M'sila University
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Department of Mathematics
Year 2024/2025
Exam: ALG4 (S4 LMD)
Duration: 1 hour 30 min

Exercise 1. (8 pts)

We consider the quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$q(x, y, z) = 3x^2 - 2xy - 2xz + 3y^2 + 3z^2 - 2yz$$

1 1. Find the matrix A of q.

2 2. Find the symmetric bilinear form φ associated with q.

2 + 2 3. Using Gauss method, find the sum of squares of q and the orthogonal basis B' for q and the formula $D = P^t AP$.

1 4. Find the rank and signature of q.

Exercise 2. (8 pts)

We consider the quadratic form q over $\mathbb{R}^3$ defined by:

$$q(x_1, x_2, x_3) = 2x_1^2 - 2x_2^2 - 6x_3^3 + 3x_1x_2 - 4x_1x_3 + 7x_2x_3$$

1 1. Find the matrix A of q.

2 2. Using Gauss method, find the sum of squares of q.

1 3. Deduce the rank and the signature of q.

2 4. Find $ker\ q$.

2 5. Find $C(q)$, the set of isotropic vectors and verify that $ker\ q \subset C(q)$.

Exercise 3. (4 pts)

We consider the hermitian form φ over $\mathbb{C}^2$ defined by the matrix

$$A = \begin{bmatrix} \sqrt{2} & 1 + i\sqrt{2} \\ 1 - i\sqrt{2} & \sqrt{2} \end{bmatrix}$$

2 1. Find the hermitian form φ and the quadratic hermitian form q.

2 2. Using Gauss method, deduce the rank and the signature of q.

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L2 - S4

Exo1. $q(x,y,z) = 3x^2 - 2xy - 2xz + 3y^2 + 3z^2 - 2yz$

1) $A = \mathcal{M}_B(q) = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$; ${}^tA = A$ 1

2) $\varphi(u,v) = {}^t u A v = (x,y,z) \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} =$

$= (x,y,z) \begin{pmatrix} 3x' - y' - z' \\ -x' + 3y' - z' \\ -x' - y' + 3z' \end{pmatrix}$ 2

$$\boxed{\begin{aligned}\varphi(u,v) = {} & 3xx' - xy' - xz' - yx' + 3yy' - yz' \\ & - zx' - zy' + 3zz'\end{aligned}}$$

$a^2 + 2ab = (a+b)^2 - b^2$

3) Gauss method:

(I): $3x^2 - 2xy - 2xz = 3\left[\underbrace{x}_{a}^2 + 2x\underbrace{\left(-\frac{1}{3}y - \frac{1}{3}z\right)}_{b}\right]$

$= 3\left[\left(x - \frac{1}{3}y - \frac{1}{3}z\right)^2 - \left(-\frac{1}{3}y - \frac{1}{3}z\right)^2\right]$ 2

$= 3\left(x - \frac{1}{3}y - \frac{1}{3}z\right)^2 - \frac{1}{3}y^2 - \frac{1}{3}z^2 + \frac{2}{3}yz.$

Then $q(x,y,z) = 3\left(x - \frac{1}{3}y - \frac{1}{3}z\right)^2 + \frac{8}{3}y^2 + \frac{8}{3}z^2 - \frac{4}{3}yz$

(II): $\frac{8}{3}y^2 - \frac{4}{3}yz = \frac{8}{3}\left[\underbrace{y}_{a}^2 + 2y\underbrace{\left(-\frac{1}{4}z\right)}_{b}\right]$

$= \frac{8}{3}\left[\left(y - \frac{1}{4}z\right)^2 - \frac{1}{16}z^2\right] = \frac{8}{3}\left(y - \frac{1}{4}z\right)^2 - \frac{1}{6}z^2$

Finally:

$$\boxed{q(x,y,z) = 3(\underbrace{x - \tfrac{1}{3}y - \tfrac{1}{3}z}_{x'})^2 + \tfrac{8}{3}(\underbrace{y - \tfrac{1}{4}z}_{y'})^2 + \tfrac{15}{6} z'^2}$$

We put $\begin{cases} x' = x - \frac{1}{3}y - \frac{1}{3}z \\ y' = y - \frac{1}{4}z \\ z' = z \end{cases} \Rightarrow \begin{cases} x = x' + \frac{1}{3}y' + \frac{5}{12}z' \\ y = y' + \frac{1}{4}z' \\ z = z' \end{cases}$ 1

$$P = (e_1' e_2' e_3') = \begin{array}{c} \begin{array}{ccc} e_1' & e_2' & e_3' \end{array} \\ \begin{pmatrix} 1 & \frac{1}{3} & \frac{5}{12} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

The new basis: $\mathcal{B}' = (e_1', e_2', e_3')$ and 1

$$\boxed{D = \mathcal{M}_{\mathcal{B}'}(q) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{8}{3} & 0 \\ 0 & 0 & \frac{15}{6} \end{pmatrix} = {}^tPAP}$$

4) The rank: $rg(q) = (3,0)$, q is non-degenerate and $Ker(q) = \{0\}$. 1

The signature: $sig(q) = (s,t) = (3,0)$.

