Chapter 3

Predicate Calculus

Definition 3.1 A "predicate" is a statement concerning objects within a mathematical theory that can be either true or false depending on the objects involved.

Example.

1) "Being an even number" is true for 2 but false for other numbers.

2) "Being smaller than" is true for (2,3) but false for (3,2).

Predicate calculus allows the construction of complex statements from predicates by using special symbols to represent variables, functions on these variables, and the relationships between them. Certain variables never change; these are constants.

In this sense, predicate calculus is richer than propositional calculus.

3.1 Syntax of Predicate Calculus

3.1.1 First-Order Alphabet

Let $v = \{x_0, x_1, x_2, ...\}$ be a set whose elements are called variables.

$*\mathcal{C} = \{c_0, c_1, c_2, ..\}$ is a set whose elements are constants.

$*\mathcal{F} = \bigcup\limits_{n>0} \mathcal{F}_n$ represents functions of arity n.

$*\mathcal{R} = \bigcup\limits_{n>0} \mathcal{R}_n$ represents a union of sets $\mathcal{R}_n$ whose elements are relations of arity n.

(It is assumed that $\mathcal{R}_2$ contains a particular element of equality)

$*$ The symbols $\forall$ and $\exists$ (for all, there exists).

Definition 3.2 A first-order alphabet is an alphabet A of the form:

$A = \cup \cup \{(,), \rceil, \wedge, \vee, \Rightarrow, \Leftrightarrow, \forall, \exists\} \cup \mathcal{C} \cup \mathcal{F} \cup \mathcal{R}$.

- The part $\cup \cup \{(,), \rceil, \wedge, \vee, \Rightarrow, \Leftrightarrow, \forall, \exists\}$ is the logical part of the alphabet or language.

- The part $\sum = \mathcal{C} \cup \mathcal{R} \cup \mathcal{F}$. is called the signature of the language.

- The logical part is common to all languages, so what characterizes a language is its non-logical part or signature.

Example. Signature of elementary arithmetic

$\sum = \{+, ., <, 0, 1\}$

$\mathcal{C} = \{0, 1\}$

$\mathcal{F} = \{+, .\} = \mathcal{F}_2$

$\mathcal{R} = \{<, =\} = \mathcal{R}_2$

3.1.2 Terms

Let A^* be the set of words formed from the alphabet A.

Definition 3.3 The set of terms T constructed from A is the smallest set of $A^*/$ such that

1) $\cup \cup \mathcal{C} \subseteq T$.

2) $t_1, t_2, ..., t_n \in T \Rightarrow f\, t_1, ..., t_n \in T, \forall f \in \mathcal{F}_n\ \forall n \geq 1$.

Example. In the theory of real numbers, the constants are $\mathbb{R}$, the variables are x,y,..., the functions = are real functions: $\cos, \sin, +, ., ..$

$*$ $\sin x$ is a term.

$*$ $.xx$ is a term usually represented by x^2.

$*$ $+xy$ is a term usually represented by $x + y$.

$*$ The expression $\sin(x + \sin(y^2 + x))$ is represented by the term: $\sin + x \sin + x.yy$

Note. As a word, each term has a unique representation.

The Height of a Term

$T = \bigcup_{n \geq 0} T_n$ *with* $T_0 = v \cup \mathcal{C}$ and $T_{n+1} = T_n \cup \{ft_1, ..., t_k,\ f \in \mathcal{F}_k \text{ and } t_1, ..., t_k \in T_n\}$

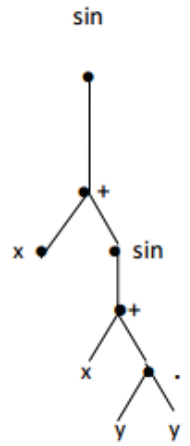
$h(t) = min\{n/t \in T_n\}$

Syntax of Predicate Calculus

$*$ We can also refer to the decomposition tree of a term. The height of a term is the height of its decomposition tree:

Example. The term: $\sin + x \sin + x.yy$

$h = 5$



3.1.3 Formula

An atomic formula is a word in A^* that is $R\,t_1...t_n$ where R is a relation of arity n $(R \in \mathcal{R})$ and $t_1....t_n$ are terms (we write $= t_1 t_2$ for $t_1 = t_2$).

