2.1 Alphabet and Word

Let A be any set (finite or infinite). The elements of A are called letters, and A itself is called an alphabet.

Definition 2.1. A word over the alphabet A is a finite sequence of elements from A:

$$U = U_1 U_2 \dots U_n$$

where n is the length of the word U.

The set of words over A is denoted by A^*.

On A^*, the concatenation operation is defined as follows:

$$A^* \times A^* \longrightarrow A^*$$

$$(U, V) \longmapsto U \cdot V = U_1 \cdot U_2 \dots U_n \cdot V_1 \cdot V_2 \dots V_m$$

where $U = U_1 \dots U_n$ and $V = V_1 \dots V_m$.

The length of a word defines a function:

$$l : A^* \longrightarrow \mathbb{N}$$

$$U \longmapsto l(U) = n = \text{length of } U$$

Concatenation is an associative operation and has the empty word ε as its neutral element:

$$u \cdot \varepsilon = \varepsilon \cdot u = u$$

In other words, A^* is a monoid.

Definition 2.2. We say that $a \in A$ has an occurrence in the word u if a is a letter of u, i.e., if $u = u_1 u_2 \ldots u_n$, then $\exists k \in \{1, 2, \ldots, n\}$ such that $a = u_k$.

Remark. There may be multiple occurrences of a in u.

Example.

$$A = \{a, b \ldots x, y, z\}$$

$$u = abaab$$

$$l(u) = 5$$

The letter a has three occurrences in u, and b has two.

Properties 2.1.

- $l(uv) = l(u) + l(v)$
- $uv = uw \Rightarrow v = w$
- $uv = vw \Rightarrow u = w$

Definition 2.3. The word u is a prefix of the word v if there exists a word w such that $v = uw$. The word u is a suffix of v if there exists a word w such that $v = wu$.

2.2 Syntax of Propositional Formulas

Definition 2.4. The propositional connectors are the symbols:

$$\neg : \text{ for negation (not)}$$

$$\wedge : \text{ for conjunction (and)}$$

$$\vee : \text{ for disjunction (or)}$$

$$\longrightarrow : \text{ for implication}$$

$$\longleftrightarrow : \text{ for equivalence}$$

Let P be a non-empty set of elementary or atomic propositions. The elements of P are denoted by p, q, r, s.

Remark.

1- In elementary logic, a proposition is a statement that communicates facts, e.g., $p =$ "It is raining," $q =$ "The weather is nice."

2- P does not contain the connectors $\neg, \wedge, \vee, \longrightarrow, \longleftrightarrow$; we consider the following alphabet: $A = P \cup \{\neg, \wedge, \vee, \longrightarrow, \longleftrightarrow\} \cup \{()\}$.

Let A^* be the set of words over A:

$$(p \longrightarrow q) \in A^*$$

$$(p \in A^*)$$

$$p \in A^*$$

$$() \in A^*$$

$$(pq\wedge) \in A^*$$

Definition 2.5. The set $\mathcal{F}$ of propositional formulas is the smallest subset of A^* that satisfies:

1- $P \subseteq \mathcal{F}$ (every elementary proposition is a formula).

2- If $F \in \mathcal{F}$, then $\neg F \in \mathcal{F}$.

3- If $F, G \in \mathcal{F}$, then $(F * G) \in \mathcal{F}$ with $* \in \{\wedge, \vee, \longrightarrow, \longleftrightarrow\}$.

Remark.

1- Formulas are sequences of symbols without any inherent meaning. Assigning a meaning, i.e., a truth value ("true" or "false") to a formula constitutes its semantics.

2- The term "smallest" is to be understood in terms of set inclusion. $\mathcal{F}$ is therefore the intersection of all subsets of A^* that satisfy properties 1, 2, and 3. This intersection is non-empty because A^* itself satisfies these properties, so $\mathcal{F} = \bigcap_{Y \subseteq A^*} Y$, where Y satisfies properties 1, 2, and 3.

Examples.

- $(\neg p \longrightarrow q)$ is a formula.

- $(p \land q \land r)$ is not a formula.
- $(\neg p \longrightarrow q)$ is a formula.
- p is a formula.
- $(p \longrightarrow q \lor r)$ is not a formula.

Definition 2.6. The length of a formula F is the number of letters in F: $l(F) = \#$ letters in F.

Example. $F = (p \land q)$, then $l(F) = 5$. If $F = p$, then $l(F) = 1$.

Remark There is no formula of length 0.

- It is possible to give a more explicit description of the set $\mathcal{F}$: we will define, by recursion, a sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of subsets of A^*. We set $\mathcal{F}_0 = P$ and for each n:

  $$\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{\neg F \mid F \in \mathcal{F}_n\} \cup \{(F * G) \mid F, G \in \mathcal{F}_n, * \in \{\land, \lor, \longrightarrow, \longleftrightarrow\}\}$$

  Note that the sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is increasing, i.e., $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for $n \leq m$, so $\mathcal{F}_n \subseteq \mathcal{F}_m$.

Proposition 2.1. $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$

Proof. Let $Z = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$:

Z is a subset of A^* that satisfies properties 1, 2, and 3, so $\mathcal{F} \subseteq Z$ (because $\mathcal{F}$ is the smallest subset of A^* that satisfies properties 1, 2, and 3).

Is $Z \subseteq \mathcal{F}$?

We show by induction that for every integer n, we have $\mathcal{F}_n \subseteq \mathcal{F}$.

If $n = 0$, then $P = \mathcal{F}_0 \subseteq \mathcal{F}$ by definition. We assume (induction hypothesis) that $\mathcal{F}_n \subseteq \mathcal{F}$, then $\mathcal{F}_{n+1} \subseteq \mathcal{F}$ according to the definition of $\mathcal{F}_{n+1}$ and the stability properties of $\mathcal{F}$.

Definition 2.7. The height of a formula $F \in \mathcal{F}$ is the smallest integer n such that $F \in \mathcal{F}$. It is denoted $h[F]$: $h(F) = \min\{n \mid F \in \mathcal{F}_n\}$.

Example.

- $F = p$, then $h(F) = 0$.

- $F = (p \wedge q)$, then $h(F) = 1$.
- $F = \neg p$, then $h(F) = 1$.
- $F = (\neg p \wedge q)$, then $h(F) = 2$.

2.3 Principle of Indication on the Set of Formulas

Suppose we want to demonstrate that a certain proposition $Q(F)$ holds for all $F \in \mathcal{F}$. We can do this by reasoning by induction (in the usual sense) on the height of F: we will then need to show, first, that $Q(F)$ is true for every formula F belonging to $\mathcal{F}_0$, and then that if $Q(F)$ is true for all $F \in \mathcal{F}_n$, then $Q(F)$ is also true for every $F \in \mathcal{F}_{n+1}$ for all $n \in \mathbb{N}$.

Principle. If Q verifies:

1) $Q(p)$ is true $\forall p \in P$ i.e., ($Q(F)$ is true for $F \in \mathcal{F}_n$).

2) $Q(F)$ is true $\Rightarrow Q(\rceil F)$ is true.

3) $Q(F)$ is true and $Q(G)$ is true $\Rightarrow Q(F * G)$ is true for $* \in \{, , \wedge, \vee, \Rightarrow, \Leftarrow\}$, then $Q(F)$ is true $\forall F \in \mathcal{F}$.

