CHAPTER 1

Topology of $\mathbb{R}^n$

Throughout this course, we designate by $\mathbb{R}^n$ ($n \geqslant 1$ being an integer) the Cartesian product

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{n\text{factors}}$$

In other words, $\mathbb{R}^n$ is the set of n-ordered tuples $(x_1, \ldots, x_n)$ of real numbers. We consider the following operations:

- **Addition :** $\forall x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n : x + y = (x_1 + y_1, \ldots, x_n + y_n)$
- **Multiplication by a scalar:** $\forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \forall \lambda \in \mathbb{R} : \ \lambda x = (\lambda x_1, \ldots, \lambda x_n)$

Given these two operations, $\mathbb{R}^n$ is a vector space on $\mathbb{R}$ of dimension n and has as canonical basis the n vectors $e_1 = (1, 0, \ldots), e_2 = (0, 1, 0, \ldots)$ and $e_n = (0, \ldots, 0, 1)$.

1.1 Normes

Definition 1.1. *A norm on $\mathbb{R}^n$ is any application $\mathcal{N} : \mathbb{R}^n \to [0, +\infty[$ having the following properties:*

1. ***Separation :*** *Pour tout $x \in \mathbb{R}^n, \mathcal{N}(x) = 0 \iff x = 0$.*
2. ***Homogeneity:*** *For all $x \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}, \mathcal{N}(\lambda x) = |\lambda| \mathcal{N}(x)$.*
3. ***Triangular inequality:*** *For all $x, y \in \mathbb{R}^n, \mathcal{N}(x + y) \leqslant \mathcal{N}(x) + \mathcal{N}(y)$.*

Remark 1.1. *Equipped with a norm, $\mathbb{R}^n$ is said to be a normed vector space. We generally use the notation $\|x\|$ for $\mathcal{N}(x)$, which recalls the analogy with the absolute value in $\mathbb{R}$.*

Example 1.1. *In dimension $n = 1$, we check that if $\mathcal{N}$ is a norm on $\mathbb{R}$, then there exists a unique $\beta > 0$ such that $\mathcal{N}(x) = \beta |x|$.*

Example 1.2 (Normes usuelles). *Let $p \in [1, +\infty[$. On $\mathbb{R}^n$, the following norms defined for $x = (x_1, x_2, \ldots, x_n)$ by*

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}, \quad \|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$

If $p = 2$, we call the Euclidean norm.

Properties 1.1. *Let $\mathcal{N}$ be a norm on $\mathbb{R}^n$. Then*

- *$\mathcal{N}$ 1-**lipschitzian** application, in other words, for all $x, y \in \mathbb{R}^n$,*

$$|\mathcal{N}(x) - \mathcal{N}(y)| \leqslant \mathcal{N}(x - y).$$

- *$\mathcal{N}$ is a **convex** application, that is to say:*

$$\forall \lambda \in [0, 1], \forall x, y \in \mathbb{R}^n : \mathcal{N}(\lambda x + (1 - \lambda)y) \leqslant \lambda\mathcal{N}(x) + (1 - \lambda)\mathcal{N}(y).$$

We will now state a fundamental theorem which states that the choice of norm does not affect the topological properties of functions in $\mathbb{R}^n$ (continuity, limit, etc.).

Definition 1.2. *Two norms $\mathcal{N}_1$ and $\mathcal{N}_2$ on $\mathbb{R}^n$ are said to be equivalent, if we can find two positive constants α and β such that*

$$\alpha\mathcal{N}_1(x) \leqslant \mathcal{N}_2(x) \leqslant \beta\mathcal{N}_1(x), \quad \forall x \in \mathbb{R}^n.$$

Theorem 1 All norms on $\mathbb{R}^n$ are equivalent.

Example 1.3 (Usual norms). *The following three norms*

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} \quad \text{and} \quad \|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\},$$

are two by two equivalent, and we have for all $x \in \mathbb{R}^n$

$$\|x\|_\infty \leqslant \|x\|_1 \leqslant n\|x\|_\infty, \quad \|x\|_\infty \leqslant \|x\|_2 \leqslant \sqrt{n}\|x\|_\infty \quad \text{and} \quad \|x\|_2 \leqslant \|x\|_1 \leqslant \sqrt{n}\|x\|_2.$$

Definition 1.3 (Scalar product). *Any symmetric positive definite bilinear form, in other words any application of*

$$\begin{aligned} p : \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x, y) &\mapsto p(x, y), \end{aligned}$$

which verifies for all $x, y, z \in \mathbb{R}^n$ and all $a, \beta \in \mathbb{R}$

- $p(x, y) = p(y, x)$.
- $p(x, x) \geqslant 0$ *and* $p(x, x) = 0 \Longleftrightarrow x = 0$.
- $p(ax + \beta y, z) = ap(x, z) + \beta p(y, z)$.