Definition3.4 The set of first-order formulas is the smallest subset $\mathcal{F}$ of $A^*/$

$*1)$ Any atomic formula $\in \mathcal{F}$.

$*2)\ F,\, G,\, \in \mathcal{F} \Rightarrow \rceil\, F,\, (F \wedge G),\, (F \vee G),\, (F \Rightarrow G),\, (F \Leftrightarrow G) \in \mathcal{F}.$

$*3)\ F,\, \in \mathcal{F} \Rightarrow \forall\, v_n\, F$ and $\exists\, v_n\, F \in \mathcal{F},\, \forall\, n.$

Note.

1) We have $\mathcal{F} = \bigcup\limits_{n \geq 0} \mathcal{F}_n,\, \mathcal{F}_0 =$atomic formula,

$\mathcal{F}_{n+1} = \mathcal{F} \cup \{\rceil\, F,\, F \in \mathcal{F}\} \cup \{(F * G),\, F,\, G \in \mathcal{F}_n,\, * =\} \cup \{\forall\, v_k\, F,\, F \in \mathcal{F}_n,\, k \in \mathcal{N}\} \cup \{\exists\, v_k\, F,\, F \in \mathcal{F}\ and\ h(F) = min\,\{n/\, F \in \mathcal{F}_n\}.$

2) As a word, every formula has a unique writing.

3) A formula also has a decomposition tree, with the leaves being the atomic formulas.

Example. Let the signature $\sum = \{p,\, Q,\, R,\, f,\, g,\, T\}$

$*\, \forall\, x\, (Rxy \Rightarrow Qxfy).$

$* \rceil \exists x(Rxy \vee Qxgyx)$.

$* \forall x\,(px \wedge \exists y(Tyx \Rightarrow sxy))$ are formulas.

The subformulas of F are defined as follows: $*$ If F is atomic, then $sf(F) = \{F\}$.

$* F = \rceil G,\ sf(F) = \{F\} \cup sf\{G\}$.

$*$ If $F = (G_1 * G_2),\ sf(F) \cup sf\{G_1\} \cup sf\{G_2\}$.

$*$ If $F = \forall x_k G,\ sf(F) = \{F\} \cup sf(G)$.

$F = \exists x_k G$.

3.2 Free and Bound Variables

A variable v_n can appear multiple times in a formula F. We say it has multiple occurrences. These occurrences are of two types: free and bound.

Definition3.5 We define by indication the free occurrences of v_n:

$\star$ If F is atomic, all occurrences of v_n in F are free.

$\star$ If $F = \rceil G$, the free occurrences of v_n in F are those of v_n in G.

$\star$ If $F = (G\,\alpha\,H)$, the free occurrences of v_n are those of v_n in G and those of v_n in H.

$\star$ If $F = \forall v_k G$ or $F = \exists v_k G$ with $k \neq n$, the free occurrences of v_n in F are those of v_n in G.

$\star$ If we say that x_n is quantified in F as $F = \forall v_n G$ or $F = \exists v_n G$ no occurrence of v_n in F is free.

Example. $\sum = \{R,\, c,\, f\}$

R: Relation, c: constant, f: function

$F = \forall x_0(\exists x_1 \forall x_0(Rx_1x_0 \Rightarrow \rceil x_0 \simeq x_3) \wedge \forall x_2(\exists x_2(Rx_1x_2 \vee fx_0 \simeq c) \wedge x_2 \simeq x_2)$

All occurrences of x_0 and x_2 are bound. The first two occurrences of x_1 are bound, but the third is free. x_3 is free.

Definition3.6 A variable in F is free if it has at least one free occurrence. A closed formula is a formula with no free variables. In the previous example, F is not closed $((x_1\, and\, x_3)$

are free). A closure of a variable in F is a formula of the form $\forall v_{i_1}, v_{i_2}, \ldots, v_{i_n} F$ where $v_{i_1}, \ldots, v_{i_n}$ are the free variables of F. An inverse closure is closed.