Exo 2: $q(x_1,x_2,x_3) = 2x_1^2 - 2x_2^2 - 6x_3^2 + 3x_1x_2 - 4x_1x_3 + 7x_2x_3$

1) $A = \mathcal{M}_{\mathcal{B}}(q) = \begin{pmatrix} 2 & \frac{3}{2} & -2 \\ \frac{3}{2} & -2 & \frac{7}{2} \\ -2 & \frac{7}{2} & -1 \end{pmatrix}$; ${}^tA = A$. 1

$a^2 + 2ab = (a+b)^2 - b^2$

2) Gauss method:

(I): $2x_1^2 + 3x_1x_2 - 4x_1x_3 = 2\left[\underbrace{x_1^2}_{a} + 2x_1(\underbrace{\tfrac{3}{4}x_2 - x_3}_{b})\right]$

$= 2\left[(x_1 + \tfrac{3}{4}x_2 - x_3)^2 - (\tfrac{3}{4}x_2 - x_3)^2\right] =$

$$= 2\left(x_1 + \frac{3}{4}x_2 - x_3\right)^2 - \frac{9}{8}x_2^2 - 2x_3^2 + 3x_2x_3$$

Then: $q(x_1,x_2,x_3) = 2\left(x_1 + \frac{3}{4}x_2 - x_3\right)^2 \underbrace{- \frac{25}{8}x_2^2} - 8x_3^2 + \underbrace{10x_2x_3}$

(II): $-\frac{25}{8}x_2^2 + 10x_2x_3 = -\frac{25}{8}\left(x_2^2 - \frac{16}{5}x_2x_3\right)$

$$= -\frac{25}{8}\left[\underbrace{x_2^2}_{a} + 2x_2\left(\underbrace{-\frac{8}{5}x_3}_{b}\right)\right]$$

$$= -\frac{25}{8}\left[\left(x_2 - \frac{8}{5}x_3\right)^2 - \frac{64}{25}x_3^2\right].$$

$$= -\frac{25}{8}\left(x_2 - \frac{8}{5}x_3\right)^2 + 8x_3^2.$$

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Finally:

$$\boxed{q(x_1,x_2,x_3) = 2\left(x_1 + \frac{3}{4}x_2 - x_3\right)^2 - \frac{25}{8}\left(x_2 - \frac{8}{5}x_3\right)^2}$$

3) $rg(q) = 2$, q is degenerate: $Ker(q) \neq \{0\}$.

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$sig(q) = (1,1)$.

4) $Ker\, q = E^{\perp}$: $AX = 0 \Rightarrow \begin{pmatrix} 2 & \frac{3}{2} & -2 \\ \frac{3}{2} & -2 & 7/2 \\ -2 & \frac{7}{2} & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{cases} 2x + \frac{3}{2}y - 2z = 0 \longrightarrow (1) \\ \frac{3}{2}x - 2y + \frac{7}{2}z = 0 \longrightarrow (2) \\ -2x + \frac{7}{2}y - 6z = 0 \longrightarrow (3) \end{cases}$$

$z = \frac{5}{8}(-8x) \Rightarrow \boxed{z = -5x}$

By (1) + (3): $z = \frac{5}{8}y$

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By (1): $2x + \frac{3}{2}y - 2\left(\frac{5}{8}y\right) = 0 \Rightarrow \boxed{y = -8x}$

Then $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -8x \\ -5x \end{pmatrix} = x\underbrace{\begin{pmatrix} 1 \\ -8 \\ -5 \end{pmatrix}}_{v}$

