ila University
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Department of Mathematics
Year: 2025/2026
Exam: ALG4
Duration: 2 hours

Exercise 1. (8pts)

We consider the following quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ defined by:

$$q(x,y,z) = x^2 - 2y^2 + z^2 + 6xy - 2yz$$

1. Find the matrix A of q.
2. Find the symmetric bilinear form φ associated with q.
3. Using Gauss method, find the sum of squares of q and the orthogonal basis B'.
4. Same question as 3 using diagonalisation: $P(\lambda) = (\lambda+4)(1-\lambda)(\lambda-3)$
   $V_1 = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix}$. $V_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $V_3 = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$. Verify that the matrix P is orthogonal.
5. Find the rank and signature of q.

Exercise 2. (6pts)

We consider the quadratic form q over $\mathbb{R}^3$ defined by:

$$q(x_1,x_2,x_3) = 2x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3$$ [redacted]

1. Find the matrix A of q.
2. Using Gauss method, find the sum of squares of q and deduce the rank and signature of q.
3. Find $ker\ q$ and $C(q)$, the set of isotropic vectors and verify that $ker\ q \subseteq C(q)$.

Exercise 3. (4pts)

We consider the hermitian form φ over $\mathbb{C}^2$ defined by the matrix:

$$A = \begin{bmatrix} 2 & 1-2i \\ 1+2i & 2 \end{bmatrix}$$

1. Find the hermitian form φ and the quadratic hermitian form q.
2. Using Gauss method, deduce the rank and signature of q.

Exercise 4. (2pts)

Let $M = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$, where $a,b,c \in \mathbb{R}$. We put $S = a+b+c$ and $\sigma = ab+bc+ca$.

- Show that M orthogonal if and only if $\sigma = 0$ and $S = \pm 1$

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EX1. $q(u) = q(x,y,z) = x^2 - 2y^2 + z^2 + 6xy - 2yz$

1) $A = \mathcal{M}_B(q) = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ symmetric ${}^tA = A$ (1)

3) Method (Gauss): $\boxed{a^2 + 2ab = (a+b)^2 - b^2}$

(I): $x^2 + 6xy = x^2 + 2x(3y)$
$= (x+3y)^2 - 9y^2$ (2)

$q(u) = (x+3y)^2 - 11y^2 + z^2 - 2yz$

(II): $z^2 - 2yz = z^2 + 2z(-y) = (z-y)^2 - y^2$

Then: $\boxed{q(u) = \underbrace{(x+3y)^2}_{x'} + \underbrace{(z-y)^2}_{y'} - 12\underbrace{y^2}_{z'}}$

we put $\begin{cases} x' = x + 3y \\ y' = z - y \\ z' = y \end{cases} \Rightarrow \begin{cases} x = x' - 3z' \\ y = z' \\ z = y' + z' \end{cases}$

$$P = \begin{array}{c} \begin{matrix} e_1' & e_2' & e_3' \end{matrix} \\ \begin{pmatrix} 1 & 0 & -3 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$, The new basis (orthogonal): $B' = (e_1', e_2', e_3')$. (1)

where $e_1' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $e_3' = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$

The formula: $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -12 \end{pmatrix} = {}^tPAP$.

$q(u') = q(x',y',z') = x'^2 + y'^2 - 12z'^2$

2) SBF φ: $\varphi(u,v) = {}^tU A V = (x,y,z)\begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$

$= (x,y,z)\begin{pmatrix} x' + 3y' \\ 3x' - 2y' - z' \\ -y' + z' \end{pmatrix} = x(x' + 3y') + y(3x' - 2y' - z')$
$+ z(-y' + z')$ (1)

A

$$\boxed{q(u,v) = xx' + 3xy' + 3yx' - 2yy' - yz' - zy' + zz'}$$

4) Diagonalization method:

we have $p(\lambda) = 0 \Rightarrow \begin{cases} \lambda_1 = -4 \longrightarrow E(\lambda_1) = \langle v_1 \rangle \\ \lambda_2 = 1 \longrightarrow E(\lambda_2) = \langle v_2 \rangle \\ \lambda_3 = 3 \longrightarrow E(\lambda_3) = \langle v_3 \rangle . \end{cases}$