3.2.1 Scope of a Quantifier

Let $\ominus = \exists$ or $\forall$. In any formula F containing Ox, the term Ox is followed by a unique subformula G in which the variable x is free under Ox.

Definition 3.7 G is the scope of the quantifier O. The occurrences of x that fall within the domain of O are the free occurrences of x in G.

Example: $F = \exists x((px \vee Q\underset{8}{y} \wedge g\underset{12}{y} = z) \Rightarrow (\exists x \forall^{\swarrow 21} \underset{22}{y}\, Rx\underset{25}{y} \wedge fxz =) y$ where y is free at positions $8\, and\, 12$, and bound at positions 22 and 25, quantified at 22 and 25 within the domain of 21.

3.2.2 Substitution in Terms

Notation

$t = t[x_{i_1}, x_{i_2}, \ldots, x_{i_n}]$ indicates that the variables with at least one occurrence in the term t are among $x_{i_1}, \ldots, x_{i_n}$. If $m = \underset{j}{max}\, x_j$ we can also write $t = t[x_0, x_1, \ldots, x_m]$.

Definition 3.8 Let $y_1, \ldots, y_k$ be variables and $t, u_1, \ldots, u_k$ be terms. The term $t_{\frac{u_1}{x_1}, \ldots, \frac{u_k}{x_k}}$ is obtained by substituting the terms $u_1, u_2, \ldots, u_k$ for the variables $y_1, \ldots, y_k$ in all occurrences of y_i in t. More precisely:

$\star t =$ constant or variable $\neq y_i$ then $t_{\frac{u_1}{y_1}, \ldots, \frac{u_k}{y_k}} = t$.

$\star t = y_i\,(1 \leq i \leq k)\, t_{\frac{u_1}{y_1}, \ldots, \frac{u_k}{y_k}} = u_i$.

$\star t = ft_1t_2 \ldots t_n\; t_{\frac{u_1}{y_1}, \ldots, \frac{u_k}{y_k}} = ft_{1\frac{u_1}{y_1}, \ldots, \frac{u_k}{y_k}} \ldots t_{n\frac{u_1}{y_1}, \ldots, \frac{u_k}{y_k}}$

Proposition 3.1 $t_{\frac{u_1}{y_1}, \ldots, \frac{u_k}{y_k}}$ is a term.

Proof. By induction on the term t.

3.2.3 Substitution in Formulas

Notation

$F = F[x_{i_1}, ..., x_{i_n}]$ means that the free variables in F are among x_{i_1}. We will substitute terms for free variables in a formula.

Definition 3.9 Let F be a formula, and $y_1, ..., y_k$ be terms. The expression $F_{\frac{u_1}{y_1} \cdots \frac{u_k}{y_k}}$ obtained by substituting the terms $u_1, ..., u_k$ for the variables $y_1, ..., y_k$ is defined as follows:

$\star If\ F = Rt_1....t_n$ is atomic, then $F_{\frac{u_1}{y_1} \cdots \frac{u_k}{y_k}} = Rt_{1\frac{u_1}{y_1} \cdots \frac{u_k}{y_k}} .....t_{n\frac{u_1}{y_1} \cdots \frac{u_k}{y_k}}$.

$\star If\ F = \rceil G$ then $F_{\frac{u_1}{y_1} \cdots \frac{u_k}{y_k}} = \rceil G_{\frac{u_1}{y_1} \cdots \frac{u_k}{y_k}}$.

$\star If\ F = (G \, \alpha \, H)$, $F_{\frac{u_1}{y_1} \cdots \frac{u_k}{y_k}} = (G_{\frac{u_1}{y_1} \cdots \frac{u_k}{y_k}} \, \alpha \, H_{\frac{u_1}{y_1} \cdots \frac{u_k}{y_k}})$.

$\star If\ F = OxG \, (x \neq y_i)$, $F_{\frac{u_1}{y_1} \cdots \frac{u_k}{y_k}} = OxG_{\frac{u_1}{y_1} \cdots \frac{u_k}{y_k}}$, $O = \exists$ *or* $\forall$.

