Chapter2

Numerical Solution of Nonlinear Equations

Introduction

In this chapter we shall some numerical method for solving equation of one real variable can be written in the from

$$f(x) = 0$$

Example

$$x^3 - 7x + 8 = 0, \qquad x - 2\sin x = 0, \qquad e^x = 4x$$

The real number α is called the real solution(root) of the equation $f(x) = 0$ if and only if $f(\alpha) = 0$

And geometrically the real solution (root) of an equation is the value of x where the graph of $f(x)$ meets the $\boldsymbol{x}$-axis in rectangular coordinate system.

Root Isolation

Let $f(x)$ be a function that is defined, continuous, and differentiable on the interval $[a, b]$. The goal is to numerically approximate the simple roots of the equation $f(x) = 0$.

The first step is to isolate the roots, i.e., to determine a closed and bounded interval $[a, b]$ where has exactly one root.

To find this interval, we can:

1.Use the graphical method: Study the variations of f, plot the curve, and locate the solution, which helps to choose $[a, b]$.

2.Use the algebraic method: Apply the intermediate value theorem.

Intermediate Value Theorem

Let f be a function defined and continuous on the interval $[a, b]$.

If $f(a) \times f(b) < 0$, then there exists at least one real number $\alpha \in [a, b]$ such that $f(\alpha) = 0$.

Furthermore, if $f(x)$ is monotonic on the interval $[a, b]$, the solution α is unique.

Bisection method

Assume that f is well-defined on the interval $[a; b]$ and α is the real root of the equation $f(x) = 0$.
Set $a_1 = a$ and $b_1 = b$. Find the midpoint c_1 of $[a_1;\ b_1]$ by

$$c_1 = \frac{a_1 + b_1}{2}$$

If $f(c_1) = 0$, set $\alpha = c_1$ and we are done.
If $f(c_1) \neq 0$, then we have

- $f(c_1)\, f(a_1) > 0 \Rightarrow \alpha \in [c_1; b_1]$. Set $a_2 = c_1$ and $b_2 = b_1$.
- $f(c_1)\, f(a_1) < 0 \Rightarrow \alpha \in [a_1; c_1]$. Set $a_2 = a_1$ and $b_2 = c_1$.

Proceeding in this manner, we find

$$c_n = \frac{a_n + b_n}{2}$$

which gives us the n th approximation of the root of $f(x) = 0$, and the root lies $[a_n;\ b_n]$ where

$$b_n - a_n = \frac{b-a}{2^{n-1}}, n \geq 1$$

Continue above process until convergence

Theorem

Suppose that $f \in C[a;\ b]$ with $f(a) \times f(b) < 0$. The Bisection method generates a sequence $\{c_n\}_{n\geq 1}$ converging to a root α of f with

$$|\alpha - c_n| \leq \frac{b-a}{2^n}, n \geq 1$$

The rate of convergence is $O(\frac{1}{2^n})$.

Proof

For each $n \geq 1$, $\alpha \in [a_n;\ b_n]$ and
Hence, we have

$$|\alpha - c_n| \leq \frac{b_n - a_n}{2} = \frac{b-a}{2^n}$$

If on repeating this process n times, the latest interval is as small as given ε then

$$\frac{b-a}{2^n} \leq \varepsilon$$

Or $n \geq \ln\left(\frac{b-a}{\varepsilon}\right) / \ln 2$.

In particular, the minimum number of iterations required for converging to a root in the interval $[0, 1]$ for a given ε are as under:

ε:	10^{-2}	10^{-3}	10^{-4}
n:	7	10	14

Stopping Criteria

In other stopping criteria can be used. Let $\varepsilon > 0$ be a given tolerance and $c_1;\ c_2; \ldots;\ c_n$ be generated by Bisection method.

