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Solution of Logic final exam

Exercise 1: (7 Pts)

1,5

Problem: Determine the height of the propositional formula

$$F = (P \wedge (Q \vee R)) \Rightarrow (S \vee T)$$

Solution:

- The height of a formula is defined as the maximum depth of nested logical operations.
- In this formula:
  - The outermost operation is $\Rightarrow$ (1 level).
  - Inside the left operand $(P \wedge (Q \vee R))$:
    - $\wedge$ (1 level).
    - $Q \vee R$ (1 additional level).
  - The right operand $(S \vee T)$ has a height of 1.
- Therefore, the height of F is 3.

$E_0 = \{P, Q, R\}$
$F_1 = E_0 \cup \{(Q \vee R), (S \vee T)\}$
$F_2 = F_1 \cup \{P \wedge (Q \vee R)\}$
$F_3 = F_2 \cup \{F\}$. $\boxed{h = 3}$

2

Set of Subformulas for the formula

$G = (A \Rightarrow (B \wedge C)) \vee D$

- The subformulas of G are:
  - A
  - B
  - C
  - $B \wedge C$
  - $A \Rightarrow (B \wedge C)$
  - D
  - $G = (A \Rightarrow (B \wedge C)) \vee D$

Thus, the set of subformulas is $\{A, B, C, B \wedge C, A \Rightarrow (B \wedge C), D, G\}$.

Substitution in a Formula

the substitution $P \to X$ in the formula

$$H = (P \vee Q) \wedge (R \Rightarrow P)$$

- Substitute P with X:

1,5

$$H' = (X \vee Q) \wedge (R \Rightarrow X)$$

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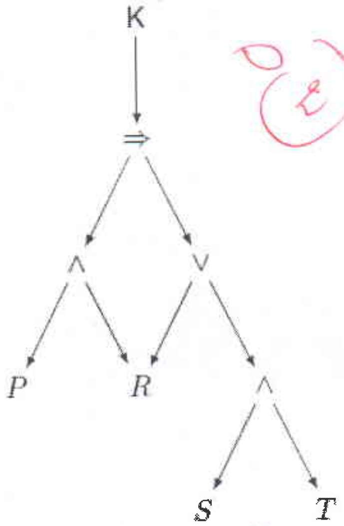
Draw the tree representation of the formula

$$K = (P \wedge Q) \Rightarrow (R \vee (S \wedge T))$$



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Exercise02 (05 Pts)

Theorem: The length of a propositional formula with n binary connectives is $2n + 1$.

Proof by Induction:

0

1

Base case: For $n = 0$, a formula without binary connectives is simply a propositional variable. Therefore, the length is 1:

$$2 \cdot 0 + 1 = 1.$$

Induction hypothesis: Assume that for a formula with k binary connectives, the length is given by:

0

2

$$L(k) = 2k + 1.$$

Induction step: Consider a formula with $k + 1$ binary connectives. This formula can be constructed by combining two formulas, each with a and b binary connectives respectively, such that $a + b = k$. The length of the new formula is:

$$L(k+1) = L(a) + L(b) + 1 = (2a+1) + (2b+1) + 1.$$

Substituting $a + b = k$:

0

2

$$L(k+1) = (2a + 2b + 2 + 1) = 2(a+b) + 1 + 1 = 2k + 1 + 1 = 2(k+1) + 1.$$

Thus, by induction, the formula holds for $n = k + 1$.

Conclusion: Therefore, the length of a propositional formula with n binary connectives is given by:

$$L(n) = 2n + 1.$$

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(8 Pts)

Given Formula:

$$\forall x \exists y (P(f(x), g(a, y)) \land Q(c))$$

Part 1: Count the Number of Distinct Terms

- **Constants:**
  - a
  - c
- **Variables:**
  - x (bound variable)
  - y (bound variable)
- **Function Terms:**
  - $f(x)$
  - $g(a, y)$

Distinct Terms: 1,5

- $f(x)$
- $g(a, y)$
- a
- c
- x
- y

Total Distinct Terms: 6 0,5

Part 2: Identify Free and Bound Variables

- **Bound Variables:**
  - x: Bound by the universal quantifier $\forall x$. 1
  - y: Bound by the existential quantifier $\exists y$. 1
- **Free Variables:**
  - There are no free variables in this formula, as both x and y are bound by quantifiers. 1

- **Distinct Terms: 6**
- **Bound Variables: 2** (x, y)
- **Free Variables: 0**

• The formula is close. 1

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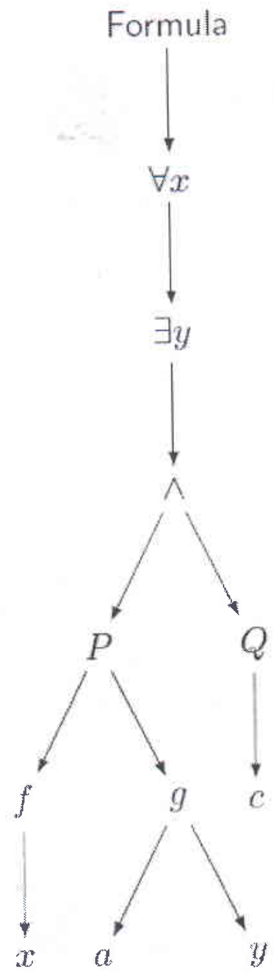
3: Draw the Tree of This Formula



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