Chapter5

Numerical Integration

Newton-cotes method

Is a numerical integration approach that approximates a function $f(x)$ using an interpolating polynomial then integrates the polynomial

We divide the time interval $[a,b]$ into equal sub-intervals $[x_i, x_{i+1}]$ where each subinterval has a length of

$h = \frac{b-a}{n}$. The Lagrange interpolate of f at the equidistant points

$(x_0 = a, x_n = b, x_i = x_0 + ih, \forall i = 0, \dots, n)$

$$P_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)\,, \text{where } L_i(x) = \prod_{j=0}^{n} \frac{x - x_j}{x_i - x_j}\ , j \neq i$$

By putting $x = x_0 + th$ we get

$$L_i(x) = \prod_{j=0}^{n} \frac{t-j}{i-j} = (-1)^{n-i} \frac{\prod_{j=0}^{n}(t-j)}{(n-i)!\, i!}, j \neq i$$

$$\int_a^b f(x)dx \approx \int_a^b P_n(x)dx = h \sum_{i=0}^{n} \frac{(-1)^{n-i} f(x_i)}{(n-i)!\, i!} \int_0^n \prod_{j=0}^{n} (t-j)\, dt \ \text{ and } j \neq i$$

This relationship is called **Newton-cotes** formula.

Simple Trapezoidal Rule(n=1)

The formula for the simple trapezoidal rule for a function $f(x)$ over the interval $[a,b]$ is given by:

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a) + f(b))$$

- If $f \in C^2([a,b])$ the error for the simple Trapezoidal Rule can be expressed as:

$$R_t(f) = -\frac{h^3}{12} f^{(2)}(c), \qquad c \in [a,b]$$

Composite Trapezoidal Rule

For the composite trapezoidal rule, we divide the interval $[a,b]$ into n equal subintervals.The formula is:

$$\int_a^b f(x)dx \approx \frac{h}{2}\left(f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right)$$

Where $h = \frac{b-a}{n}$ and $x_i = a + ih, \forall i = 1, \dots, n-1$.

- If $f \in C^2([a,b])$ the error for the Composite Trapezoidal Rule can be expressed as:

$$R_t(f) = -\frac{(b-a)^3}{12n^2} f^{(2)}(c), \qquad c \in [a,b]$$

Example

We will approximate the following integral:

$$\int_0^1 e^{-x^2} dx$$

using both the simple trapezoidal rule and the composite trapezoidal rule (with $n = 2$), and then estimate the integration error for each method.

Simple Trapezoidal Rule

The formula for the simple trapezoidal rule for a function $f(x)$ over the interval $[a,b]$ is given by:

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a) + f(b))$$

In our case $fx) = e^{-x^2}, a = 0, b = 1$

$$\int_0^1 e^{-x^2} dx \approx \frac{1}{2}\left(1 + \frac{1}{e}\right) \approx 0.683939$$

Composite Trapezoidal Rule (n = 2)

$$h = \frac{b-a}{2} = 0.5$$

$$x_0 = 0, x_1 = 0.5, x_2 = 1$$

$$\int_a^b f(x)dx \approx \frac{h}{2}(f(a) + 2f(x_1) + f(b))$$

$$\int_a^b e^{-x^2} dx \approx \frac{1}{4}(1 + 2e^{-0.25} + e^{-1}) \approx 0.73137$$

Error for the Simple Trapezoidal Rule:

$$R_t(f) = -\frac{h^3}{12} f^{(2)}(c), \qquad c \in [0,1]$$

$$f^{(2)}(x) = (4x^2 - 2)e^{-x^2}$$

$$|R_t(f)| = \frac{1}{12}\left|f^{(2)}(c)\right| \leq \frac{1}{12}\max_{0\leq x\leq 1}\left|f^{(2)}(x)\right| = \frac{1}{6e}$$

Error for the Composite Trapezoidal Rule (n = 2):

$$R_t(f) = -\frac{(b-a)^3}{12n^2} f^{(2)}(c)$$

$$|R_t(f)| = \frac{1}{48}\left|f^{(2)}(c)\right| \leq \frac{1}{48}\max_{0\leq x\leq 1}\left|f^{(2)}(x)\right| = \frac{1}{24e}$$

Simple Simpson's Rule(n=2)

Approximates f(x) with a quadratic polynomial.

The Simpson's Rule is a special case of the Newton-Cotes formula where $n = 2$, we obtain

$$\int_a^b f(x)dx \approx \frac{h}{3}\left(f(a) + 4f(\frac{a+b}{2}) + f(b)\right)$$

using $h = \frac{b-a}{2}$ (the step size for two subintervals)

- If $f \in C^4([a,b])$, the error for the Simple Simpson's Rule is given by:

$$R_s(f) = -\frac{(b-a)^5}{180} f^{(4)}(c)$$

Where

- $R_s(f)$ is the approximation error of the integral.
- $f^{(4)}(c)$ is the fourth derivative of f, evaluated at some point c in the interval $[a,b]$.

Composite Simpson's Rule Formula:

If the interval $[a,b]$ is divided into $n = 2k$ equal subintervals, the Composite Simpson's Rule is given by:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left(f(a) + 4\sum_{i=1(i\text{ odd})}^{n-1} f(x_i) + 2\sum_{i=2(i\text{ even})}^{n-2} f(x_i) + f(b)\right)$$

$h = \frac{b-a}{n}$ is the width of each subinterval, and $x_i = a + ih, \forall i = 1, \ldots, n-1$.

- If $f \in C^4([a,b])$, the error for the composite Simpson's Rule is given by:

$$R_s(f) = -\frac{(b-a)^5}{180n^4} f^{(4)}(c)$$