Numerical analysis final exam correction 2024/2025

Solution of exercise 1: (6pts)

1. Newton's method………………………………………………………………………….(2pts)

x	$f(x)$			
-1	0			
		2/3		
1/2	1		-4/3	
		-2		8/15
1	0		0	
		-2		
3/2	-1			

$p(x) = \frac{2}{3}(x+1) - \frac{4}{3}(x+1)\left(x+\frac{1}{2}\right) + \frac{8}{15}(x+1)\left(x+\frac{1}{2}\right)(x-1)$ …………………(0.5pts)

$$p(x) = \frac{8}{15}x^3 - \frac{8}{5}x^2 - \frac{8}{15}x + \frac{8}{5}$$

2. $p\left(\frac{1}{4}\right) = 1.3750, f\left(\frac{1}{4}\right) = 0.7071 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . \dots\dots . (1\text{pts})$

$$e = \left|f\left(\frac{1}{4}\right) - p\left(\frac{1}{4}\right)\right| = 0.6679 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots ... (0.5\text{pts})$$

3. Interpolation error

$E = |f(x) - p(x)| = \frac{|f^{(4)}(c)|}{4!}|(x-x_0)(x-x_1)(x-x_2)(x-x_3)|, c \in [-1, \frac{3}{2}]$ ……..(0.5pts)

$$f^{(4)}(c) = \pi^4 \sin \pi c$$

$E = \left|f\left(\frac{1}{4}\right) - p\left(\frac{1}{4}\right)\right| \leq \frac{\pi^4}{24}\left|\left(\frac{1}{4}+1\right)\left(\frac{1}{4}-\frac{1}{2}\right)\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-\frac{3}{2}\right)\right| = 1.1891$ …………………..(1.5pts)

Solution of exercise 2:(8pts)

1. Study of function changes ………………………………………………………………………….(1.5pts)

$$f(x) = e^x - 4x, \qquad x \in [0, +\infty[$$

$f'(x) = e^x - 4, f'(x) = 0 \Rightarrow x = \ln 4$

x	0	$\ln 4$	$+\infty$
$f'(x)$	$-$	0	$+$
$f(x)$	1 ↘	$f(\ln 4)$	↗ $+\infty$

$f(\ln 4) = 4 - 8\ln 2 = -1.5452$

2. f decreasing, continuous on $[0\,,1]$ and $f(0)f(1) < 0$ ………………………………………….(1pts)

According I.V.T the equation $f(x) = 0$ has only one solution in$[0\,,1]$

3. the iterations in the bisection method for $\varepsilon = 0.08$ ………………………………………………(1pts)

$$n \geq \frac{\ln \frac{b-a}{\varepsilon}}{\ln 2} = 3.64, \text{ so } n = 4$$

4. Study of convergence of the sequence ……………………………………………………………….(1pts)

$$\begin{cases} x_{n+1} = g(x_n), \quad n = 0,1,2, \dots \dots \dots \\ \quad x_0 \in [a,b], \quad g(x) = \frac{e^x}{4} \end{cases}$$

$$g'(x) = \frac{e^x}{4} > 0, \quad \max_{x\in[0\,,1]} |g'(x)| = \frac{e}{4} < 1, \qquad g([0,1]) \subset [\frac{1}{4}, \frac{e}{4}] \subset [0,1]$$

Then there is exactly one fixed-point α in $[0,1]$.

So the sequence is converges to the root α and the convergence method is linear(first-order).

5. Newton's method …………………………………………………………………………………….(1pts)

$$\begin{cases} x_{n+1} = x_n - \dfrac{e^{x_n} - 4x_n}{e^{x_n} - 4}, \quad n = 0,1,2, \dots \\ x_0 \in [a,b] \end{cases}$$

6. Calculate approximate root α of a function…………………………………………………….(1pts+1pts)

Fixed-point method	Newton's method
$x_0 = 0$ $x_1 = \frac{e^{x_0}}{4} = 0.25$ $x_2 = \frac{e^{x_1}}{4} = 0.321$ $x_3 = \frac{e^{x_2}}{4} = 0.3446$ $x_4 = \frac{e^{x_3}}{4} = 0.3529$ $x_5 = \frac{e^{x_4}}{4} = 0.3558$ $x_6 = \frac{e^{x_5}}{4} = 0.3568$ $\lvert x_6 - x_5 \rvert = 0.001$ So the root is 0.3568	$x_0 = 0$ $x_1 = x_0 - \frac{e^{x_0} - 4x_0}{e^{x_0} - 4} = 0.3333$ $x_2 = x_1 - \frac{e^{x_1} - 4x_1}{e^{x_1} - 4} = 0.3572$ $x_3 = x_2 - \frac{e^{x_2} - 4x_2}{e^{x_2} - 4} = 0.3574$ $\lvert x_6 - x_5 \rvert = 0.002 > 0.001$ So The root is 0.3572

Comparison ………………………………………………………………………………(0.5pts)

Newton's method is more efficient than the fixed-point method because it converges faster, and its order of convergence is 2, while the fixed-point method has order 1.

Solution of exercise 3: (6pts)

$$I = \int_0^1 \sin \pi x \, dx$$

1.Simple trapezoidal method ………………………………………………………………(0.5pts)

$$a = 0, b = 0, f(x) = \sin \pi x$$

$$I = \int_a^b f(x)dx \approx \frac{b-a}{2}\big(f(a) + f(b)\big) = 0$$

2. Simple simpson's method ……………………………………………...(1pts)

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left(f(a) + 4f(\frac{a+b}{2}) + f(b)\right) = 0.6667$$

3.Composite trapezoidal method (n=4) ……………………………………(1.5pts)

$$h = \frac{b-a}{4} = \frac{1}{4}, x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, x_4 = 1$$

$$\int_a^b f(x)dx \approx \frac{h}{2}\left(f(a) + f(b) + 2\sum_{i=1}^{3} f(x_i)\right) = 0.6036$$

$$R_t(f) = -\frac{(b-a)^3}{12n^2} f^{(2)}(c), c \in [a,b] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (0.5\text{pts})$$

$$f^{(2)}(c) = -\pi^2 \sin \pi c$$

$$|R_t(f)| = \frac{(b-a)^3}{12n^2}\left|f^{(2)}(c)\right| = \frac{\pi^2}{12n^2}|\sin \pi c| \leq \frac{\pi^2}{12n^2} \ldots\ldots\ldots.. (0.5\text{pts})$$

$$\frac{\pi^2}{12n^2} \leq 10^{-5} \Leftrightarrow n \geq 286.81, \text{ so } n = 287 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (0.5\text{pts})$$

$$5.\text{ exact solution}: I = [-\frac{\cos \pi x}{\pi}]_0^1 = \frac{2}{\pi} \approx 0.6366 \ldots\ldots\ldots\ldots\ldots. \ldots (0.5\text{pts})$$

Absolute error: $E_A = |0.6366 - 0.6036| = 0.03$ ……………………….(0.5pts)

Relative error: $E_R = \frac{0.03}{0.6366} = 0.05184$ ………………………………….(0.5pts)