1. $|c_n - c_{n-1}| \leq \varepsilon$
2. $\frac{|c_n - c_{n-1}|}{|c_n|} \leq \varepsilon, c_n \neq 0$
3. $|f(c_n)| \leq \varepsilon$

Rate of Convergence. As the error decreases with each step by a factor of $\frac{1}{2}$, (i.e., $\frac{e_{n+1}}{e_n} = \frac{1}{2}$), the convergence in the bisection method is linear(first-order).

with $e_n = |\alpha - c_n|$

Example

1) Show that the equation

$$e^x - 3x = 0$$

has exactly one root in $[1.5;\ 1.6]$.
2) Use Bisection method to determine an approx. root which is accurate to at least within 10^{-3}.
The root is $\alpha = 1.512134625$ correct to 9 decimal places

Solution:

$$f(1.5) = -0.02, f(1.6) = 0.15$$

$f'(x) = e^x - 3x$ for $x \in [1.5, 1.6]$, only one root of $f(x) = 0$ lies between 1.5 and 1.6

Applying Theorem, it follows that

$$n \geq \ln\left(\frac{0.5}{10^{-3}}\right) / \ln 2$$

or, equivalently, $n \geq 6.64$
So,7 iterations will ensure the required accuracy

n	a_n	b_n	$C_n = \dfrac{a_n + b_n}{2}$	$f(c_n)$
1	1.5	1.6	1.55	0.06
2	1.5	1.55	1.525	0.02
3	1.5	1.525	1.5125	0.00056
4	1.5	1.5125	1.50625	-0.0090
5	1.50625	1.5125	1.50935	-0.00426
6	1.50935	1.5125	1.5109	-0.0018
7	1.50935	1.5109	1.5117	-0.0006

Fixed-Point Iteration method
Definition
The number α is called a fixed point of a real-valued function g if $g(\alpha) = \alpha$.
Example

$$g(x) = x^2 - 2$$
$$g(2) = 2, \quad g(-1) = -1$$

$-1,2$ two fixed points of g.

- The fixed point α is intersection of $y = g(x)$ and $y = x$.

Note: A root-finding problem of the form

$$f(\alpha) = 0$$

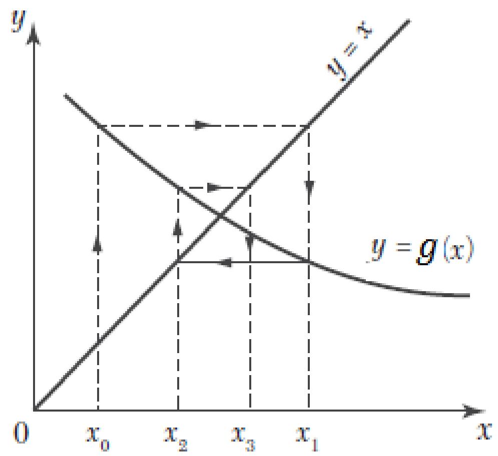
Where α is a root of f, can be transformed to a fixed-point form

$$g(\alpha) = \alpha$$

Create a sequence of iterations
Let $x = x_0$ be an initial approximation of the desired root α. Then the first approximation x_1 is given by $x_1 = g(x_0)$. Now treating x_1 as the initial value, the second approximation is

$$x_2 = g(x_1)$$

Proceeding in this way, the n th approximation is given by $x_n = g(x_{n-1})$



Theorem
i) If $g \in C[a; b]$ and $g([a; b]) \subseteq [a; b]$, then g has at least one fixed point α in $[a; b]$.
ii) If, in addition, $g'(x)$ exists on $]a; b[$ and satisfies

$$|g'(x)| \leq k < 1 \text{ for all } x \in]a, b[\text{ with } k > 0$$
then there is exactly one fixed point α in [a; b].
then the sequence of approximations $x_0, x_1, x_2, \dots, x_n$ will converge to the root α provided the initial approximation x_0 is chosen in [a; b].