$\star\ If\ F = Oy_iG \, (i = 1, 2, ..., k)$, $F_{\frac{u_1}{y_1} \cdots \frac{u_k}{y_k}} = Oy_iG_{\frac{u_1}{y_1} \cdots \frac{u_{i-1}}{y_{i-1}}, \frac{u_{i+1}}{y_{i+1}} \cdots \frac{u_k}{y_k}}$

Example. $F = \forall x_0(\exists x_1 \forall x_0(Rx_1x_0 \Rightarrow \rceil x_0 = x_3) \wedge \forall x_2(\exists x_2(Rx_1x_2 \vee fx_0 = c) \wedge v_2 = v_2))$

$t = ffc$

$F_{\frac{t}{x_1}} = \forall x_0(\exists x_1 \forall x_0(Rx_1x_0 \Rightarrow \rceil x_0 = x_3) \wedge \forall x_2(\exists x_2(\exists x_2(Rffcx_0 \vee fx_0 = c) \wedge x_2 = x_0))$

3.3 Semantics of Predicate Calculus

We aim to assign truth values to formulas of predicate calculus. To do so, it is necessary to specify a domain in which variables take values, and where the symbols of relation and function have meaning.

Example 1. Consider the formula $\exists y \forall x \, Ryx$. To assign a truth value to this formula, we must assign values to the variables x, y and specify the definition of the "relation" R. Thus, it depends on the set in which the variables x, y take their values and on the definition of R in that set.

$\star$ If $R = \leqslant$ over $\mathbb{N}$, then the formula is true.

$\star$ If $R = <$ over $\mathbb{N}$, the formula is false.

⋆ If $R = \leqslant$ or $<$ over $\mathbb{Z}$, the formula is false.

Therefore, we need to specify a set in which the variables take values and in which the symbols of relations and functions become true relations and functions. This set will be a structure (or a realization of the language or alphabet A of the predicate calculus considered).

Example 2. *Rcx*

⋆ True for $\mathbb{N}$, $c = 0$, $x = 1$, $R = \leqslant$.

⋆ False for $\mathbb{N}$, $c = 0$, $x = -1$, $R = \leqslant$.

3.3.1 Definition of a Structure

Let A be an alphabet of first-order.

Definition 3.10 An A-structure is given by a set S such that:

⋆ Every constant symbol c in A is associated with an element $\bar{c}$ in S.

⋆ Each function symbol f of arity n is associated with a function $\bar{f} : S^n \longrightarrow S$.

⋆ Each relation symbol R of arity n is associated with a relation $\bar{R}$ of arity n on S ($\bar{R} \subseteq S^n$) (the equality relation corresponds to equality in S).

Example. Let the signature language be $\Sigma = \{R, f, c_0, c_1\}$. A possible structure is $S = \mathbb{N}, \bar{R} = \leqslant, \bar{f} = +, \bar{c_0} = 0, \bar{c_1} = 1$. Another possible structure is $S = \mathbb{R}, \bar{R} = <, \bar{f} = x, \bar{c_0} = e, \bar{c_1} = T$.

Remark. When considering multiple structures associated with the same language, and for any arbitrary structure, we write $\bar{c_s}, \bar{f_s}$, and $\bar{R_s}$.

Definition 3.11 Let S and S' be two structures of the same language (alphabet). We say that S is a substructure of S' if:

⋆ $S \subseteq S'$.

⋆ $\bar{c_s} = \bar{c_{s'}}$ for every constant c.

⋆ $\bar{f_s} = \bar{f_{s'}}$ for every function symbol f of arity n.

⋆ $\bar{R_s} = \bar{R_{s'}} \cap S^n$ for every relation R of arity n.

Example. $\Sigma = \{R, f, g, c_0, c_1\}$

$\star$ $S' = \mathbb{Z}, \bar{R_{s'}} = \leqslant, \bar{f_{s'}} = +, \bar{g_{s'}} = +, \bar{c_{0_{s'}}} = 0, \bar{c_{1_{s'}}} = 1$

$\star$ $S = \mathbb{N}, \bar{R_s} = \leqslant, \bar{f_s} = +, \bar{g_s} = +, \bar{c_{0_s}} = 0, \bar{c_{1_s}}$

Let S and S' be two structures of the same alphabet. A structure morphism between S and S' is a function $\varnothing : S \longrightarrow S'$.

