Chapter3

Polynomial Interpolation

The Concept of Interpolation:

Interpolation is a mathematical method used to find an approximate value of an unknown function at a certain point using a set of known values of the function at specific points.

The goal of interpolation is to construct an approximate function (called the interpolation function) that passes through all the known points and is used to estimate unknown function values within the given range.

Let f be an unknown function whose images $y_0, y_1, y_2, \dots, y_n$ are known at the points $x_0, x, x_2, \dots, x_n$

If the interpolation function is a polynomial $p(x)$, then:

$$p(x_i) = y_i, \quad i = 0,1,2, \dots . n$$

Theorem : Polynomial Interpolation

Given $n + 1$ distinct points $x_0, x, x_2, \dots, x_n$ and corresponding values $y_0, y_1, y_2, \dots, y_n$, there exists a unique polynomial p of degree less than or equal to n such that $P(x_i) = y_i$ for $i = 0, \dots, n$.

Lagrange Formula:

Theorem :

Given $n + 1$ distinct points $x_0, x, x_2, \dots, x_n$ and corresponding values $y_0, y_1, y_2, \dots, y_n$, there exists a unique polynomial P_n such that $P_n(x_i) = y_i$. This polynomial can be written in the form:

$$P_n(x) = \sum_{i=0}^{n} L_i(x) y_i \text{ , where } L_i(x) = \prod_{j=0}^{n} \frac{x - x_j}{x_i - x_j} \text{ ,} j \neq i$$

This relation is called the Lagrange interpolation formula, and the polynomials L_i are the characteristic polynomials (of Lagrange).

$L_i(x)$ have the property that

$$L_i(x_j) = \begin{cases} 1 \text{ if } j = i \\ 0 \text{ if } j \neq i \end{cases}$$

Example

Estimate the value of $f(3.3)$ using the Lagrange interpolation formula based on the following data table:

	x_0	x_1	x_2	x_3
x	**0**	**1**	**3**	**4**
f(x)	**−1**	**0**	**8**	**15**

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_3)(x_0-x_3)} = \frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)} = \frac{-x^3}{12} + \frac{2x^2}{3} - \frac{19x}{12} + 1$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{x(x-3)(x-4)}{(1)(-2)(-3)} = \frac{x^3}{6} - \frac{7x^2}{6} + 2x$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{x(x-1)(x-4)}{(3)(2)(-1)} = \frac{-x^3}{6} + \frac{5x^2}{6} - \frac{2}{3}x$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{x(x-1)(x-3)}{(4)(3)(1)} = \frac{x^3}{12} - \frac{x^2}{3} + \frac{x}{4}$$

$$p(x) = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2 + L_3(x)y_3$$

$$p(x) = x^2 - 1$$

$$f(3.3) \approx p(3.3) = 9.89$$

Newton's method

Let (x_i, y_i) for $i = 0,1,2,\dots,n$ be a set of $n+1$ distinct points.

- The divided difference of order 1 is defined as:

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

- The divided difference of order 2 is defined as:

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

- The divided difference of order n is defined as:

$$f[x_0, x_1, x_2, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$

The divided differences can be calculated by arranging them in a table as follows:

i	x_i	y_i	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-4}, x_{i-3}, x_{i-2}, x_{i-1}, x_i]$	$\dots$
0	x_0	$\boxed{y_0}$					
1	x_1	y_1	$\boxed{f[x_0, x_1]}$				
2	x_2	y_2	$f[x_1, x_2]$	$\boxed{f[x_0, x_1, x_2]}$			
3	x_3	y_3	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$\boxed{f[x_0, x_1, x_2, x_3]}$		
4	x_4	y_4	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$	$\boxed{f[x_0, x_1, x_2, x_3, x_4]}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\boxed{\ddots}$