$\boxed{Ker(q) = \langle v \rangle}$ and $\boxed{\dim Ker(q) = 1.}$

5) we have $C(q) = \{ u \in \mathbb{R}^3 \, / \, q(u) = 0 \}$

$$q(u) = 0 \Rightarrow 2\left(x_1 + \frac{3}{4}x_2 - x_3\right)^2 - \frac{25}{8}\left(x_2 - \frac{8}{5}x_3\right)^2 = 0$$

$$\Rightarrow 2\left(x_1 + \frac{3}{4}x_2 - x_3\right)^2 = \frac{25}{8}\left(x_2 - \frac{8}{5}x_3\right)^2$$

$$\Rightarrow \begin{cases} \sqrt{2}\left(x_1 + \frac{3}{4}x_2 - x_3\right) = \frac{5}{2\sqrt{2}}\left(x_2 - \frac{8}{5}x_3\right). \\ \text{or} \\ \sqrt{2}\left(x_1 + \frac{3}{4}x_2 - x_3\right) = -\frac{5}{2\sqrt{2}}\left(x_2 - \frac{8}{5}x_3\right). \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = \frac{1}{2}x_2 - x_3 \longrightarrow X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \underbrace{\begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}}_{V_1} + x_3 \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{V_2} \\ \text{or} \\ x_1 = -\frac{1}{2}x_2 + x_3 \longrightarrow X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x_2 + x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \underbrace{\begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix}}_{V_3} + x_3 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{V_4} \end{cases}$$

Then $C(q) = P_1 \cup P_2$,

Where $P_1 = \langle v_1, v_2 \rangle$ and $P_2 = \langle v_3, v_4 \rangle$.

* $\mathrm{Ker}(q) \subset C(q)$: $\begin{pmatrix} 1 \\ -8 \\ -5 \end{pmatrix} = -8V_1 - 5V_2 \in C(q)$

$\in \mathrm{Ker}(q)$.

Exo3 : $A = \begin{pmatrix} \sqrt{2} & 1+i\sqrt{2} \\ 1-i\sqrt{2} & \sqrt{2} \end{pmatrix}$

1) $\varphi(U,V) = {}^tU A \overline{V} = (x_1, x_2) \begin{pmatrix} \sqrt{2} & 1+i\sqrt{2} \\ 1-i\sqrt{2} & \sqrt{2} \end{pmatrix} \begin{pmatrix} \overline{y_1} \\ \overline{y_2} \end{pmatrix}$

$$\boxed{\varphi(U,V) = \sqrt{2}\, x_1 \overline{y_1} + (1+i\sqrt{2}) x_1 \overline{y_2} + (1-i\sqrt{2}) x_2 \overline{y_1} + \sqrt{2}\, x_2 \overline{y_2}}$$

$$q(U) = \varphi(U,U) = \sqrt{2}|x_1|^2 + \sqrt{2}|x_2|^2 + 2\,\mathrm{Re}\left[(1+i\sqrt{2}) x_1 \overline{y_2}\right]$$

2) Gauss method:

We have $q(U) = q(x_1,x_2) = \sqrt{2}|x_1|^2 + 2\,\mathrm{Re}\left[(1+i\sqrt{2}) x_1 \overline{x_2}\right] + \sqrt{2}|x_2|^2.$

Using the formula:

$$\boxed{|z_1|^2 + 2\,\mathrm{Re}\, z_1 \overline{z_2} = |z_1+z_2|^2 - |z_2|^2}$$

$$q(U) = \sqrt{2}\left[\underbrace{|x_1|^2}_{z_1} + 2\,\mathrm{Re}\left(\underbrace{x_1}_{z_1} \underbrace{\left(\frac{1-i\sqrt{2}}{\sqrt{2}}\right) \overline{x_2}}_{z_2} \right) \right] + \sqrt{2}|x_2|^2$$

$$= \sqrt{2}\left[\left|x_1 + \left(\frac{\sqrt{2}}{2} - i\right) x_2\right|^2 - \left|\left(\frac{\sqrt{2}}{2} - i\right) x_2\right|^2 \right] + \sqrt{2}|x_2|^2$$

$$= \sqrt{2} \cdot \left|x_1 + \left(\frac{\sqrt{2}}{2} - i\right) x_2\right|^2 - \sqrt{2} \cdot \frac{6}{4} |x_2|^2 + \sqrt{2}|x_2|^2$$

$$\boxed{q(U) = q(x_1,x_2) = \sqrt{2}\left|x_1 + \left(\frac{\sqrt{2}}{2} - i\right) x_2\right|^2 - \frac{\sqrt{2}}{2}|x_2|^2}$$

$rg(q) = 2$, q is non-degenerate ($\ker(q) = \{0\}$), $sig(q) = (1,1)$.