$\star$ $\varnothing(\bar{c_s}) = \bar{c_{s'}}$ for all constants c

$\star$ $\varnothing(\bar{f_{a_1,...,a_n}}) = \bar{f_{s'}}(\varnothing(a_1), ..., \varnothing(a_n))$

for every function symbol f of arity n, for all $a_i \in S$, $\star$ $(a_1), ..., a_n) \in \bar{R_s^n} \Rightarrow (\varnothing(a_1), ..., \varnothing(a_n) \in \bar{R_{s'}^n})$

for all $a_1, ..., a_n \in S$, and for all relation symbols R of arity n.

Example. $\Sigma = \{f, c\}$

$S =< \mathbb{R}_+^*, 1, x >$; $S' =< \mathbb{R}, 0, + >$

$$\varnothing : \mathbb{R}_+^* \longrightarrow \mathbb{R}$$

$$x \longmapsto \log x$$

Definition 3.12 A structure morphism $\varnothing : S \longrightarrow S'$ is a monomorphism if, for every relation R of arity n, if $a_1, \ldots, a_n \in S$, then $(a_1, a_2, \ldots, a_n) \in \bar{R_s} \Leftrightarrow (\varnothing(a_1), \ldots, \varnothing(a_n) \in \bar{R_{s'}}$.

Example. Every substructure defines a monomorphism (every injection $S \circlearrowleft S'$); the converse is also true.

Proposition 3.2 Every monomorphism is injective.

Proof. Among the relation symbols, we have the equality relation $=$, which defines equality on S and S': "=S" = "=s", "==". Therefore, if $a_1, a_2 \in S$ and $\varnothing(a_1) = \varnothing(a_2)$, since $\varnothing$ is a monomorphism, we have: $\varnothing(a_1) = \varnothing(a_2) \Leftrightarrow a_1 = a_2$, thus $\varnothing$ is injective.

Definition 3.13 An isomorphism of structures $\varnothing : S \longrightarrow S'$ is a surjective monomorphism; in particular, $\varnothing$ must be a bijection. An automorphism is an isomorphism $\varnothing : S \longrightarrow S$.

Example. $S =< \mathbb{R}_+^*, 1, \times >$ and $S' =< \mathbb{R}, 0, + >$.

$\varnothing : \mathbb{R}_+^* \longrightarrow \mathbb{R}, x \longmapsto \log x$ is an automorphism.

The next step is the interpretation of terms.

Definition 3.14 Let $t = t[x_0, \ldots, x_{n-1}]$ be a term and S a structure of first-order logic. I

$a_0, \ldots, a_{n-1}$ be elements of S.

The interpretation $\bar{t_s}$ of t in S when $x_i = a_i$, $0 \le i \le n$, is defined as follows:

$\star$ If $t = x_i$, then $\bar{t_s} = a_i$.

$\star$ If $t = c$, then $\bar{t_s} = \bar{c_s}$.

$\star$ If $t = ft_1 \cdots t_k$, then $\bar{t_s} = \bar{f}_s(\bar{t_{1_s}}, \ldots, \bar{t_{k_s}})$.

Example. $\Sigma = \{f, g, c_0, c_1\}$.

$S = \mathbb{N}, \bar{f} = +, \bar{g} = \times, \bar{c_0} = 0, \bar{c_1} = 1.$

$t = gyfxc_1 = t[x, y]$

$$\begin{pmatrix} x = x_0 \\ y = x_1 \end{pmatrix}$$

Let $a_0 = 2, a_1 = 3$.