Equipped with a scalar product, $\mathbb{R}^n$ is said to be a Euclidean space.

Example 1.4. *The usual scalar product of $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ denoted by $\langle x, y\rangle$ or $x \cdot y$ is defined by*

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$$

The norm associated with this scalar product is the Euclidean norm on $\mathbb{R}^n$, that is, for all

$x \in \mathbb{R}^n$:

$$\|x\|_2 = \sqrt{\langle x, x \rangle}.$$

1.2 Open and closed parts of $\mathbb{R}^n$.

Definition 1.4 (Balls). *In $\mathbb{R}^n$ equipped with any norm $\|.\|$. We call open ball with center $a \in \mathbb{R}^n$ and radius $r > 0$, the set*

$$B(a, r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}.$$

We call the closed ball with center $a \in \mathbb{R}^n$ and radius $r > 0$, the set

$$\bar{B}(a, r) = \{x \in \mathbb{R}^n : \|x - a\| \leqslant r\}.$$

Finally, we call the sphere with centre $a \in \mathbb{R}^n$ and radius $r > 0$, the set

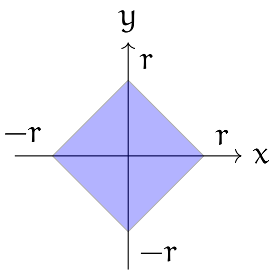
$$S(a, r) = \{x \in \mathbb{R}^n : \|x - a\| = r\}.$$

In the case where $a = 0$ and $r = 1$, we have what are known as unit balls or spheres.

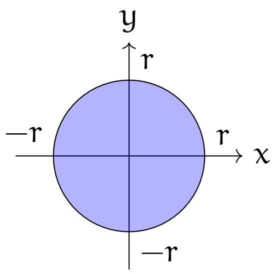
Remark 1.2. *In dimension $n = 1$, the open ball $B(a, r)$ is the open interval $]\, a - r; a + r[$ while the closed ball is $[a - r; a + r]$.*

Example 1.5. *We will graphically represent in $\mathbb{R}^2$ unit balls with center $O = (0,0)$ and radius r for the three usual norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$.*
The closed ball with center $O = (0,0)$ and radius $r > 0$ is the set of points

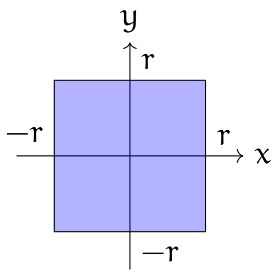
$$\bar{B}(O, r) = \{(x,y) \in \mathbb{R}^2 : \ \|(x,y) - (0,0)\| \leqslant r\} = \{(x,y) \in \mathbb{R}^2 : \|(x,y)\| \leqslant r\}.$$



Relating to $\|\cdot\|_1 : |x| + |y| \leqslant r.$



Relating to $\|\cdot\|_2 : \ \sqrt{x^2 + y^2} \leqslant r.$



Relating to
$\|\cdot\|_\infty : \ \max(|x|, |y|) \leqslant r.$

Definition 1.5 (Bounded part). *We say that A is a bounded part in $\mathbb{R}^n$ if it is contained in a closed ball of $\mathbb{R}^n$. In other words:*

$$A \text{ is bounded} \Longleftrightarrow \exists a \in \mathbb{R}^n, \exists r > 0 : \ A \subset \bar{B}(a, r).$$

Or in an equivalent way

$$A \text{ is bounded} \Longleftrightarrow \exists M > 0, \forall x \in \mathbb{R}^n : |x| \leqslant M.$$

Definition 1.6 (Neighbourhood). *Let $\mathcal{V}$ be a part of $\mathbb{R}^n$ and $a \in \mathcal{V}$. We say that $\mathcal{V}$ is a neighbourhood of a, if V contains an open ball centred at a. In other words:*

$$\mathcal{V} \text{ is neighbourhood of } a \iff \exists r > 0 : B(a, r) \subset \mathcal{V}.$$