The new q-orthogonal basis $\mathcal{B}' = (v_1', v_2', v_3')$ is:

$$v_1' = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{35}}\begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix} = \begin{pmatrix} 3/\sqrt{35} \\ -5/\sqrt{35} \\ -1/\sqrt{35} \end{pmatrix}$$

$$v_2' = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{10} \\ 0 \\ 3/\sqrt{10} \end{pmatrix}$$

1

$$v_3' = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{14}}\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3/\sqrt{14} \\ -2/\sqrt{14} \\ 1/\sqrt{14} \end{pmatrix}$$

The formula: $D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} = P^{-1}AP = {}^tPAP$.

where $P = (v_1', v_2', v_3') = \begin{pmatrix} 3/\sqrt{35} & 1/\sqrt{10} & -3/\sqrt{14} \\ -5/\sqrt{35} & 0 & -2/\sqrt{14} \\ -1/\sqrt{35} & 3/\sqrt{10} & 1/\sqrt{14} \end{pmatrix}$

P is orthogonal: ${}^tPP = P\,{}^tP = I_3$

$$P\,{}^tP = \begin{pmatrix} 3/\sqrt{35} & 1/\sqrt{10} & -3/\sqrt{14} \\ -5/\sqrt{35} & 0 & -2/\sqrt{14} \\ -1/\sqrt{35} & 3/\sqrt{10} & 1/\sqrt{14} \end{pmatrix}\begin{pmatrix} 3/\sqrt{35} & -5/\sqrt{35} & -1/\sqrt{35} \\ 1/\sqrt{10} & 0 & 3/\sqrt{10} \\ -3/\sqrt{14} & -2/\sqrt{14} & 1/\sqrt{14} \end{pmatrix}$$

1

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

5) The rank of q:
$rg(q) = 3$
$\mathrm{Ker}\, q = \{0\}$
q is nondegenerate

1

The signature: $(2,1)$

2

Exo2. $q(u) = q(x_1, x_2, x_3) = 2x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3$

1) $A = \mathcal{M}_B(q) = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ symmetric ${}^tA = A$. 1

2) Gauss method: $\boxed{a^2 + 2ab = (a+b)^2 - b^2}$

(I) $\cancel{x_2^2 + 2x_1x_2 = x_2^2 + 2x_2(x_1) = (x_2+x_1)^2 - x_1^2}$

$$x_2^2 + 2x_1x_2 + 2x_2x_3 = x_2^2 + 2x_2(x_1 + x_3)$$

$$= (x_2 + x_1 + x_3)^2 - (x_1 + x_3)^2$$

$$= (x_1 + x_2 + x_3)^2 - x_1^2 - x_3^2 - 2x_1x_3$$ 2

$$q(u) = (x_1 + x_2 + x_3)^2 + x_1^2 + x_3^2 - 2x_1x_3$$

Finally: $\boxed{q(u) = q(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^2 + (x_1 - x_3)^2}$

The rank $rg(q) = 2$, q is degenerate.

$\ker q \neq \{0\}$ 1

The signature $(s, t) = (2, 0)$.

3) $\ker q$: $AX = 0 \Rightarrow \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ 1

$\Rightarrow \begin{cases} 2x_1 + x_2 = 0 & \rightarrow (1) \\ x_1 + x_2 + x_3 = 0 & \rightarrow (2) \\ x_2 + 2x_3 = 0 & \rightarrow (3) \end{cases}$