$\bar{t_s} = \bar{g}(3, \bar{f}(2, 1)) = 3 \times (2 + 1) = 3 \times 3 = 9$

Proposition 3.3 Let $t = t[x_0, \ldots, x_{n-1}]$ and $t' = t'[y, x_0, \ldots, x_{n-1}]$ be two terms, and

$a_0, \ldots, a_{n-1}$ be elements of the structure S. Then we have $\bar{u_s} = \bar{t_{s'}}$, with $u = \bar{t'}_{\frac{t}{y}}$ a

$y \rightsquigarrow \bar{t_s} = b \in S$.

Proof. By induction on the term t'

$$\star t' = c, u = t' = c$$

$$\bar{u_s} = \bar{c_s} = \bar{t'_s}$$

$$\star t' = x_i, u = t' = x_i$$

$$\bar{u_s} = a_i = \bar{t'_s}$$

$$\star t' = y, u = t$$

$$\bar{u_s} = \bar{t_s} = \bar{t'_s}$$

$$\star t' = ft_1 \cdots t_k, \text{ then } u = fu_1 \cdots u_k \text{ with } u_i = t_{i\frac{t}{y}}$$

By the hypothesis $\bar{u_{i_s}} = \bar{t_{i_s}}$

Thus: $\bar{u_s} = \bar{f}(\bar{u_{1_s}}, \ldots, \bar{u_{k_s}}) = f(\bar{t_{1_s}}, \ldots, \bar{t_{k_s}}) = \bar{t'_s}$

3.3.2 Satisfaction of a Formula in a Structure

Let A be a first-order alphabet and S an A-structure. Let $F = F[x_0, ..., x_{n-1}]$ be a formula (this notation means that the free variables of F are among the x_i).

The satisfaction of a formula F in S is defined when the variables x_i are interpreted by the elements $a_0, a_1, \ldots, a_{n-1}$ of S.

This is written as:

$$S \models F[a_0, a_1, \ldots, a_{n-1}]$$

Definition 3.15 $\star$ If F is the atomic formula $Rt_1t_2 \ldots t_k$ with $t_i = t_i[x_0, \ldots, x_{n-1}]$, then

$$S \models F[a_0, \ldots, a_{n-1}] \Leftrightarrow (\bar{t_{1s}}, \ldots, \bar{t_{ks}}) \in \bar{R}_s$$

$\star$ If $F = \neg G$

$$S \models F[a_0, \ldots, a_{n-1}] \Leftrightarrow S \not\models G[a_0, \ldots, a_{n-1}]$$

$\star$ If $F = (G \wedge H)$, then

$$S \models F \Leftrightarrow S \models G \text{ and } S \models H$$

$\star$ If $F = (G \Rightarrow H)$, then

$$S \models F \Leftrightarrow S \not\models G \text{ or } S \models H$$

$\star$ If $F = (G \vee H)$, then

$$S \models F \Leftrightarrow S \models G \text{ or } S \models H$$

$\star$ If $F = (G \Leftrightarrow H)$, then

$$S \models F \Leftrightarrow (S \models G \text{ and } S \models H) \text{ or } (S \not\models G \text{ and } S \models H)$$

$\star$ If $F = \forall x G$ (where $x \neq x_i$), then

$$S \models F[a_0, \ldots, a_{n-1}] \Leftrightarrow S \models G[a, a_0, \ldots, a_{n-1}] \text{ for all } a \text{ where } x \text{ is interpreted by } a$$

$\star$ If $F = \exists x G$ (where $x \neq x_i$), then

$$S \models F[a_0, \ldots, a_{n-1}] \Leftrightarrow \text{ if there exists } a \in S \text{ such that } S \models G[a, a_0, \ldots, a_{n-1}] \text{ with } x \mapsto a$$

$\star$ If $F = \forall x_i G$, then

$$S \models F[a_0, \ldots, a_{n-1}] \Leftrightarrow S \models G[a_0, \ldots, a_{i-1}, x_i \mapsto a, a_{i+1}, \ldots, a_{n-1}] \text{ for all } a \in S$$

$\star$ If $F = \exists x_i G$, then

$$S \models F[a_0, \ldots, a_{n-1}] \Leftrightarrow \text{ there exists } a \in S \text{ such that } S \models G[a_0, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_{n-1}]$$

Remark If F is closed (without free variables), we write $S \models F$; F is satisfied in S.