Example. $Q(F)$: "F has an equal number of opening and closing parentheses" i.e., $Q(F) =$ "$O(F) = f(F)$". We show that $Q(F)$ is true, $\forall F \in \mathcal{F}$. For this, let us define: $O(F) = \sharp$ opening parentheses, $f(F) = \sharp$ closing parentheses.

1) Let $F = p \in P$, $O(F) = f(F) = 0$, therefore $Q(p)$ is true.

2) We assume that $Q(F)$ is true $\Rightarrow Q(\rceil F)$ is true.

$$O(F) = f(F)$$

$$O(\rceil F) = O(F) = f(F) = f(\rceil F) \text{ i.e., } O(\rceil F) = f(\rceil F) \text{ i.e., } Q(\rceil F) \text{ is true.}$$

3) Let us assume that:

$$\left.\begin{array}{l} O(F) = f(F) \text{ and } O(G) = f(G) \\ O((F * G)) = O(F) + O(G) + 1 \\ f((F * G)) = f(F) + f(G) + 1 \end{array}\right\} \Rightarrow O((F{*}G)) = f((F{*}G)) \text{ hence } Q((F{*}G)) \text{ is true.}$$

Its Formulas.

We define the set $sf(F)$ of the subformulas of F by:

- If $F = p$, $sf(F) = \{F\}$
- If $F = \rceil G$, $sf(F) = \{sf(G)\} \cup \{F\}$.
- If $F = (G * H)$, $sf(F) = \{sf(G)\} \cup \{sf(H)\} \cup \{F\}$.

Example.

$$F = \rceil(((\underbrace{(p \Rightarrow q)}_{G} \wedge r) \iff \underbrace{s}_{H})$$

$$= \rceil\,(\underbrace{G \iff H}_{K}) = \rceil\,K$$

$$sf(F) = sf(\rceil K) = sf(K) \cup \{F\}$$

$$K = (G \iff H)$$

$$sf(K) = sf(G) \cup sf(H) \cup \{K\}$$

$$H = s,\ therefore:\ sf(H) = \{s\}$$

$$G = ((\underbrace{p \Rightarrow q)}_{G_1} \wedge \underbrace{r}_{G_2}) = (G_1 \wedge G_2)$$

$$sf(G) = sf(G_1 \wedge G_2) = sf(G_1) \cup sf(G_2) \cup \{G\}$$

$$sf(G_2) = \{r\}$$

$$G_1 = (p \Rightarrow q)\ therefore,\ sf(G_1) = sf(p \Rightarrow q) = \{p, q\} \cup \{p \Rightarrow q\}$$

$$sf(F) = \{p, q, r, s, (p \Rightarrow q), (p \Rightarrow q) \wedge r, ((p \Rightarrow q) \wedge r) \iff s, F\}$$

2.4 The Interpretation of a Logical Formula

2.4.1 Decomposition Tree of a Formula

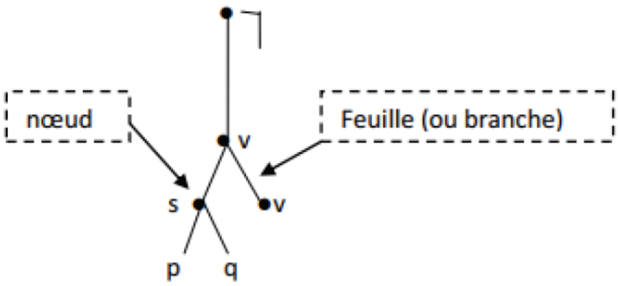
The tree A_F of the formula F is defined by recursion on F.

- • If $F = p$, then $A_F = {}^0p$.

- • If $F = \rceil G$, then $A_F = q^{\rceil}_{A_G}$.
- • If $F = (G * H)$, then $A_F = \begin{array}{c} * \\ /\backslash \\ A_G \quad A_H \end{array}$

  Where: $*$ can be any of the logical operators: $=, \wedge, \vee, \Rightarrow, \Longleftrightarrow$

Example. $F = \rceil((p \wedge q) \vee q)$



Example. $M = (((A \wedge (\rceil B \Rightarrow \rceil A)) \wedge (\rceil B \vee \rceil C)) \Rightarrow (C \Rightarrow \rceil A))$

Let us define: $M_0 = ((A \wedge (\rceil B \Rightarrow \rceil A)) \wedge (\rceil B \vee \rceil C))$ *and* $M_1 = (C \Rightarrow \rceil A)$

We first observe that M can be written as $(M_0 \Rightarrow M_1)$

Next, define: $M_{00} = (A \wedge (\rceil B \Rightarrow \rceil A))$, $M_{01} = (\rceil B \vee \rceil C)$, $M_{10} = C$, $M_{11} = \rceil A$

We write $M_0 = (M_{00} \wedge M_{01})$ *and* $M_1 = (M_{10} \Rightarrow M_{11})$ and proceed by successively defining:

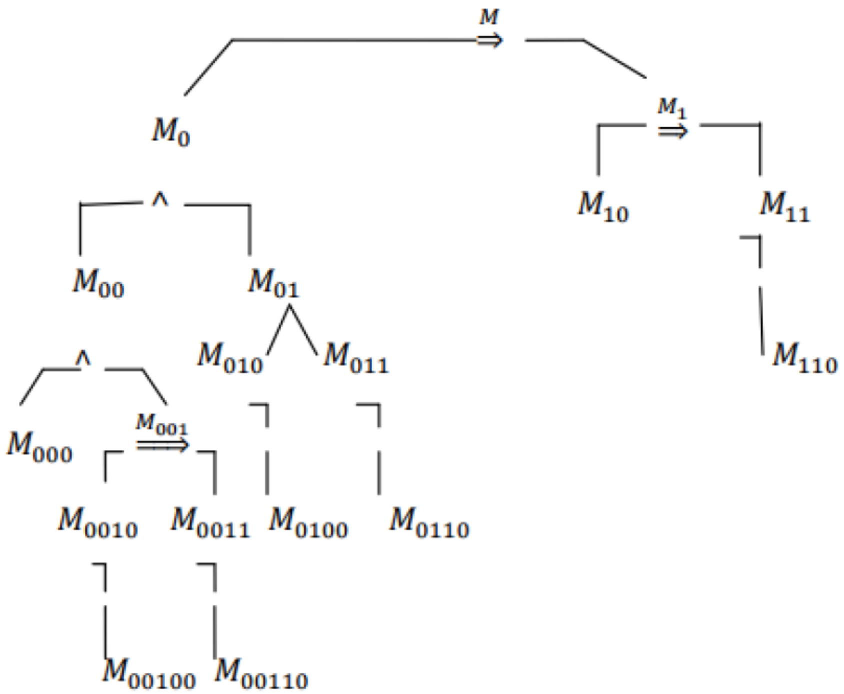
- $M_{000} = A$
- $M_{001} = (\rceil B \Rightarrow \rceil A)$
- $M_{010} = \rceil B$
- $M_{011} = \rceil C$
- $M_{110} = A$
- $M_{0010} = \rceil B$
- $M_{0011} = \rceil A$
- $M_{0100} = B$
- $M_{0110} = C$

- $M_{00100} = B$
- $M_{00110} = A$

So that:

$M_{00} = (M_{000} \wedge M_{001})$, $M_{01} = (M_{010} \vee M_{011})$, $M_{11} = \urcorner M_{110}$, $M_{001} = (M_{0010} \Rightarrow M_{0011})$, $M_{010} = \urcorner M_{0100}$, $M_{011} = \urcorner M_{0110}$, $M_{0010} = \urcorner M_{00100}$ *and* $M_{0011} = \urcorner M_{00110}$

$h(M) = 5$ i.e., $M \in \mathcal{F}_s$.