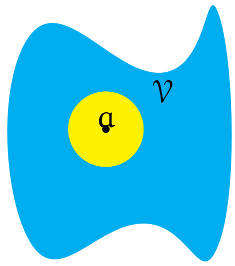
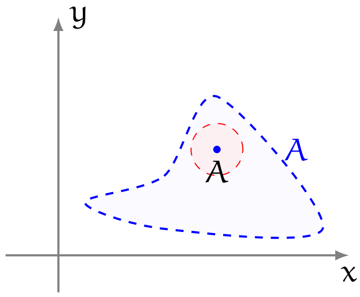


Fig. 1.1 . A neighbourhood

Definition 1.7 (Open part). *A part $A \subset \mathbb{R}^n$ is said to be open, if for any point $x \in A$, A contains an open ball centred at x. We also say that A is an open of $\mathbb{R}^n$.*

In other words:

$$A \text{ is an open} \iff \forall x \in A, \exists r > 0 : \; B(x, r) \subset A.$$



Properties 1.2.

- *Any union of any family (even infinite) of openings is still open.*
- *Any finite intersection of openings is still open.*

Definition 1.8 (Closed part). *A part $A \subset \mathbb{R}^n$ is said to be closed if and only if its complementary $A^c = \mathbb{R}^n \backslash A$ is an open.*

1.3 Interior - Adhesion - Boundary - Isolated point - Accumulation point

Definition 1.9 (The interior). *Let A be any part of $\mathbb{R}^n$. We call the interior of A and we denote $\mathring{A}$, the set defined by*

$$\mathring{A} = \{x \in A : \exists r > 0, B(x, r) \subset A\}.$$

Properties 1.3. *Let A be a part of $\mathbb{R}^n$.*

- *$\mathring{A} \subseteq A$ and furthermore $\mathring{A}$ is the largest open set contained in A.*
- *A is open if and only if $A = \mathring{A}$.*

1.4 Numerical sequences in $\mathbb{R}^n$

Definition 1.10 (The adhesion). *Let A be any part of $\mathbb{R}^n$. We call* ***adherence of*** *A (also called closure), the set defined by*

$$\overline{A} = \{x \in \mathbb{R}^n : \forall \delta > 0, B(x, \delta) \cap A \neq \varnothing\}.$$

Properties 1.4. *Let A be a part of $\mathbb{R}^n$.*

- *$A \subseteq \overline{A}$ and furthermore $\overline{A}$ is the smallest closed which contains A.*
- *A is closed if and only if $A = \overline{A}$.*

Definition 1.11 (The boundary). *Let A be any part of $\mathbb{R}^n$. We call* ***boundary of*** *A (also called edge) and we denote ∂A, the set defined by*

$$\partial A = \overline{A} \setminus A.$$

Note that ∂A is a closed set.

Definition 1.12 (Isolated point). *Let A be a non-empty subset of $\mathbb{R}^n$, and $a \in A$. We say that the point a is isolated in A if there exists a neighborhood $\mathcal{V}$ of a in $\mathbb{R}^n$ such that*

$$\mathcal{V} \cap A = \{a\}.$$

Definition 1.13 (Accumulation point). *Let A be a non-empty part of $\mathbb{R}^n$, and $a \in \mathbb{R}^n$. The point a is said to be a* ***accumulation point of*** *A if it adheres to A without being isolated in A, in other words, if a is an adherence point of $A \setminus \{a\}$. The set of accumulation points of A, denoted A', is a closed set.*

Example 1.6. *Let $A =]0, 1] \cup \{2\}$. For the usual topology on $\mathbb{R}$, we have,*

$$\overline{A} = [0, 1] \cup \{2\} \qquad \mathring{A} =]0, 1[\qquad A' = [0, 1] \qquad \partial A = \{0, 1\}$$

2 is an isolated point in A, but 0 is not.