- **Theorem: Newton's Formula**

  Let (x_i, y_i) for $i = 0,1,2,\dots,n$ be a set of $n+1$ distinct points. The interpolation polynomial in the form of **Newton's** is given by:

$$p(x) = f(x_0) + \sum_{i=0}^{n} f[x_0, x_1, x_2, \dots, x_i](x - x_0)(x - x_1)\dots\dots(x - x_{i-1})$$

Where

Example:

To find the Newton interpolation polynomial for the function f passing through the points $(0,1), (1,2), (2,9), and\ (3,28)$, we construct the divided difference table and use it to form the. Estimate the value of $f(2.3)$.

$$p(x) = x^3 + 1$$

$$f(2.3) \approx p(2.3) = 13.167$$

Step 1: Divided Difference Table

x_i	$f[x_i]$	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$
0	1			
1	2	1		
2	9	7	3	
3	28	19	6	1

Explanation:

- $f[x_0, x_1] = \frac{2-1}{1-0} = 1$
- $f[x_1, x_2] = \frac{9-2}{2-1} = 7$
- $f[x_2, x_3] = \frac{28-9}{3-2} = 19$
- $f[x_0, x_1, x_2] = \frac{7-1}{2-0} = 3$
- $f[x_1, x_2, x_3] = \frac{19-7}{3-1} = 6$
- $f[x_0, x_1, x_2, x_3] = \frac{6-3}{3-0} = 1$

Step 2: Construct the Newton Polynomial

Using the divided differences, the Newton interpolation polynomial is:

$$P(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2).$$

Substitute the values:

$$P(x) = 1 + 1(x - 0) + 3(x - 0)(x - 1) + 1(x - 0)(x - 1)(x - 2).$$

Final Polynomial

$$P(x) = 1 + (x) + 3(x)(x - 1) + (x)(x - 1)(x - 2).$$

$$p_3(x) = x^3 + 1$$

$$f(2.3) = 13.1670$$

Theorem: Error of interpolation

Suppose $x_0, x_1, \dots, x_n$ are distinct numbers in the interval $[a,b]$ and $f \in C^{n+1}[a,b]$. Then, for each x in $[a,b]$, a number ξ (generally unknown) between $x_0, x_1, \dots, x_n$, and hence $in\ ]a,b[$, exists with

$$E_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\ldots..(x - x_n)$$

where $P_n(x)$ is the interpolating polynomial.

Consequences: If $f^{(n+1)}$ is continuous on $[a,b]$ then:

$$|E_n(x)| = \frac{|f^{(n+1)}(\xi)|}{(n+1)!}\left|\prod_{i=0}^{n}(x - x_i)\right| \le \frac{M_{n+1}}{(n+1)!}\left|\prod_{i=0}^{n}(x - x_i)\right|$$

Where

$$M_{n+1} = \max_{x\in[a,b]}\left|f^{(n+1)}(x)\right|$$

Example:

Find the polynomial that fits the function y=1/x defined by the following table

Then, find the approximate value for f(3) and calculate the error incurred

x	2	2.5	4
y=f(x)	0.5	0.4	0.25

x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_0, x_1, x_2]$
2	0.5		
		-0.2	
2.5	0.4		0.05
		-0.1	
4	0.25		

$$p(x) = 0.5 - 0.2(x - 2) + 0.05(x - 2)(x - 2.5)$$

$$p(x) = 0.05x^2 - 0.425x + 1.15$$

$$f(3) \approx p(3) = 0.325$$

Estimate the error

$$f'(x) = \frac{-1}{x^2}, f''(x) = \frac{2}{x^3}, f^{(3)}(x) = \frac{-6}{x^4}$$

$$\max_{x\in[2,4]}\left|f^{(3)}(x)\right| = \max_{x\in[2,4]}\frac{6}{x^4} = \frac{6}{2^4} = \frac{3}{8}$$

$$|E_n(3)| = \frac{|f^{(3)}(\xi)|}{3!}\left|\prod_{i=0}^{2}(3 - x_i)\right| \le \frac{\frac{3}{8}}{3!}\ |(3-2)(3-2.5)(3-4)| = \frac{0.5}{16} = 0.03125$$