Example Let $\sum = \{\underset{2}{\underset{\uparrow}{R}}, \underset{1}{\underset{\uparrow}{f}}, c\}$ be the structure, and $S = \mathbb{R}, \bar{R}_s = \leqslant, \bar{f}_s = \cos, \bar{c}_s = \pi$.

$\star$ $F = Rcx \; (= Rt_1t_2)$

$$\begin{aligned} S \models F[a] &\Leftrightarrow \bar{t_{1s}}, \bar{t_{2s}} \in \bar{R}_s \\ &\Leftrightarrow \bar{t_{1s}} \leqslant \bar{t_{2s}} \\ &\Leftrightarrow \pi \leqslant a \end{aligned}$$

Thus: $S \models F[a] \Leftrightarrow a \in [\pi, +\infty[$

$\star$ $F = \text{``}fx_0 = c\text{``} = cfx_0$

$$S \models F[a] \Leftrightarrow \bar{c_s} = fx_{0_s} \Leftrightarrow \pi = \cos a$$

Thus: $\forall a \in \mathbb{R}, S \not\models F[a]$

$\exists x_1 \underbrace{fx_1}_{G} = x_0$, free variable x_0.

$$\begin{aligned} S \models F[a_0] &\Leftrightarrow S \models G[a_0, a_1] \text{ for some } a_1 \\ &\Leftrightarrow \exists a_1 / \cos a_1 = a_0 \\ &\Leftrightarrow a_0 \in [-1, 1] \end{aligned}$$

$\star$ $\forall x_1 \underbrace{Rx_0fx_1}_{G}$, free variable x_0

$$\begin{aligned} S \models F[a_0] &\Leftrightarrow S \models G[a_0, a_1] / \forall a_1 \\ &\Leftrightarrow \forall a_1 / a_0 \leqslant \cos a_1 \\ &\Leftrightarrow a_0 \in]-\infty, -1] \end{aligned}$$

$\star$ $\forall x_1 \underbrace{\exists x_2 (Rx_1x_2 \wedge fx_2 = x_0)}_{G}$, free variable x_0

$$S \models F[a_0] \Leftrightarrow \forall a_1 \exists a_2 / S \models G[a_0, a_1, a_2]$$
$$\Leftrightarrow \forall a_1 / \exists a_2 / a_1 \leqslant a_2 \text{ and } a_0 = \cos a_2$$
$$\Leftrightarrow a_0 \in [-1, 1]$$

$\star$ $x_0 \exists x_1 \underbrace{fx_1 = x_0}_{G}$ closed formula

$$\forall a_0, \exists a_1 / \cos a_1 = a_0 \quad S \models F$$

$\star$ $\exists x_1 \forall x_2 \underbrace{Rfx_2x_1}_{G}$ closed formula

$$\exists a_1 / \forall a_2, \cos a_2 \leqslant a_1 \quad S \models F$$

Proposition 3.4 Let $t = t[x_0, \ldots, x_{n-1}]$ be a term and $F = F[z, x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}]$ a formula such that no occurrence of z is within the scope of $\forall x_i$ or $\exists x_i$. Then, for any structure S and $\forall a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1} \in S$, we have:

$$S \models F_{\frac{t}{z}}[a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1}] \Leftrightarrow S \models F[\bar{t_s}, a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1}]$$

Proof By induction on the formula F

$\star$ $F = Rt_1t_2 \ldots t_k \quad t_i = t_i[z, x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}]$

$$F_{\frac{t}{z}} = Rt_{1_{\frac{t}{z}}} \ldots t_{k_{\frac{t}{z}}} = Rr_1 \ldots r_k$$
$$S \models F_{\frac{t}{z}} \Leftrightarrow (\bar{r_{1_s}}, \ldots, \bar{r_{k_s}}) \in \bar{R_s}$$

Let $c_i = \bar{t_{i_s}}[\bar{t}, a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1}]$

Then $\bar{r_{i_s}} = c_i$, so $S \models F_{\frac{t}{z}} \Leftrightarrow (\bar{t_{i_s}}[\bar{t_s}, a_0, ..., a_{n-1}, b_0, ..., b_{m-1}]) \in \bar{R_s}$, $i = 1, .., k \Leftrightarrow S \models F[\bar{t_s}, a_0, ..., a_{n-1}, b_0, ..., b_{m-1}]$

Consequence and Universal Equivalence

Let A be a first-order alphabet:

$\star$ A closed formula F is universally valid if $S \models F$ for every A-structure S. We denote this by

$\models F$.