Proof. Since α is a root of g, we have
$$\alpha = g(\alpha)$$
If x_{n-1} and x_n be 2 successive approximations to α, we have $x_n = g(x_{n-1})$
By mean value theorem,
$$g(x_{n-1}) - g(\alpha) = g'(c)(x_{n-1} - \alpha)$$
where c lies between x_{n-1} and α
becomes
$$x_n - \alpha = g'(c)(x_{n-1} - \alpha)$$
$|g'(c)| \leq \text{k} < 1$ then
$$|x_n - \alpha| \leq k|x_{n-1} - \alpha|$$
Similarly
$$|x_{n-1} - \alpha| \leq k|x_{n-2} - \alpha|$$
i.e., $|x_n - \alpha| \leq k^2|x_{n-2} - \alpha|$
Proceeding in this way, $|x_n - \alpha| \leq k^n|x_0 - \alpha|$
As $n \to +\infty$, the R.H.S. tends to zero, therefore, the sequence of approximations converges to the root α. the convergence method is linear(first-order).
Error estimation:
And we have the following error estimation:
$$|x_n - \alpha| \leq \frac{k^n}{1-k}|x_1 - x_0|$$

This relation allows us to calculate the number of iterations required to approximate α:
$$|x_n - \alpha| \leq \varepsilon \Longrightarrow \frac{k^n}{1-k}|x_1 - x_0|$$
$$n \geq \frac{\ln\left(\frac{\varepsilon(1-k)}{|x_1 - x_0|}\right)}{\ln k}, k = \max_{\text{x}\in[\text{a,b}]}|g''(x)|$$
Example
Find a real root of the equation $\cos x = 3x - 1$ correct to three decimal places using Iteration method for $n = 6$.
Solution
We have $f(x) = \cos x - 3x + 1 = 0$
$f(0) = 2$ and $f\left(\frac{\pi}{2}\right) = -\frac{3\pi}{2} + 1 = -3.71$
So A root lies between 0 and $\frac{\pi}{2}$.
Rewriting the given equation as $x = \frac{1}{3}(\cos x + 1)$, we have
$g'(x) = \frac{-\sin x}{3}$ and $|g'(x)| = \frac{|\sin x|}{3} \leq \frac{1}{3} < 1$ in $[0, \frac{\pi}{2}]$
Hence the iteration method can be applied and we start with $x_0 = 0$.
Then the successive approximations are,
$x_1 = g(x_0) = \frac{1}{3}(\cos 0 + 1) = \frac{2}{3}$=0.6667
$$x_2 = g(x_1) = \frac{1}{3}(\cos 0.6667 + 1) = 0.5953$$
$$x_3 = g(x_2) = \frac{1}{3}(\cos 0.5953 + 1) = 0.6093$$
$$x_4 = g(x_3) = \frac{1}{3}(\cos 0.6093 + 1) = 0.6067$$

$x_5 = g(x_4) = \frac{1}{3}(\cos 0.6067 + 1) = 0.6072$
$x_6 = g(x_5) = \frac{1}{3}(\cos 0.6072 + 1) = 0.6071$
Hence x_5 and x_6 being almost the same, the root is 0.607 correct to three decimal places

Newton-Raphson Method

let's start with a differentiable function $f: [a, b] \to \mathbb{R}$ and a point $x_0 \in [a, b]$ we call $(x_1, 0)$ the intersection of the tangent to the graph $y = f(x)$ at $(x_0, f(x_0))$ with the $x - axis$,if $x_1 \in [a, b]$ we repeat the process with the tangent at the point of abscissa x_1 . the process leads to the definition of a recursive sequence

$$\begin{cases} x_0 \in [a, b] \\ x_{n+1} = x_n - \dfrac{f(x_n)}{f^{'}(x_n)}, n = 0,1,2, \ldots \\ f^{'}(x_n) \neq 0 \end{cases}$$

Proof: The tangent at the point of abscissa x_n has the equation:

$$y = f^{'}(x_n)(x - x_n) + f(x_n)$$

Thus, the point $(x, 0)$, which lies on the tangent line (and on the $x - axis$), satisfies:

$$0 = f^{'}(x_n)(x - x_n) + f(x_n)$$

Solving for x, we get:

$$x = x_n - \frac{f(x_n)}{f^{'}(x_n)}$$

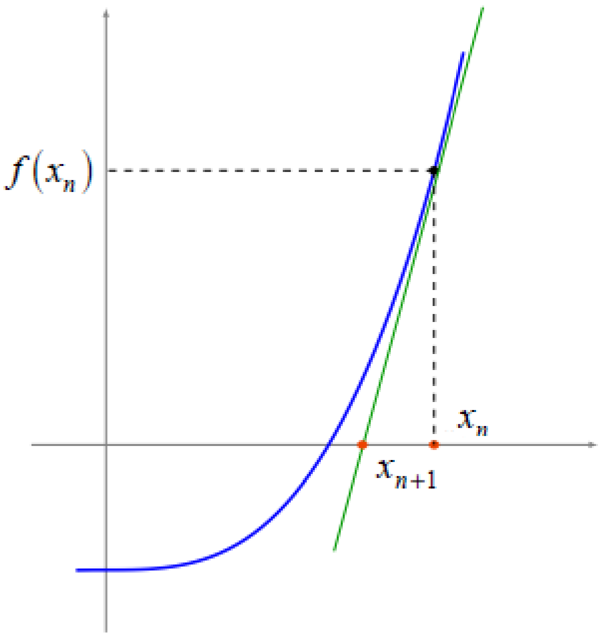
Study of Convergence

Theorem : (Global convergence of the Newton-Raphson method)

Let f be a C^2-class function on an interval $[a, b]$ such that:

i) $f(a) \times f((b) < 0$
ii) $\forall x \in [a, b]: f^{'}(x) \neq 0$
iii) $f^{'}(x)$ and $f^{''}(x)$ are non-zero and retain constant signs on $[a, b]$

Then the sequence of Newton-Raphson iterations converges to the unique zero $\alpha \in [a, b]$ of f.



Error estimation:
we have the following error estimation:

$$|x_n - \alpha| \leq \frac{M}{2m}|x_{n-1} - \alpha|^2$$
$$M = \max_{x\in[a,b]}|f''(x)| \text{ and } m = \min_{x\in[a,b]}|f''(x)|$$

- According to the previous estimate
$$\lim_{n\to\infty}\frac{|x_n - \alpha|}{|x_{n-1} - \alpha|^2} = l < \infty$$
So, the Newton-Raphson method is a second-order iterative method(then the convergence is said to be quadratic)

Example:
Find the approximation of the value $\sqrt{10}$ using Newton's method with a precision of $\varepsilon = 10^{-5}$.
Solution:
We consider the function f, which has $\sqrt{10}$ as a solution, defined by:
$$f(x) = x^2 - 10$$
$$f(x) = 0 \Longrightarrow x = \sqrt{10}$$
We seek to find the interval $[a, b]$:
The square root of 10 is approximately three, so we can use the interval [3,4].
We have:
$9 < 10 < 16 \Longrightarrow 3 < \sqrt{10} < 4$
Now, let us verify the conditions of Newton's theorem on the interval [3,4]:

1) $f(x) = x^2 - 10$ is of class C^2 on the interval [3,4].
$$f(3) = -1, f(4) = 6 \Longrightarrow f(3) \times f(4) < 0$$
2) $\forall x \in [3,4]: f'(x) = 2x > 0$ and $f''(x) = 2 > 0$

The function f verifies the convergence conditions.
The choice of x_0:
$$f(4) = 6 > 0, f''(4) = 2 \Longrightarrow f(4) \times f''(4) > 0$$
So, the Newton-Raphson sequence

$$\begin{cases} x_0 = 4 \\ x_{n+1} = x_n - \dfrac{{x_n}^2 - 10}{2x_n} = \dfrac{1}{2}(x_n + \dfrac{10}{x_n}), n = 0,1,2, \ldots \end{cases}$$
converges to the solution α.
Calculation of the approximate solution:

i	x_i	$f(x_i)$	$f'(x_i)$	$f(x_i)/f'(x_i)$
0	4	6	8	0.75
1	3.25	0.5625	6.5	0.086583
2	3.163462	0.007491	6.326924	0.001183
3	3.162279	0.000008	6.324558	0.000001
4	3.162278			

We stop the calculations until $|x_4 - x_3| = 0.000001 < \varepsilon$
Then the approximate solution obtained by Newton's method is:
$$\alpha \approx 3.162278$$