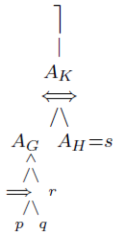


Definition 2.8. The height of a tree is the maximum number of leaf nodes from the root to one end of the tree; in Example 1, $h(F) = 3$.

Example. $F = \urcorner(((\underbrace{p \Rightarrow q}_{G}) \wedge r) \Longleftrightarrow \underbrace{s}_{H})$

$F = \urcorner(\underbrace{G \Longleftrightarrow H}_{K}) = \urcorner K$

$A_F = \underset{A_K}{\overset{\urcorner}{|}}$; $K = (G \Longleftrightarrow H)$



$$h(F) = 4$$

Each node n determines a subtree A_n corresponding to a subformula F. Conversely, the tree of each subformula is a subtree of the tree of the formula. Thus:

Subformulas of F = Subtrees of A_F

2.4.2 Substitution in a Formula

Let F be a formula, and let $p_1, p_2, \ldots, p_n$ be elementary propositions. The notation $F[p_1, p_2, \ldots, p_n]$ means that the propositional variables in F are among the p_i, where $i = 1, 2, \ldots, n$.

Example. If $F = (p \iff (p \wedge q))$, we write $F[p, q]$.

Definition 2.9. We denote $F_{\frac{G_1}{p_1}, \frac{G_2}{p_2}, \ldots, \frac{G_n}{p_n}}$ as the formula obtained by substituting G_i in place of p_i.

Alternative Notation. $F_{\frac{G_1}{p_1}, \ldots, \frac{G_n}{p_n}} = F[G_1, \ldots, G_n]$

Example. If $F = (p \iff (p \wedge q))$ and $G = (q \Rightarrow p)$, then $F_{\frac{G}{p}} = F(G, q) = ((q \Rightarrow p) \iff ((p \Rightarrow q) \wedge q))$

Proposition. The expression $F_{\frac{G_1}{p_1}, \ldots, \frac{G_n}{p_n}}$ is also a formula.

Proof. We proceed by induction on the formula F:

- If $F = p$:
  - If $F = p_K$, then $F_{\frac{G_1}{p_1}, \ldots, \frac{G_n}{p_n}} = G_K$
  - If $F = p \neq p_1, p_2, \ldots, p_n$, then $F_{\frac{G_1}{p_1}, \ldots, \frac{G_n}{p_n}} = F$
- If $F = \neg G$, assume $G_{\frac{G_1}{p_1}, \ldots, \frac{G_n}{p_n}}$ is a formula. Then $F_{\frac{G_1}{p_1}, \ldots, \frac{G_n}{p_n}} = \neg G_{\frac{G_1}{p_1}, \ldots, \frac{G_n}{p_n}}$ is also a formula.

- If $F = (G * H)$, where $*$ is $\wedge, \vee, \Rightarrow, \Leftrightarrow$, the reasoning is similar.

Theorem 2.1 (Substitution and Evaluations). Let v be a valuation, $F, G_1, G_2, \ldots, G_n$ be formulas, and $p_1, p_2, \ldots, p_n$ be elementary propositions. Define v' as follows:

$$v'(p) = \begin{cases} v(p) & \text{if } p \neq p_1, p_2, \ldots, p_n, \\ \bar{v}(G_i) & \text{if } p = p_i \text{ for } 1 \leq i \leq n. \end{cases}$$

Then $\bar{v}(F_{\frac{G_1}{p_1},\ldots,\frac{G_n}{p_n}}) = \bar{v'}(F)$.

Proof.
- If $F = p$:
  - If $p \neq p_i$, then $F_{\frac{G_1}{p_1},\ldots,\frac{G_n}{p_n}} = F$ and $\bar{v}(F_{\frac{G_1}{p_1},\ldots,\frac{G_n}{p_n}}) = \bar{v}(F) = v(F) = v'(F) = \bar{v'}(F)$.
  - If $p = p_i$, then $F_{\frac{G_1}{p_1},\ldots,\frac{G_n}{p_n}} = G_i$ and $\bar{v}(F_{\frac{G_1}{p_1},\ldots,\frac{G_n}{p_n}}) = \bar{v}(G_i) = v'(p_i) = v'(F) = \bar{v'}(F)$.
- If $F = \neg G$ and $\bar{v}(G_{\frac{G_1}{p_1},\ldots,\frac{G_n}{p_n}}) = \bar{v'}(G)$, then $\bar{v}(F_{\frac{G_1}{p_1},\ldots,\frac{G_n}{p_n}}) = \bar{v}(\neg G_{\frac{G_1}{p_1},\ldots,\frac{G_n}{p_n}}) = 1 + \bar{v}(G_{\frac{G_1}{p_1},\ldots,\frac{G_n}{p_n}}) = 1 + \bar{v'}(G) = \bar{v'}(\neg G) = \bar{v'}(F)$.
- If $F = (G * H)$ with $*$ being $\wedge, \vee, \Rightarrow, \Leftrightarrow$, similar reasoning applies.

Corollary. If F is a tautology, then $F_{\frac{G_1}{p_1},\ldots,\frac{G_n}{p_n}}$ is also a tautology.

Proof. For any valuation v, we have $\bar{v}(F_{\frac{G_1}{p_1},\ldots,\frac{G_n}{p_n}}) = \bar{v'}(F_{\frac{G_1}{p_1},\ldots,\frac{G_n}{p_n}})$.

Theorem 2.2. Let F be a formula, G a sub-formula of F, and H a formula equivalent to G. Then $F' = F_{\frac{H}{G}}$ is logically equivalent to F.

Proof. By induction on formulas:

- If $F = p$ and $G = F$, then $F' = H$, and hence $F' \sim F$.
- If $F = \neg F_1$, then $G = F$, and hence $H = F'$, and $F \sim F'$. If G is a sub-formula of F_1, then $F_1' = F_{1\frac{H}{G}} \sim F_1$, so $F' = \neg F_1' \sim F$.
- If $F = F_1 * F_2$, where $* = \wedge, \vee, \Rightarrow, \Leftrightarrow$, there are three possibilities:
  - If $G = F$ and $F' = H$, then $F' \sim F$.

– If G is a sub-formula of F_1, then by induction hypothesis, the formula F_1', resulting from substituting H for G in F_1, is logically equivalent to F_1. The formula F' is then $(F_1' * F_2)$, which is logically equivalent to F, because for any valuation v, $\bar{v}(F') = \bar{v}(F_1') \cdot \bar{v}(F_2) = \bar{v}(F_1) \cdot \bar{v}(F_2) = \bar{v}(F)$. Similar reasoning applies to other cases: $* = \vee, \Rightarrow, \Leftrightarrow$.

2.5 Semantics

Definition 2.10. A truth value distribution or valuation v is a function: $v : P \longrightarrow \{0, 1\}$ where P is the set of elementary propositions. We say that v defines a model $\mathcal{M}$ of propositional calculus. The values 0 and 1 represent "true" and "false" and can also be denoted by v and F, where $1 = v$ and $0 = F$. If P has a cardinality n, the number of different truth values is exactly $2^n = 2^{\sharp P}$.

