Third series in Mathematical Logic

Second year Mathematics

2025/2026

Exercise 1:

Prove that every well-formed formula has a balanced number of left and right parentheses.

Exercise 2:

Define the **complexity** $c(\phi)$ of a formula ϕ recursively:

- $c(\phi) = 0$ if ϕ is a propositional variable
- $c(\neg\phi) = 1 + c(\phi)$
- $c((\phi \circ \psi)) = 1 + \max(c(\phi), c(\psi))$ for $\circ \in \{\wedge, \vee, \rightarrow\}$

Prove: For all formulas ϕ, the number of connectives in ϕ is $\leq 2^{c(\phi)} - 1$.

Exercise 3:

Let σ be a substitution. Prove: For any formula ϕ, ϕ^σ is a well-formed formula.

Exercise 4:

Let σ and τ be substitutions. Define their **composition** $\sigma \circ \tau$ as:

$$(\sigma \circ \tau)(p) = (\sigma(p))^\tau$$

Exercise 5:

. Let σ be a substitution with $\sigma(p) = (q \wedge r)$ and $\sigma(q) = \neg p$. Compute:

(a) $(p \wedge q)^{\sigma}$

(b) $(\neg p \rightarrow q)^{\sigma}$

(c) $((p \vee r) \wedge q)^{\sigma}$

exercise 6:

Let σ and τ be substitutions defined as:

$$\begin{array}{lll} \sigma(p) = (q \vee r), & \sigma(q) = \neg p, & \sigma(r) = r \\ \tau(p) = p, & \tau(q) = (p \wedge q), & \tau(r) = \neg r \end{array}$$

Compute:

(a) $((p \rightarrow q) \wedge r)^{\sigma}$

(b) $((p \rightarrow q) \wedge r)^{\tau}$

(c) $(((p \rightarrow q) \wedge r)^{\sigma})^{\tau}$

(d) $(((p \rightarrow q) \wedge r)^{\tau})^{\sigma}$