06 For Logic of first order

Second Year Mathematics

2025/2026

Exercise 01

Given signature: $\Sigma = (c, f^{(1)}, g^{(2)})$
Consider term: $t = g(f(c), g(c, f(c)))$

Tasks:

(a) Draw the term tree for t.

(b) List all subterms of t.

Exercise 02

Same signature $\Sigma = (c, f^{(1)}, g^{(2)})$, term $t = f(g(c, f(c)))$.

Task: Compute depth of each subterm and height of term.

Exercise 03

Given: $\varphi = \forall x(P(x) \to \exists y Q(f(x), y))$

Tasks:

(a) Is this a well-formed formula if $P^{(1)}, Q^{(2)}, f^{(1)}$ are in the signature?

(b) Draw the formula tree.

(c) List all subformulas.

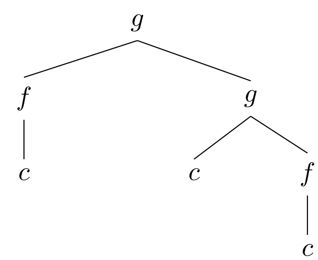
Exercise 04

Given: $\psi = \forall x(R(x, y) \to \exists x S(x, z))$

Task: Identify free and bound occurrences of variables.

Solution:

(a) Term tree:



(b) Subterms (from smallest to largest):

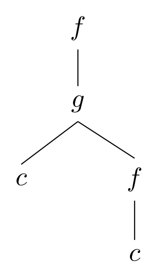
(a) c

(b) $f(c)$

(c) $g(c, f(c))$

(d) $g(f(c), g(c, f(c)))$

Solution:
Term: $f(g(c, f(c)))$



Depth of a node = length of path from root to that node.

- Node f (root): depth 0
- Node g: depth 1
- Left child c: depth 2
- Right child f: depth 2
- Node c under that f: depth 3

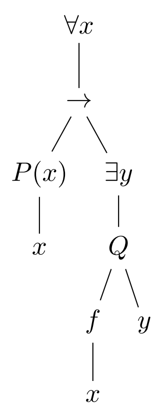
Height of term = maximum depth of a node = 3.
Solution:

(a) Yes, it's well-formed:

- $P(x)$ is an atomic formula

- $Q(f(x), y)$ is an atomic formula ($f(x)$ is a term, y is a term)
- $\exists y Q(f(x), y)$ is a formula
- $P(x) \to \exists y Q(f(x), y)$ is a formula
- $\forall x(\dots)$ is a formula

(b) Formula tree:



(c) Subformulas:

(a) $P(x)$
(b) $Q(f(x), y)$
(c) $\exists y Q(f(x), y)$
(d) $P(x) \to \exists y Q(f(x), y)$
(e) $\forall x(P(x) \to \exists y Q(f(x), y))$

Solution:

- $\forall x$ binds x in $R(x, y) \to \exists x S(x, z)$
- Inside, $\exists x$ binds x in $S(x, z)$ (this x is different from outer x – *variable shadowing*)
- Free variables: y in $R(x, y)$ is free
- z in $S(x, z)$ is free

Thus:

- Bound variables: x (twice, different scopes)
- Free variables: y, z

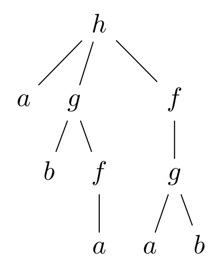
Exercise Set 3: Complex Examples

Exercise 3.1

Signature: constants a, b, functions $h^{(3)}, g^{(2)}, f^{(1)}$.
Term: $t = h(a, g(b, f(a)), f(g(a, b)))$

Task: Draw term tree.

Solution:



Exercise 3.2

$\varphi = \exists x(P(x) \wedge \forall y(Q(x, y) \rightarrow R(f(y))))$

Tasks:

(a) Draw the formula tree.

(b) List free variables (if any).

(c) List all subformulas.

Solution:

(a) Tree:

(b) x bound by $\exists x$

y bound by $\forall y$

(c) Subformulas:

(a) $P(x)$

(b) $Q(x, y)$

(c) $R(f(y))$

(d) $Q(x, y) \rightarrow R(f(y))$

(e) $\forall y(Q(x, y) \rightarrow R(f(y)))$

(f) $P(x) \wedge \forall y(Q(x, y) \rightarrow R(f(y)))$

(g) $\exists x(P(x) \wedge \forall y(Q(x, y) \rightarrow R(f(y))))$

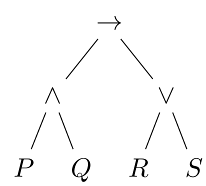
Exercise Set 4: Parsing and Ambiguity

Exercise 4.1

Given string in propositional logic: $P \land Q \to R \lor S$

Task: Insert parentheses to reflect tree: $((P \land Q) \to (R \lor S))$

Draw tree:

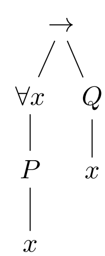


Exercise 4.2

String: $\forall x Px \to Qx$

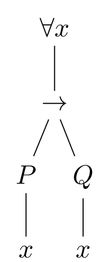
Task: This is ambiguous. Two possible readings:

(a) $(\forall x Px) \to Qx$:



Free variable: x in Qx

(b) $\forall x(Px \to Qx)$:



No free variables.

Usually, $\forall x Px \to Qx$ means $(\forall x Px) \to Qx$, because quantifier binding extends only to the smallest formula.

Exercise Set 5: Substitution Visualization

Exercise 5.1

Term $t = g(x, f(y))$, substitution $\sigma = \{x \mapsto h(a), y \mapsto b\}$.
Result: $t\sigma = g(h(a), f(b))$.

Task: Draw tree of t and tree of $t\sigma$.

Solution:

Tree of t:

```
   g
  / \
 x   f
     |
     y
```

Tree of $t\sigma$:

```
   g
  / \
 h   f
 |   |
 a   b
```

Exercise 5.2

Formula $\phi = \forall y P(x, y)$, substitution $\sigma = \{x \mapsto f(y)\}$.

Task: Apply substitution (careful with variable capture).

Solution:

Naively replacing x by $f(y)$ inside $\forall y P(x, y)$ would cause y in $f(y)$ to be captured by $\forall y$. We must rename bound variable y first:

$\phi\sigma$ after renaming bound y to z: $\forall z P(f(y), z)$

Tree of $\forall z P(f(y), z)$:

```
   ∀z
   |
   P
  / \
 f   z
 |
 y
```