Example. Let $P = \{p, q\}$, then $2^2 = 4$

1 1

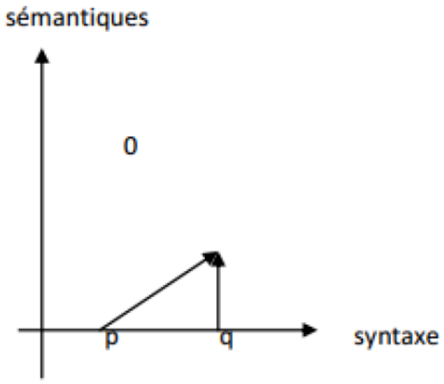
1 0

0 1

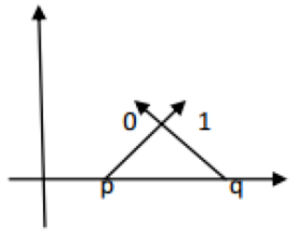
0 0

$$
\begin{array}{ll}
v_1 : P \longrightarrow \{0, 1\} & v_2 : P \longrightarrow \{0, 1\} \\
\quad p \longrightarrow 1 & \quad p \longrightarrow 1 \\
\quad q \longrightarrow 1 & \quad q \longrightarrow 0 \\
& \\
v_3 : P \longrightarrow \{0, 1\} & v_4 : P \longrightarrow \{0, 1\} \\
\quad p \longrightarrow 0 & \quad p \longrightarrow 0 \\
\quad q \longrightarrow 1 & \quad q \longrightarrow 0
\end{array}
$$

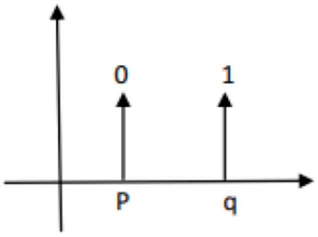
The goal of semantics is to assign truth values to the formulas of propositional calculus for the different valuations defined on the elementary propositions.



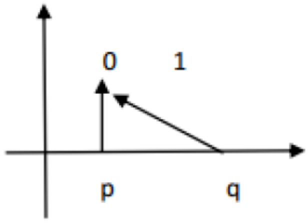
M_1



M_2



M_3



M_4

Theorem 2.3. For any valuation $v\ :\ P \longrightarrow \{0\,,\,1\}$, there exists a unique extension $\bar{v}\ :$ $\mathcal{F} \longrightarrow \{0\,,\,1\}$ $(i.e.\ \bar{v} = v\ \ on\ \ P)$ such that:

1) $\bar{v}(\neg F) = 1 \Longleftrightarrow \bar{v}(F) = 0.$

2) $\bar{v}((F \wedge G)) = 1 \Longleftrightarrow \bar{v}(F) = \bar{v}(G) = 1.$

3) $\bar{v}((F \vee G)) = 0 \Longleftrightarrow \bar{v}(F) = \bar{v}(G) = 0.$

4) $\bar{v}((F \Rightarrow G)) = 0 \Longleftrightarrow \bar{v}(F) = 1\, and\, \bar{v}(G) = 0.$

5) $\bar{v}((F \Leftrightarrow G)) = 1 \Longleftrightarrow \bar{v}(F) = \bar{v}(G).$

Proof. Let $\bar{v}_1$ and $\bar{v}_2$ be two extensions of v, and let $Q(F)$ be the proposition "$\bar{v}_1(F) = \bar{v}_2(F)$". We need to show that $Q(F)$ is true for all $F \in \mathcal{F}$

• $If\, F = p\ :\ \bar{v}_1(F) = \bar{v}_2(F) = v(F)$ then $Q(F)$ is true.

• *If* $F = \neg G$ *and* $Q(G)$ is true, then:

$$\left.\begin{array}{l} \bar{v}_1(F) = 1 \Longleftrightarrow \bar{v}_1(G) = 0 \\ \bar{v}_1(F) = 0 \Longleftrightarrow \bar{v}_1(G) = 1 \end{array}\right\} \Longrightarrow \bar{v}_1(F) = \bar{v}_2(F)$$

Hence, $Q(F)$ is also true.

• The same holds for $F = (G * H)$, $* = \wedge, \vee, \Rightarrow, \Leftrightarrow$.

Remark. If we define $+$ *and* $\times$ in $\frac{\mathbb{Z}}{\mathbb{Z}} = \{0\,,\,1\}$ by

$0 + 0 = 0 \qquad 0 \times 0 = 0$

$0 + 1 = 1 \qquad 0 \times 1 = 0$

$1 + 0 = 1 \qquad 1 \times 0 = 0$

$1 + 1 = 0 \qquad 1 \times 1 = 1$

the conditions 1) *and* 5) become:

1) $\bar{v}(\neg F) = 1 + \bar{v}(F)$.

2) $\bar{v}((F \wedge G)) = \bar{v}(F)\,\bar{v}(G)$.

3) $\bar{v}((F \vee G)) = \bar{v}(F) + \bar{v}(G) + \bar{v}(F)\,\bar{v}(G)$.

4) $\bar{v}((F \Rightarrow G)) = 1 + \bar{v}(F) + \bar{v}(F)\,\bar{v}(G)$.

5) $\bar{v}((F \Leftrightarrow G)) = 1 + \bar{v}(F) + \bar{v}(G)$.

These conditions are often written in truth table form for the connectives $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$.

F	$\neg F$
0	1

F	G	$(F \wedge G)$
1	1	1
1	0	0
0	1	0
0	0	0

F	G	$(F \vee G)$
1	1	1
1	0	1
0	1	1
0	0	0

F	G	$(F \Rightarrow G)$
1	1	1
1	0	0
0	1	1
0	0	1

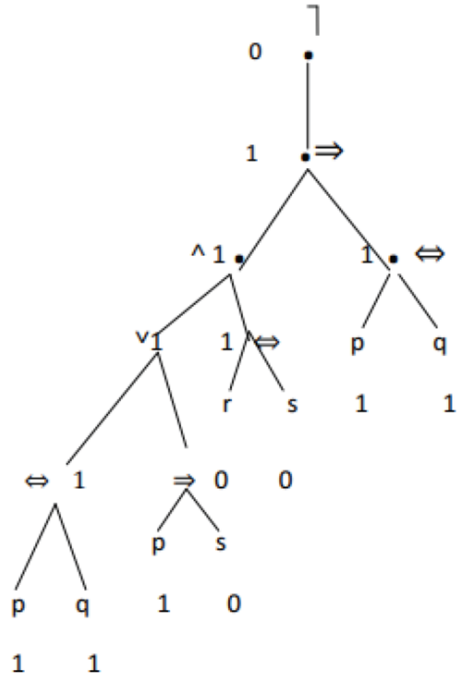
F	G	$(F \Leftrightarrow G)$
1	1	1
1	0	0
0	1	0
0	0	1

Example. $F = \rceil(((p \Leftrightarrow q) \vee (p \Rightarrow q) \wedge (r \Leftrightarrow s)) \Rightarrow (p \Rightarrow q))$

Assume that $P = \{p, q, r, s\}$

$$v : P \longrightarrow \{0\,,\,1\}$$
$$p \longrightarrow 1$$
$$q \longrightarrow 1$$
$$r \longrightarrow 0$$
$$s \longrightarrow 0$$

Compute $\bar{v}(F)$, hence: $\bar{v}(F) = 0$



2.6 Tautologies and Logical Equivalences

Let v be a truth value defining a model $\mathcal{M}$ of propositional logic, and let $\bar{v}$ be its extension to formulas.

