r^2 \ln(r^2) = 0.$$

So $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. *Thus f is extendable by continuity at the point $(0,0)$ and the extension by continuity $\tilde{f}: \mathbb{R}^2 \to \mathbb{R}$ is given by*

$$\tilde{f}(x) = \begin{cases} xy \ln(x^2 + y^2) & : (x,y) \neq (0,0), \\ 0 & : (x,y) = (0,0). \end{cases}$$

Example 2.27. *Let $f: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be the function defined by*

$$f(x,y) = \frac{\sin(x^2) - \sin(y^2)}{x^2 + y^2}.$$

- *If we take the path $\gamma_1(t) = (t,0)$, the restriction of f along γ_1 is*

$$f \circ \gamma_1(t) = f(t,0) = \frac{\sin(t^2)}{t^2} \xrightarrow[t\to 0]{} 1.$$

- *If we take the path $\gamma_2(t) = (0,t)$, the restriction of f along γ_2 is*

$$f \circ \gamma_2(t) = f(0,t) = \frac{-\sin(t^2)}{t^2} \xrightarrow[t\to 0]{} -1.$$

We conclude that $\lim_{(x,y)\to(0,0)} f(x,y)$ *does not exist. Thus f is not extendable by continuity in $(0,0)$.*

2.7 Topological properties associated with continuity

Continuity can be characterised using open or closed sets.

Theorem 7 Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a function. The following assertions are equivalent

1. f is continuous on Ω.
2. For any open V of $\mathbb{R}^p$, there exists an open U of $\mathbb{R}^n$ such that $f^{-1}(V) = U \cap \Omega$ (in particular if Ω is an open then $f^{-1}(V)$ is an open of $\mathbb{R}^n$).
3. For any closed set F of $\mathbb{R}^p$, there exists a closed set E of $\mathbb{R}^n$ such that $f^{-1}(F) = E \cap \Omega$ (in particular if Ω is a closed set then $f^{-1}(F)$ is a closed set of $\mathbb{R}^n$).

Example 2.28. *The set* $U = \{(x,y) \in \mathbb{R}^2 : x^4 - y^2 > 0\}$ *is an open set of* $\mathbb{R}^2$. *Indeed, the function*

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$(x,y) \to f(x,y) = x^4 - y^2.$$

is continuous on $\mathbb{R}^2$ *and* $U = f^{-1}(]0, +\infty[)$ *with* $]0, +\infty[$ *is an open interval of* $\mathbb{R}$. *We deduce that* U *is open of* $\mathbb{R}^2$.

Theorem 8 Let $\mathcal{K}$ be a compact (closed and bounded) subset of $\mathbb{R}^n$. If $f : \mathcal{K} \to \mathbb{R}^p$ is continuous on $\mathcal{K}$, then $f(\mathcal{K})$ is a compact subset of $\mathbb{R}^p$.

The inverse image of a compact by a continuous function is not compact in general as the following example shows.

Example 2.29. *The null function*

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f(x) = 0, \end{aligned}$$

is continuous on $\mathbb{R}$ *and* $\{0\}$ *is compact, but* $f^{-1}(\{0\}) = \mathbb{R}$ *is not compact.*

Corollary 2.4. *Let* $\mathcal{K}$ *be a compact subset of* $\mathbb{R}^n$. *If* $f : \mathcal{K} \to \mathbb{R}^p$ *be a continuous function on* $\mathcal{K}$, *then*

- f *is bounded on* $\mathcal{K}$, *that is,*

$$\exists M \geqslant 0, \forall x \in \mathcal{K}, \|f(x)\| \leqslant M.$$

- f *achieves its bounds, that is, there exist* $a, b \in \mathcal{K}$ *such that*

$$f(a) = \min_{x \in \mathcal{K}} f(x) and f(b) = \max_{x \in \mathcal{K}} f(x).$$