solution of series

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October 2025

Exercise 01

We proceed by structural induction on the definition of a well-formed formula.

Let $L(\varphi)$ denote the number of left parentheses '(' in φ, and $R(\varphi)$ the number of right parentheses ')' in φ. We show $L(\varphi) = R(\varphi)$ for all wffs φ.

Base case: If φ is a propositional variable (e.g., P, Q, R), then $L(\varphi) = 0$ and $R(\varphi) = 0$, so $L(\varphi) = R(\varphi)$.

Inductive steps:

1. **Negation:** If $\varphi = \neg A$ and $L(A) = R(A)$, then

$$L(\varphi) = L(A), \quad R(\varphi) = R(A),$$

   so $L(\varphi) = R(\varphi)$.

2. **Binary connectives:** If $\varphi = (A \circ B)$ where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, and $L(A) = R(A)$, $L(B) = R(B)$, then

$$L(\varphi) = L(A) + L(B) + 1, \quad R(\varphi) = R(A) + R(B) + 1.$$

   Since $L(A) = R(A)$ and $L(B) = R(B)$, we have $L(\varphi) = R(\varphi)$.

By structural induction, $L(\varphi) = R(\varphi)$ for all wffs φ.

Exercise 02

For every formula ϕ in propositional logic, the number of connectives in ϕ is at most $2^{c(\phi)} - 1$.

We proceed by structural induction on ϕ. Let $N(\phi)$ denote the number of connectives in ϕ.

Base case: ϕ is a propositional variable.
Then $c(\phi) = 0$ and $N(\phi) = 0$.
Check: $0 \leq 2^0 - 1 = 0$. True.
Inductive step 1: $\phi = \neg\psi$.
By induction hypothesis: $N(\psi) \leq 2^{c(\psi)} - 1$.

We have $c(\phi) = 1 + c(\psi)$ and $N(\phi) = 1 + N(\psi)$.
Then:

$$N(\phi) = 1 + N(\psi) \leq 1 + (2^{c(\psi)} - 1) = 2^{c(\psi)}.$$

We need to show $2^{c(\psi)} \leq 2^{c(\phi)} - 1 = 2^{1+c(\psi)} - 1 = 2 \cdot 2^{c(\psi)} - 1$.
That is, $2^{c(\psi)} \leq 2 \cdot 2^{c(\psi)} - 1$, i.e., $1 \leq 2^{c(\psi)}$, which is true since $c(\psi) \geq 0$.
So $N(\phi) \leq 2^{c(\phi)} - 1$.
Inductive step 2: $\phi = (\psi \circ \chi)$ where $\circ \in \{\wedge, \vee, \rightarrow\}$.
By induction: $N(\psi) \leq 2^{c(\psi)} - 1$, $N(\chi) \leq 2^{c(\chi)} - 1$.
We have $c(\phi) = 1 + \max(c(\psi), c(\chi))$ and $N(\phi) = 1 + N(\psi) + N(\chi)$.

Let $m = \max(c(\psi), c(\chi))$. Then $c(\phi) = 1 + m$.

Now:

$$N(\phi) = 1 + N(\psi) + N(\chi) \leq 1 + (2^{c(\psi)} - 1) + (2^{c(\chi)} - 1) = 2^{c(\psi)} + 2^{c(\chi)} - 1.$$

Since $c(\psi) \leq m$ and $c(\chi) \leq m$, we have $2^{c(\psi)} \leq 2^m$ and $2^{c(\chi)} \leq 2^m$. Thus:

$$N(\phi) \leq 2^m + 2^m - 1 = 2^{m+1} - 1 = 2^{c(\phi)} - 1.$$

This completes the induction.

Exercise 03:

Let σ be a substitution mapping propositional variables to formulas. For any formula ϕ, the result $\phi\sigma$ of applying σ to ϕ is a well-formed formula.

By structural induction on ϕ:

Base case: $\phi = P$ (a propositional variable). Then $\phi\sigma = \sigma(P)$, which is a well-formed formula by the definition of substitution.

Inductive step 1: $\phi = \neg\psi$. By the induction hypothesis, $\psi\sigma$ is a well-formed formula. Then $\phi\sigma = \neg(\psi\sigma)$ is a well-formed formula by the negation formation rule.

Inductive step 2: $\phi = (\psi \circ \chi)$ where $\circ \in \{\wedge, \vee, \rightarrow\}$. By the induction hypothesis, $\psi\sigma$ and $\chi\sigma$ are well-formed formulas. Then $\phi\sigma = (\psi\sigma \circ \chi\sigma)$ is a well-formed formula by the binary connective formation rule.

By structural induction, $\phi\sigma$ is a well-formed formula for all ϕ.

Exercise 04

Let σ and τ be substitutions. Their **composition** $\sigma \circ \tau$ is defined by:

$$(\sigma \circ \tau)(p) = (\sigma(p))\tau$$

for any propositional variable p.

For any formula ϕ and substitutions σ, τ,

$$\phi(\sigma \circ \tau) = (\phi\sigma)\tau.$$

By structural induction on ϕ:

Base case: $\phi = p$ (propositional variable). Then $p(\sigma \circ \tau) = (\sigma \circ \tau)(p) = (\sigma(p))\tau$ by definition of composition. Also $(p\sigma)\tau = (\sigma(p))\tau$. So equality holds.

Inductive step 1: $\phi = \neg\psi$. Assume $\psi(\sigma \circ \tau) = (\psi\sigma)\tau$ (IH). Then $(\neg\psi)(\sigma \circ \tau) = \neg(\psi(\sigma \circ \tau)) = \neg((\psi\sigma)\tau)$ by IH. Also $((\neg\psi)\sigma)\tau = (\neg(\psi\sigma))\tau = \neg((\psi\sigma)\tau)$. So equality holds.

Inductive step 2: $\phi = (\psi \circ \chi)$ where $\circ \in \{\wedge, \vee, \rightarrow\}$. Assume $\psi(\sigma \circ \tau) = (\psi\sigma)\tau$ and $\chi(\sigma \circ \tau) = (\chi\sigma)\tau$ (IH). Then $(\psi \circ \chi)(\sigma \circ \tau) = (\psi(\sigma \circ \tau) \circ \chi(\sigma \circ \tau)) = ((\psi\sigma)\tau \circ (\chi\sigma)\tau)$. Also $((\psi \circ \chi)\sigma)\tau = ((\psi\sigma \circ \chi\sigma))\tau = ((\psi\sigma)\tau \circ (\chi\sigma)\tau)$. So equality holds.

By structural induction, the theorem is proved.

Exercise 05:

Given $\sigma(p) = (q \wedge r)$, $\sigma(q) = \neg p$, and $\sigma(r) = r$:

[label=()]

1. $(p \wedge q)\sigma = (q \wedge r) \wedge \neg p$
2. $(\neg p \rightarrow q)\sigma = \neg(q \wedge r) \rightarrow \neg p$
3. $((p \vee r) \wedge q)\sigma = ((q \wedge r) \vee r) \wedge \neg p$

Exercise 06:

Let σ and τ be substitutions defined as: $\sigma(p) = (q \vee r), \sigma(q) = \neg p, \sigma(r) = r$
$\tau(p) = p, \tau(q) = (p \wedge q), \tau(r) = \neg r$

Compute:

[label=()]

1. $((p \rightarrow q) \wedge r)\sigma$
2. $((p \rightarrow q) \wedge r)\tau$
3. $(((p \rightarrow q) \wedge r)\sigma)\tau$
4. $(((p \rightarrow q) \wedge r)\tau)\sigma$