Definition 2.11.

1) The formula F is said to be satisfied in the model $\mathcal{M}$ if $\bar{v}(F) = 1$, and we denote this as $\mathcal{M} \models F$. Otherwise, it is said to be unsatisfied $\bar{v}(F) = 0$, and we denote this as $\mathcal{M} \nvDash F$.

2) F is a tautology if for every model $\mathcal{M}$, we have $\mathcal{M} \models F$, denoted as $\vDash F$.

Example. $F = (p \vee \neg p)$, $P = \{p\}$. The truth value $\bar{v}_2(F) = 1$. F is an anti-tautology if for every model $\mathcal{M}$, we have $\mathcal{M} \nvDash F$, denoted as $\nvDash F$.

Example. $F = (p \wedge \neg q)$, $P = \{p\}$, $v_1 : p \longrightarrow 1$, $\bar{v}_1(F) = 0$

, $v_2 : p \longrightarrow 0$, $\bar{v}_2(F) = 0$.

- A tautology is therefore a formula that is always true ($\forall$ valuation).

- An anti-tautology is a formula that is always false.

3) F is logically equivalent to G if $(F \Leftrightarrow G)$ is a tautology, denoted as $F \sim G$. In other words, $\bar{v}(F) = \bar{v}(G)$ for every valuation v.

Example. $F = p$, $F \sim G$, because $(p \Leftrightarrow \neg\neg p)$ is a tautology $G = \neg\neg p$.

Remark.

1) In terms of truth tables, a tautology is a formula that has 1 everywhere in its last column.

- An anti-tautology has 0 everywhere in its last column.

- Two logically equivalent formulas have the same truth tables.

2) $\sim$ defines an equivalence relation on $\mathcal{F}$, with the quotient set $\dfrac{\mathcal{F}}{\sim} = \{[F], F \in \mathcal{F}\}$, $[F] = \{G \in \mathcal{F} / F \sim G\}$ being the equivalence classes of F. When comparing two formulas "for logical equivalence," it means comparing the corresponding classes in $\dfrac{\mathcal{F}}{\sim}$.

$$\left.\begin{array}{l} F \text{ is a tautology} \Longleftrightarrow \forall v,\, \bar{v}(F) = 1 \\ G \text{ is a tautology} \Longleftrightarrow \forall v,\, \bar{v}(G) = 1 \end{array}\right\} \Longleftrightarrow F \sim G$$

Thus, all tautologies are logically equivalent and form the class 1.

Similarly, all anti-tautologies are logically equivalent and form the class 0.

3) $F = G \Rightarrow F \sim G$.

$F \sim G \not\Rightarrow F = G$.

$F \sim G \Rightarrow [F] = [G]$.

Example. Here are some tautologies in the form of equivalences:

1) Idempotency of conjunction and disjunction

$((p \wedge p) \Leftrightarrow p)$

$((p \vee p) \Leftrightarrow p)$

2) Commutativity of conjunction, disjunction, and equivalence:

$((p \wedge q) \Leftrightarrow (q \wedge p))$, $((p \vee q) \Leftrightarrow (q \vee p))$, $((p \Leftrightarrow q) \Leftrightarrow (q \Leftrightarrow p))$

3) Associativity of conjunction, disjunction, equivalence:

$(((p \vee q) \vee r) \Leftrightarrow (p \vee (q \vee r)))$, $(((p \wedge q) \wedge r) \Leftrightarrow (p \wedge (q \wedge r)))$,

$(((p \Leftrightarrow q) \Leftrightarrow r) \Leftrightarrow (p \Leftrightarrow (q \Leftrightarrow r)))$

4) Distributivity of disjunction/conjunction and vice versa:

$(p \vee (q \wedge r)) \Leftrightarrow ((p \vee q) \wedge (p \vee r))$, $(p \wedge (q \vee r)) \Leftrightarrow ((p \wedge q) \vee (p \wedge r))$

5) Absorption:

$((p \wedge (p \vee q)) \Leftrightarrow p)$, $((p \vee (p \wedge q)) \Leftrightarrow p)$

6) De Morgan's Laws:

$(\neg(p \vee q)) \Leftrightarrow (\neg p \wedge \neg q)$, $(\neg(p \wedge q)) \Leftrightarrow (\neg p \vee \neg q)$

7) Contrapositive:

$((p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p))$

8) $(\neg\neg p \Leftrightarrow p)$, $((p \Rightarrow q) \Leftrightarrow (\neg p \vee q))$

Here are non-equivalent formulas:

$(p \wedge p)$ and $(q \wedge q)$ (taking $v(p) = 1$ and $v(q)$)

$(p \Rightarrow p)$ and p (taking $v(p) = 0$)

$(p \Leftrightarrow q)$ and $(p \Rightarrow q)$ (taking $v(p) = 0$, $v(q) = 1$)

$(p \Rightarrow (p \Rightarrow p))$ and $((p \Rightarrow p) \Rightarrow p)$ (taking $v(p) = 0$)

Remark. Thanks to the associativity of $\wedge$ and $\vee$ the following notations can be used: The formula $((F \wedge G) \wedge H)$ will be denoted as $(F \wedge G \wedge H)$.

The formula $((F \vee G) \vee H)$ will be denoted as $(F \vee G \vee H)$. More generally, for any natural number k if $F_1, F_2, ..., F_k$ are formulas, we will represent them as:

$$\underbrace{(F_1 \wedge F_2 \wedge ... \wedge F_k)}_{\bigwedge_{i=1}^{k} F_i} \underset{def}{=} F_1 \wedge (F_2 \wedge (... \wedge F_k...))$$

$$\underbrace{(F_1 \vee F_2 \vee ... \vee F_k)}_{\bigvee_{i=1}^{k} F_i} \underset{def}{=} F_1 \vee (F_2 \vee (... \vee F_k..))$$

In the list below, formulas on the same line are pairwise logically equivalent:

1) $(A \Rightarrow B)$, $(\neg A \vee B)$, $((A \wedge B) \Leftrightarrow A)$, $((A \vee B) \Leftrightarrow B)$.

2) $\neg(A \Rightarrow B)$, $(A \wedge \neg B)$.

3) $(A \Leftrightarrow B)$, $((A \wedge B) \vee (\neg A \wedge \neg B))$, $((\neg A \vee B) \wedge (\neg B \vee A))$.

4) $(A \Leftrightarrow B)$, $((A \Rightarrow B) \wedge (B \Rightarrow A))$, $(\neg A \Leftrightarrow \neg B)$, $(B \Leftrightarrow A)$.

5) $(A \Leftrightarrow B)$, $((A \vee B) \Rightarrow (A \wedge B))$.

6) $\neg(A \Leftrightarrow B)$, $(A \Leftrightarrow \neg B)$, $(\neg A \Leftrightarrow B)$.

7) A, $(A \wedge T)$, $(A \vee T)$, $(A \Leftrightarrow T)$, $(T \Rightarrow A)$.