By (1): $\boxed{x_2 = -2x_1}$

By (2): $x_1 - 2x_1 + x_3 = 0 \Rightarrow \boxed{x_3 = x_1}$

$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -2x_1 \\ x_3 \end{pmatrix} = x_1 \underbrace{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}_{v}$, Then $\boxed{\ker q = \langle v \rangle}$

and $\boxed{\dim \ker q = 1}$

The cone isotropic: $C'(q) = \{ u \in \mathbb{R}^3 \ / \ q(u) = 0 \}$

$$q(u) = 0 \Rightarrow (x_1 + x_2 + x_3)^2 + (x_1 - x_3)^2 = 0$$

$$\Rightarrow \begin{cases} x_1 + x_2 + x_3 = 0 \longrightarrow x_1 + x_2 + x_1 = 0 \longrightarrow \boxed{x_2 = -2x_1} \\ x_1 - x_3 = 0 \longrightarrow \boxed{x_1 = x_3} \end{cases}$$

1

Then $u = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -2x_1 \\ x_3 \end{pmatrix} = x_1 \underbrace{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}_{V}$

i.e. $\boxed{C'(q) = \langle V \rangle = \ker q}$.

EXO3: $A = \begin{pmatrix} 2 & 1-2i \\ 1+2i & 2 \end{pmatrix}$

1) $\varphi(u,v) = {}^t u A \bar{v} = (x_1 \ x_2) \begin{pmatrix} 2 & 1-2i \\ 1+2i & 2 \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$

$$= (x_1, x_2) \begin{pmatrix} 2\bar{y}_1 + (1-2i)\bar{y}_2 \\ (1+2i)\bar{y}_1 + 2\bar{y}_2 \end{pmatrix}$$

1

$$\boxed{\varphi(u,v) = 2x_1\bar{y}_1 + (1-2i)x_1\bar{y}_2 + (1+2i)x_2\bar{y}_1 + 2x_2\bar{y}_2}$$

$$q(u) = 2|x_1|^2 + (1-2i)x_1\bar{x}_2 + (1+2i)x_2\bar{x}_1 + 2|x_2|^2$$

1 $\boxed{q(u) = 2|x_1|^2 + 2\,\mathrm{Re}\left[x_1 \overline{(1+2i)x_2}\right] + 2|x_2|^2}$

2) Gauss method:
Using the formula $|z_1|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2) = |z_1 + z_2|^2 - |z_2|^2$

$$2|x_1|^2 + 2\,\mathrm{Re}\left[x_1\,\overline{(1+2i)x_2}\right] =$$

$$2\left[|x_1|^2 + 2\,\mathrm{Re}\Big[\underbrace{x_1}_{z_1}\,\underbrace{\overline{\tfrac{1+2i}{2}x_2}}_{z_2}\Big]\right]$$

$$= 2\left[|x_1 + (\tfrac{1}{2}+i)x_2|^2 - |(\tfrac{1}{2}+i)x_2|^2\right]$$

2

$$= 2|x_1 + (\tfrac{1}{2}+i)x_2|^2 - \tfrac{5}{2}|x_2|^2$$

Then $\boxed{q(u) = 2|x_1 + (\tfrac{1}{2}+i)x_2|^2 - \tfrac{1}{2}|x_2|^2}$

The rank of q: $rg(q) = 2$, q is nondegenerate

for $q = 1$)

The signature of q: $(s,t) = (1,1)$.

<u>Exo4:</u> $M = \begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$
$$M = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}, \qquad s = a+b+c, \qquad \sigma = ab+bc+ca.$$

M orthogonal $\Leftrightarrow$ (v_1, v_2, v_3) q-orthonormal.

$$\Leftrightarrow \begin{cases} v_1\cdot v_2 = v_1\cdot v_3 = v_2\cdot v_3 = 0 \Rightarrow ab+bc+ca=0 \\ \Rightarrow \boxed{\sigma = 0} \\ \|v_1\| = \|v_2\| = \|v_3\| = 1 \Rightarrow \sqrt{a^2+b^2+c^2} = 1 \\ \Rightarrow a^2+b^2+c^2 = 1 \end{cases}$$

1

$$s^2 = (a+b+c)^2 = a^2+b^2+c^2 + 2ab + 2ac + 2bc$$

$$= \underbrace{a^2+b^2+c^2}_{1} + 2\underbrace{(ab+ac+bc)}_{0} = 1$$

1

$$s^2 = 1 \Rightarrow \boxed{s = \pm 1}$$