Solution for Series 01: Mathematical Logic 2025/2026

Exercise 01

Show, by proof by contradiction, that D and D' are parallel:

— Assume D and D' are not parallel.

— Arrive at a contradiction by noting that if D and D' intersect, they have a common point with coordinates (x_0, y_0) satisfying the equations $y_0 = x_0 + 1 = x_0 - 1$; which leads to the equality $1 = -1$, contradicting the true statement $-1 \neq 1$ in the theory of real numbers on which analytic plane geometry depends.

Conclude that the hypothesis (D and D' are not parallel) is invalid, therefore D and D' are parallel.

Exercise 02

Show $(\forall n \in \mathbb{N}, \exists k \in \mathbb{N}, n^2 = 4k$ or $n^2 = 4k + 1)$:

— Take $n \in \mathbb{N}$.

— Show, by proof by cases, $(\exists k \in \mathbb{N}, n^2 = 4k$ or $n^2 = 4k + 1)$:

Case 1: Assume n is even (natural auxiliary statement) (thus $n = 2p$).

- Set (simple analysis) $k = p^2$ (note that $k \in \mathbb{N}$).
- Verify that ($n^2 = 4k$ or $n^2 = 4k + 1$) by observing that we have $n^2 = 4k$.

Case 2: Assume n is odd (not even) (thus $n = 2p + 1$).

- Set (simple analysis) $k = p^2 + p$ (note that $k \in \mathbb{N}$).
- Verify that ($n^2 = 4k$ or $n^2 = 4k+1$) by observing that we have $n^2 = 4k+1$.

— Conclude that $(\exists k \in \mathbb{N}, n^2 = 4k$ or $n^2 = 4k + 1)$.

Conclude that $(\forall n \in \mathbb{N}, \exists k \in \mathbb{N}, n^2 = 4k$ or $n^2 = 4k + 1)$.

Exercise 03

Let $P(n)$ be the statement: 7 divides $3^{2n} - 2^n$. Show by induction the statement $(\forall n \in \mathbb{N}^*, P(n))$.

— Show that $P(1)$ is true (base case), by noting that 7 indeed divides $3^2 - 2 = 7$.

— Take $k \in \mathbb{N}^*$ and assume $P(k)$ is true (thus $3^{2k} - 2^k = 7 \times N$ for some N) (induction hypothesis).

— Show that $P(k + 1)$ is true (inductive step), by noting that:

$$\begin{aligned}
3^{2(k+1)} - 2^{k+1} &= 3^2 \cdot 3^{2k} - 2 \cdot 2^k \\
&= 9 \cdot 3^{2k} - 2 \cdot 2^k \\
&= 9(3^{2k} - 2^k) + 9 \cdot 2^k - 2 \cdot 2^k \\
&= 9(7 \times N) + 7 \cdot 2^k \\
&= 7(9N + 2^k)
\end{aligned}$$

Conclude that for all $n \in \mathbb{N}^*$, 7 divides $3^{2n} - 2^n$.

Exercise 04

Show $(\forall \varepsilon > 0, |a| < \varepsilon) \Rightarrow a = 0$ by contraposition; to do this, verify the statement $(a \neq 0 \Rightarrow (\exists \varepsilon > 0, |a| \geq \varepsilon))$:

— Assume $a \neq 0$.

— Prove $(\exists \varepsilon > 0, |a| \geq \varepsilon)$: • Set (simple analysis) $\varepsilon = \frac{|a|}{2}$ (verify that $\varepsilon > 0$).
• Verify (obvious) that $|a| \geq \varepsilon$.

— Conclude that $(\exists \varepsilon > 0, |a| \geq \varepsilon)$.

Conclude that $(a \neq 0 \Rightarrow (\exists \varepsilon > 0, |a| \geq \varepsilon))$, therefore:

$$(\forall \varepsilon > 0, |a| < \varepsilon) \Rightarrow a = 0.$$