8) $\neg A$, $(A \Rightarrow \neg A)$, $((A \Rightarrow B) \wedge (A \Rightarrow \neg B))$.

9) $\neg A$, $(A \Rightarrow \perp)$, $(A \Leftrightarrow \perp)$

10) $\perp$, $(A \wedge \perp)$, $(A \Leftrightarrow \neg A)$.

11) T, $(A \vee T)$, $(A \Rightarrow T)$, $(\perp \Rightarrow A)$.

12) $(A \Rightarrow (B \wedge C))$, $((A \Rightarrow B) \wedge (A \Rightarrow C))$.

13) $(A \Rightarrow (B \vee C))$, $((A \Rightarrow B) \vee (A \Rightarrow C))$.

14) $((A \wedge B) \Rightarrow C)$, $((A \Rightarrow C) \vee (B \Rightarrow C))$.

15) $((A \vee B) \Rightarrow C)$, $((A \Rightarrow C) \wedge (B \Rightarrow C))$.

Note. From lines 12) to 15) it can be observed that there is no distributivity of implication with respect to conjunction or disjunction. However, distributivity is present on the left side in 12) and 13), i.e., when the "$\wedge$" or "$\vee$" is on the right side of the $\Rightarrow$. In cases 14) and 15), there is a sort of false distributivity where "$\wedge$" (or $\vee$) is transformed into "$\vee$" (or $\wedge$).

Theorem 2.4 (Substitutions and Evaluation) Let v be an evaluation, $F, G_1, G_2, \ldots, G_n$ be formulas, and $p_1, p_2, \ldots, p_n$ be elementary propositions.

Let v' be the evaluation defined by:

$$v' = \begin{cases} v(p) \text{ if } p \neq p_1, p_2, \ldots, p_n. \\ \bar{v}(G_i) \text{ if } p = p_i \ \ (1 \leq i \leq n) \end{cases}$$

Then: $\bar{v}(F_{\frac{G_1}{p_1} \cdots \frac{G_n}{p_n}}) = \bar{v'}(F)$.

Proof. We proceed by induction on formulas:

* If $F = p$,

  - If $p \neq p_i$ then $F_{\frac{G_1}{p_1}\cdots\frac{G_n}{p_n}} = F$ and $\bar{v}(F_{\frac{G_1}{p_1}\cdots\frac{G_n}{p_n}}) = \bar{v}(F) = v(F) = v'(F) = \bar{v'}(F)$.

  - If $p = p_i$ then $F_{\frac{G_1}{p_1}\cdots\frac{G_n}{p_n}} = G_i$ and $\bar{v}(F_{\frac{G_1}{p_1}\cdots\frac{G_n}{p_n}}) = \bar{v}(G_i) = v'(p_i) = v'(F) = \bar{v'}(F)$.

* If $F = \neg G$,

  $\bar{v}(G_{\frac{G_1}{p_1}\cdots\frac{G_n}{p_n}}) = \bar{v'}(G)$

  $\bar{v}(F_{\frac{G_1}{p_1}\cdots\frac{G_n}{p_n}}) = \bar{v}(\neg G_{\frac{G_1}{p_1}\cdots\frac{G_n}{p_n}}) = 1 + \bar{v}(G_{\frac{G_1}{p_1}\cdots\frac{G_n}{p_n}}) = 1 + \bar{v'}(G) = \bar{v'}(\neg G) = \bar{v'}(F)$.

* If $F = G \wedge H$,

  $\bar{v}(G_{\frac{G_1}{p_1}\cdots\frac{G_n}{p_n}}) = \bar{v'}(G)$ and $\bar{v}(H_{\frac{G_1}{p_1}\cdots\frac{G_n}{p_n}}) = \bar{v'}(H)$

  $\bar{v}(F_{\frac{G_1}{p_1}\cdots\frac{G_n}{p_n}}) = \bar{v}((G_{\frac{G_1}{p_1}\cdots\frac{G_n}{p_n}}) \wedge (H_{\frac{G_1}{p_1}\cdots\frac{G_n}{p_n}})) = \bar{v}(G) \cdot \bar{v}(H) = \bar{v'}(G) \cdot \bar{v'}(H) = \bar{v'}(G \wedge H) = \bar{v'}(F)$.

* Similar reasoning applies to other cases.

Corollary. If F is a tautology, then the formula $F_{\frac{G_1}{p_1}\cdots\frac{G_n}{p_n}}$ is also a tautology.

Proof: For any evaluation v we have: $\bar{v}(F_{\frac{G_1}{p_1}\cdots\frac{G_n}{p_n}}) = \bar{v'}(F) = 1$.

2.6.1 Normal Forms

2.7 Complete Systems of Connectors

Definition 2.12.

1) For any n-tuple $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n) \in \{0, 1\}^n$, we denote by $V_{\varepsilon_1,\varepsilon_2,...,\varepsilon_n}$ the valuation defined by $V_{\varepsilon_1,\varepsilon_2,...,\varepsilon_n}^{(p_i=\varepsilon_i)}$ for all $i \in \{1, 2, ..., n\}$

2) For each propositional variable p and each element $\varepsilon \in \{0, 1\}$, we denote by ε_p the formula:

$$\varepsilon_p = \begin{cases} p \text{ if } \varepsilon = 1 \\ \neg p \text{ if } \varepsilon = 0 \end{cases}$$

3) For any formula F we denote by $\Delta(F) = \{v \in \{0\,,\,1\}^P \,|\, \bar{v}(F) = 1\}$ the set of valuations such that the formula F defines a function:

$$\varnothing_F : \{0\,,\,1\}^n \longrightarrow \{0\,,\,1\}$$
$$(\varepsilon_1,\, \varepsilon_2,\, \dots,\, \varepsilon_n) \longmapsto \bar{v}_{\varepsilon_1,\, \varepsilon_2,\, \dots,\, \varepsilon_n}(F)$$

$\varnothing_F$ is compatible with the logical equivalence relation. In other words, $F \sim G \Leftrightarrow \varnothing_F = \varnothing_G$. Thus, $\varnothing_F$ defines, by quotienting, a function:

$$\varnothing : \frac{\mathcal{F}}{\sim} \longrightarrow \{0\,,\,1\}^{(0\,,\,1)^n}$$
$$.[F] \longmapsto \varnothing_F$$

where $[F]$ is the equivalence class of the formula F with respect to the relation $\sim$.

Theorem 2.5 $\varnothing$ is a bijection.

Proof.

1) $\oslash$ **Injectivity:** Let $[F]$, $[G]$ be two classes of formulas

$\varnothing([F]) = \varnothing([G]) \Rightarrow \varnothing_F = \varnothing_G \Leftrightarrow F \sim G \Leftrightarrow [F] = [G]$.

Thus: $\varnothing$ is injective.

2) $\varnothing$ **Surjectivity:** Let $\varnothing : \{0\,,\,1\}^n \longrightarrow \{0\,,\,1\}$. We need to find if there exists an $F \in \mathcal{F}$ such that $\varnothing = \varnothing_F$?

$*$ If $\varnothing$ takes only the value 0, then any anti-tautology F satisfies $\varnothing = \varnothing_F$, for example, $F = (p_1 \wedge \neg p_1)$

$*$ Otherwise, the set $x = \varnothing^{-1}(\{1\}) = \{(\varepsilon_1,\, \varepsilon_2, \dots, \varepsilon_n) \in \{0, 1\}^n \,|\, \varnothing(\varepsilon_1,\, \varepsilon_2, \dots, \varepsilon_n) = 1\}$ is non-empty.