1.5 Numerical sequences in $\mathbb{R}^n$

Definition 1.14. *We call* ***sequence of elements of*** *$\mathbb{R}^n$ any application of $\mathbb{N}$ in $\mathbb{R}^n$ which to all k makes x_k correspond. We denote such an application $(x_k)_{k\in\mathbb{N}}$. We will denote $x_k = (x_{k,1}, \dots, x_{k,n})$ the n coordinates of the element x_k in the canonical basis of $\mathbb{R}^n$.*

Definition 1.15. *Let $A \subseteq \mathbb{R}^n$. We say that $(x_k)_{k\in\mathbb{N}}$ is a* ***sequence of elements of*** *A if, and only if*

$$\forall k \in \mathbb{N}, \quad x_k \in A.$$

Definition 1.16 (Convergent sequence). *We say that a sequence $(x_k)_{k\in\mathbb{N}}$ of elements of $\mathbb{R}^n$ converges to a limit $l \in \mathbb{R}^n$ if and only if*

$$\forall \epsilon > 0, \exists k_\epsilon \in \mathbb{N} : \ k \geqslant k_\epsilon \implies \|x_k - l\| < \epsilon.$$

On note alors

$$\lim_{k\to+\infty} x_k = l.$$

Remark 1.3. *If a sequence converges, its limit is unique.*

Proposition 1.1. *A sequence $(x_k)_{k\in\mathbb{N}}$ of elements of $\mathbb{R}^n$ converges to $l=(l_1,\ldots,l_n)\in\mathbb{R}^n$ if and only if for all $1\leqslant i\leqslant n$ the sequence of the i-th coordinate of x_k converges in $\mathbb{R}$ to the i-th coordinate of l, that is*

$$\lim_{k\to+\infty} x_k = l \iff \lim_{k\to+\infty} x_{k,i} = l_i, \quad \forall i\in\{1,\ldots,n\}.$$

Example 1.7. *Consider in the sequence $\mathbb{R}^2$ the sequence $(x_k)_{k\in\mathbb{N}}$ defined by*

$$x_k = \left(\frac{k-1}{2k+1}, \frac{1}{k+3}\right).$$

Since,

$$\lim_{k\to+\infty}\frac{k-1}{2k+1} = \lim_{k\to+\infty}\frac{k-1}{2k+1} = \frac{1}{2} \quad \textit{and} \quad \lim_{k\to+\infty}\frac{1}{k+3} = \lim_{k\to+\infty}\frac{1}{k+3} = 0.$$

Then $(x_k)_{k\in\mathbb{N}}$ is convergent and we have

$$\lim_{k\to+\infty} x_k = \left(\frac{1}{2}, 0\right).$$

Proposition 1.2 (Characterization of closed sets by sequences)**.** *A subset $A\subset\mathbb{R}^n$ is closed if and only if, for any sequence $(x_k)_{k\in\mathbb{N}}$ of elements of A which converges we have*

$$\lim_{k\to+\infty} x_k \in A.$$

Definition 1.17 (Bounded sequence)**.** *A sequence $(x_k)_{k\in\mathbb{N}}$ of elements of $\mathbb{R}^n$ is bounded if, and only if*

$$\exists M>0, \forall k\in\mathbb{N}, \|x_k\|\leqslant M.$$

Theorem 2 (Bolzano-Weierstrass) Any sequence of elements of $\mathbb{R}^n$ admits a convergent extracted sequence.

Definition 1.18 (Cauchy sequence)**.** *A sequence $(x_k)_{k\in\mathbb{N}}$ of elements of $\mathbb{R}^n$ is called a Cauchy sequence if it verifies the following property:*

$$\forall\epsilon>0, \exists N_\epsilon\in\mathbb{N}, \forall(p,q)\in\mathbb{N}^2:\ p\geqslant N_\epsilon, \quad \textit{et} \quad q\geqslant N_\epsilon \implies \|x_p-x_q\|<\epsilon.$$

Proposition 1.3. *Any convergent sequence $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence.*

Theorem 3 Any Cauchy sequence in $\mathbb{R}^n$ converges. Therefore, the normed space $(\mathbb{R}^n,\|\cdot\|)$ is complete.

1.6 Compact parts of $\mathbb{R}^n$.

Definition 1.19. *Let A be any subset of $\mathbb{R}^n$. We say that A is compact (or that A is a compact) if for any sequence $(x_k)_{k\in\mathbb{N}}$ of elements of A, we can extract a subsequence $\left(x_{\varphi(k)}\right)_{k\in\mathbb{N}}$ that converges to an element of A.*

By the Bolzano-Weierstrass theorem, we have the following result

Proposition 1.4. *A subset A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Properties 1.5. *Let A be a compact of $\mathbb{R}^n$. If F is a closed set contained in A, then F is compact.*

- *The intersection of a closed set and a compact set is a compact set.*
- *Any arbitrary intersection of compact sets is compact.*
- *Any finite union of compact sets is compact.*