$\star$ A closed formula F is contradictory if and only if $\neg F$ is universally valid.

$\star$ A formula F with free variables is universally valid if its universal closure is.

$\star$ F is universally equivalent to G (denoted $F \sim G$) if and only if $(F \Leftrightarrow G)$ is universally valid.

$\star$ A theory T over A is a set of closed formulas.

$\star$ The structure S is a model of the theory T if and only if $S \models F$ for all $F \in T$ (we also say that S and T satisfy $S \models T$).

$\star$ A theory T is consistent if it has at least one model; otherwise, it is contradictory.

$\star$ A theory is finitely consistent if every finite subset is consistent.

$\star$ A closed formula F is a consequence of T if every model of T satisfies F. We denote this by $T \models F$.

(If F is not closed, consider its universal closure.)

$\star$ T_1 is equivalent to T_2 if and only if every model of T_1 is a model of T_2 and vice versa.

Proposition 3.5

1) If $F \sim F'$ and $G \sim G'$, then $\neg F \sim \neg F'$, $(F \alpha G) \sim (F' \alpha G')$, where α is $\wedge, \vee, \Rightarrow, \Leftrightarrow$

   $\forall x\, F \sim \forall x\, F'$

   $\exists x\, F \sim \exists x\, F'$

2) Let F be a formula, G a subformula of F, and $G' \sim G$. Then $F' = F_{\frac{G'}{G}} \sim F$.

Proof.

1) Consider the closed formula $\forall$. We need to show that the closure of $(\forall x\, F \Leftrightarrow \forall x\, F')$ is satisfied in any structure S. Write

$$F = F[x_0, ..., x_{n-1}, x]$$

$$F' = F'[x_0, ..., x_{n-1}, x]$$

Let $x \to a$

$$\begin{aligned} S \models \forall x\, F[a_0, ..., a_{n-1}] &\Leftrightarrow \forall a\, S \models F[a_0, ..., a_{n-1}, a] \\ &\Leftrightarrow \forall a\, S \models F'[a_0, ..., a_{n-1}, a] \\ &\Leftrightarrow S \models \forall x\, F'[a_0, ..., a_{n-1}] \\ &\Leftrightarrow S \models (\forall x\, F \Leftrightarrow \forall x\, F') \end{aligned}$$

2) By induction on the formula F.

Example.

$$\left.\begin{array}{c} \neg\forall x\, F \sim \exists x\, \neg F \\ \forall x(F \wedge G) \sim (\forall x\, F \wedge \forall x\, G) \\ \exists x(F \vee G) \sim (\exists x\, F \vee \exists x\, G) \\ \exists x(F \Rightarrow G) \sim (\forall x\, F \Rightarrow \exists x\, G) \\ \forall x\, \forall y\, F \sim \forall y\, \forall x\, F \\ \exists x\, \exists y\, F \sim \exists y\, \exists x\, F \end{array}\right\} \text{Equivalent formulas}$$

$$\left.\begin{array}{c} \exists x(F \wedge G) \Rightarrow (\exists x\, F \wedge \exists x\, G) \\ (\forall x\, F \vee \forall x\, G) \Rightarrow \forall x(F \vee G) \\ \exists x\, \forall y\, F \Rightarrow \forall y\, \exists x\, F \end{array}\right\} \text{Universally valid formulas}$$

If x is not free in F, we have: $\forall x\, F \sim \exists x\, F \sim F$.

Corollary. Every first-order formula is equivalent to a formula using only $\neg, \wedge, \vee$.

Proof. $\{\neg, \vee\}$ is complete for logical connectives, and $\exists$ can be expressed in terms of $\neg$ and $\forall$.