Let $F_x = \bigvee_{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in X} \left(\bigwedge_{1 \le i \le n} \varepsilon_i\, p_i\right)$, then $\Delta(F_x) = \{\bigvee_{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in X},\, (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in X\}$ $\circledast$ i.e., $\bar{v}_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}(F_x) = 1 \Leftrightarrow \varnothing(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) = 1$, thus: $\varnothing = \varnothing_{F_x}$ for, $\circledast$ for all $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, $\Delta\left(\bigwedge_k \varepsilon_k\, p_k\right) = \bigvee_{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in x},\, \Delta\left(\bigvee_{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in x} \left(\bigwedge_i \varepsilon_i\, p_i\right)\right) = \{v_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n},\, (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in x\}$, $\bar{v}\left(\bigwedge_k \varepsilon_k\, p_k\right) = 1 \Leftrightarrow \bar{v}\left(\bigwedge_k \varepsilon_k\, p_k\right) = 1 \Leftrightarrow v(p_k) = v_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}(p_k)$.

Corollary. If $\sharp p = n$ then there are exactly 2^{2^n} classes of formulas, each corresponding to a function $\varnothing : \{0\,,\,1\}^n \longrightarrow \{0\,,\,1\}$

Definition 2.12. A function $\varnothing : \{0\,,\,1\}^n \longrightarrow \{0\,,\,1\}$ is called a propositional connector with n places.

Example. We have already seen the connector

1) According to the previous definition, it corresponds to the 2-place connectors.

$$\begin{aligned}\varnothing : \{0\,,\,1\}^2 &\longrightarrow \{0\,,\,1\}\\ (0\,,\,0) &\longmapsto 0\\ (0\,,\,1) &\longmapsto 0\\ (1\,,\,0) &\longmapsto 0\\ (1\,,\,1) &\longmapsto 1\end{aligned}$$

Or equivalently, to the class of the formula $p_1 \wedge p_2$.

2) An example of a 1-place connector is:

$$\begin{aligned}\varnothing : \{0\,,\,1\} &\longrightarrow \{0\,,\,1\}\\ 0 &\longmapsto 1\\ 1 &\longmapsto 0\end{aligned}$$

It corresponds to the class of $\neg p_1$ and is thus the usual connector $\neg$.

3) The following 2-place connector is called the "NAND" connector:

$$\begin{aligned}\varnothing : \{0\,,\,1\}^2 &\longrightarrow \{0\,,\,1\}\\ (0\,,\,0) &\longmapsto 1\\ (0\,,\,1) &\longmapsto 0\\ (1\,,\,0) &\longmapsto 0\\ (1\,,\,1) &\longmapsto 0\end{aligned}$$

which corresponds to the formula $\neg(p_1 \wedge p_2)$.

2.7.1 Normal Forms

Definition 2.13. A formula F is said to be in canonical disjunctive normal form (CDNF) if there exists a non-empty subset $X \subseteq \{0\,,\,1\}^n$ such that $F = \bigvee_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)\in X} (\bigwedge_{1\leq i\leq n} \varepsilon_i\, p_i)$.

It is said to be in disjunctive normal form (DNF) if there exist:

* An integer $m \geq 1$

* Integers $k_1, ..., k_m \geq 1$

* For $1 \leq i \leq m$, variables $p_{i_1}, p_{i_2}, ..., p_{ik_i}$ and k_i elements $\varepsilon_{i_1}, ..., \varepsilon_{k_i}$ from $\{0\,,\,1\}$ such that:

$$F = \bigvee_{1 \leq i \leq m} (\varepsilon_{i_1} p_{i_1} \wedge \varepsilon_{i_2} p_{i_2} \wedge \ldots \wedge \varepsilon_{ik_i} p_{ik_i})$$

Similarly, conjunctive normal forms (CNF) and canonical conjunctive normal forms (CCNF) are defined (by swapping the symbols for disjunction and conjunction).

Remark. A CDNF is a DNF. Similarly, a CCNF is a CNF

$(n = k_i, \forall\, i, p_{ij} = p_j)$

Theorem 2.6 Every formula F is logically equivalent to a CNF and a DNF. The NAND connector: "or" $\rceil (p_1 \vee p_2) = (p_1 \vee p_2)$ $\vee$: NAND connector "or".

p_1	p_2	F
1	1	0
1	0	0
0	1	0
0	0	1

$\varnothing_F : \{0\,,\,1\}^2 \longrightarrow \{0\,,\,1\}$

$\{0\,,\,1\}^2 = \{(0,0)\,,\,(0,1)\,,\,(1,0)\,,\,(1,1)\}$

$(0,0) \longmapsto 1$, $(0,1) \longmapsto 0$

$F = \rceil (p_1 \wedge p_2) \underset{def}{=} (p_1\; p_2)$

$\wedge$: NAND connector "and"

$$\begin{aligned} \varnothing = \varnothing_F : \{0\,,\,1\}^2 &\longrightarrow \{0\,,\,1\} \\ (0,0) &\longmapsto 1 \\ (0,1) &\longmapsto 1 \end{aligned}$$

An n-place connector is a function $\varnothing\{0\,,\,1\}^n \longrightarrow \{0\,,\,1\}$.

p_1	p_2	$\rceil(p_1 \wedge p_2)$
1	1	0
1	0	1
0	1	1
0	0	1

Example. $F = ((p_1 \wedge p_2) \Rightarrow p_3)$

$\varnothing\{0\,,\,1\}^3 \longrightarrow \{0\,,\,1\}$

$(0,0,0) \longmapsto 1$

$(0,0,1) \longmapsto 1$

$(1,1,0) \longmapsto 0$

$\varnothing$ is a 3-place connector.

Theorem 2.7 (Normal Form) Every formula F is logically equivalent to at least one formula in disjunctive normal form (DNF) and at least one formula in conjunctive normal form (CNF).

Proof. $*$ If F is a tautology, it is logically equivalent to $p_1 \wedge \rceil p_1$ which is both a DNF and a CNF. $*$ If F is neither a tautology nor an anti-tautology, then by the previous theorem, there exists $x \neq \varnothing / \varnothing_F = \varnothing_{F_x}$, $x \in \{0\,,\,1\}^n$ i.e.: $F \sim F_x$ which is a CDNF and hence also a DNF. For $\rceil F$, there also exists $x'/\rceil F \sim F_{x'}$, so: $F = \rceil\rceil F \sim \rceil(\vee(\wedge)) = \wedge(\vee) \sim CCNF$ (according to De Morgan's law).

Example. $G = (A \Rightarrow (((B \wedge \rceil A) \vee (\rceil C \wedge A)) \Leftrightarrow (A \vee (A \Rightarrow \rceil B)))$ Let $H = (B \wedge \rceil A)$, $I = (\rceil C \wedge A)$, $J = (A \Rightarrow \rceil B)$, $K = (H \vee I)$, $L = (A \vee J)$ and $M = (K \Leftrightarrow L)$. Then we have $G = (A \Rightarrow M)$. The truth table for G is:

A	B	C	$\rceil A$	$\rceil B$	$\rceil C$	H	I	J	K	L	M	G
1	1	1	0	0	0	0	0	0	0	1	0	0
1	1	0	0	0	1	0	1	0	1	1	1	1
1	0	1	0	1	0	0	0	1	0	1	0	0
1	0	0	0	1	1	1	1	1	1	1	1	1
0	1	1	1	0	0	1	0	0	1	1	1	1
0	1	0	1	0	1	1	1	0	1	1	1	1
0	0	1	1	1	0	0	0	1	0	1	0	1
0	0	0	1	1	1	0	1	1	1	1	1	1

According to the truth table, G is satisfied by the valuations

$(0,0,0)$, $(0,0,1)$, $(0,1,1)$, $(1,0,0)$, $(1,1,0)$, while $\rceil G$ is satisfied by $(1,0,1)$ and $(1,1,1)$.

We derive the CDNF of G:

$(\rceil A \wedge \rceil B \wedge \rceil C) \vee (\rceil A \wedge \rceil B \wedge C) \vee (\rceil A \wedge B \wedge C) \vee (A \wedge \rceil B \wedge \rceil C) \vee (A \wedge B \wedge \rceil C)$.

Next, the CDNF of $\rceil G$: $(A \wedge \rceil B \wedge C) \vee (A \wedge B \wedge C)$.

Finally, the CCNF of G: $(\rceil A \wedge B \wedge \rceil C) \wedge (\rceil A \wedge \rceil B \wedge \rceil C)$.

Example. $F = \rceil((\rceil p \Rightarrow q) \Rightarrow \rceil(q \Leftrightarrow p))$ We use the equivalences:

$(p \Rightarrow q) \Leftrightarrow (\rceil p \vee q)$

$(p \Leftrightarrow q) \Leftrightarrow ((\rceil p \vee q) \wedge (\rceil q \vee p))$

$$\begin{aligned} F &\sim \rceil(\rceil(\rceil p \Rightarrow q) \vee \rceil(q \Leftrightarrow p)) \\ &\sim \rceil(\rceil(\rceil\rceil p \vee q) \vee \rceil((\rceil q \vee p) \wedge (\rceil p \vee q))) \\ &\sim \rceil(\rceil(p \vee q) \vee \rceil((\rceil q \vee p) \wedge (\rceil p \vee q))) \end{aligned}$$

We then use De Morgan's laws:

$\rceil(p \vee q) \Leftrightarrow (\rceil p \wedge \rceil q)$

$\rceil(p \vee q) \Leftrightarrow (\rceil p \vee \rceil q)$

Therefore:

$$\begin{aligned} F &\sim \rceil\rceil(p \vee q) \wedge \rceil\rceil((\rceil q \vee p) \wedge (\rceil p \vee q)) \\ &\sim (p \vee q) \wedge (\rceil q \vee p) \wedge (\rceil p \vee q) \\ &\sim (p \vee q) \wedge (\rceil q \vee p) \wedge (\rceil p \vee q) \end{aligned}$$

2.8 Complete Systems of Connectives

Definition 2.14.

1) Let $\alpha_1, \alpha_2, ..., \alpha_k$ be a set of connectives of arbitrary arity. The set $\alpha_1, \alpha_2, ..., \alpha_k$ is a complete system if and only if: for any formula $F \in \mathcal{F}$, there exists a formula G based on the alphabet $P \cup \{\alpha_1, ..., \alpha_k\} \cup \{(,)\}$ such that $F \sim G$.

2) $\alpha_1, \alpha_2, ..., \alpha_k$ is a minimal complete system if no subset $A \subsetneq \{\alpha_1, ..., \alpha_k\}$ is a complete system.

   Example: $\{\rceil, \vee, \wedge\}$ is a complete system of connectives.

   Proposition:

   1) The system $\{\rceil, \vee, \wedge\}$ is not minimal.

   2) The system $\{\rceil, \vee\}$ is minimally complete.

3) The system $\{\rceil, \wedge\}$ is minimally complete.

Proof.

1) $(p \wedge q) \sim \rceil(\rceil p \vee \rceil q)$, hence $(p \wedge q)$ can be expressed in terms of $\rceil$ and $\vee$.

2) Suppose $\vee$ can be expressed in terms of $\rceil$. Then, every formula is $\sim \rceil\rceil....\rceil p$ and therefore either p or $\rceil p$, which is not the case for $(p \wedge q)$.

2.8.1 Theories

Definition 2.15. A theory $\mathcal{Z}$ of propositional calculus is a set of formulas $T \subseteq \mathcal{F}$.

Let $\mathcal{M}$ be a model defined by the valuation v. We say that:

1) T is satisfied in m if $\mathcal{M} \models F$ for all $F \in T$. We write $\mathcal{M} \models T$.

2) T is consistent or non-contradictory or satisfiable if there exists a model $\mathcal{M} \models T$.

3) T is finitely satisfiable if and only if every finite sub-theory $T' \subseteq T$ is satisfiable (this definition is of interest only for infinite T).

4) T is contradictory if and only if it is not satisfiable, i.e., it has no model.

5) The formula F is a consequence of T if and only if every model of T is a model of F, i.e., $\mathcal{M} \vDash T \Rightarrow \mathcal{M} \vDash F$. We denote this as $T \vDash F$ or $(T \underset{\vdash}{*} F)$.

6) T and T' are two equivalent theories if and only if they have exactly the same models (or every formula in T is a consequence of T' and every formula in T' is a consequence of T).

Examples. Consider distinct propositional variables $p, q, p_1, p_2, ..., p_m ...$: the set $\{p, q, (\rceil p \vee q)\}$ is satisfiable; $\{p, \rceil q\}$ is contradictory; the empty set is satisfied by any valuation. $\{p, q\} \vDash (p \wedge q)$, $\{p, (p \Rightarrow q)\} \vDash q$, the set $\{p, q\}$ and $\{(p \wedge q)\}$ are equivalent, as well as $\{p_1, p_2, ..., p_m, ..\}$ and $\{p_1 \wedge p_2 \wedge ...p_m \wedge ...\}$.

Lemma. For any theories $T \; and \; T'$, integers $p \geq 1$, formulas $G, H, F_1, F_2, ...F_m$, and $G_1, G_2, ...G_p$, the following properties are verified:

$*T \vDash G$ if and only if $T \cup \{\rceil G\}$ is contradictory.

$*$ If T is satisfiable and $T' \subseteq T$, then T' is satisfiable.

$*$ If T is satisfiable, then T is finitely satisfiable.

$*$ If T is contradictory and $T \subseteq T'$, then T' is contradictory.

$*$ If $T \vDash G$ and $T \subseteq T'$, then $T' \vDash G$.

$*T \cup \{G\} \vDash H$ if and only if $T \vDash (G \Rightarrow H)$.

$*T \vDash (G \wedge H)$ if and only if $T \vDash G \; and \; T \vDash H$.

$*\{F_1, F_2, ..., F_m\} \vDash G$ if and only if $\vDash ((F_1 \wedge F_2 \wedge ... \wedge F_m) \Rightarrow G)$.

$*G$ is a tautology if and only if G is a consequence of the empty set.

$*G$ is a tautology if and only if G is a consequence of any set of formulas.

$*G$ is contradictory if and only if $T \vDash (G \wedge \rceil G)$.

$*G$ is contradictory if and only if every anti-tautology is a consequence.