Chapter 1

Basic Notions of Errors

Numerical analysis deals with approximating mathematical problems and computations. Since exact solutions are not always feasible, errors arise in the results. Here are the fundamental types of errors and their definitions:

Absolute Error

The absolute error is the difference between the true (exact) value x and the approximate value x^*:

$$E_A = |x - x^*|$$

Relative Error

The relative error compares the absolute error to the magnitude of the true value, showing how significant the error is relative to the size of the true value:

$$E_R = \frac{E_A}{|x|}$$

Often expressed as a percentage:

$$E_p = 100E_R$$

Example

Let $x = 1.257$ and $x^* = 1.26$

$$E_A = |1.257 - 1.26| = 0.003$$

$$E_R = \frac{E_A}{|x|} = \frac{0.003}{1.257} = 0.00239$$

$$E_p = 100E_R = 0.24\%$$

Significant Digits:

Any positive number x^* can be written in the form:

$$x^* = a_m 10^m + a_{m-1} 10^{m-1} + \cdots \dots \dots .. + a_{m-n+1} 10^{m-n+1}$$

Where $a_i \in \{0,1,2,3,4,5,6,7,8,9\}$.

The digits $a_m, a_{m-1}, \dots \dots \dots .., a_{m-n+1}$, are called significant digits (figures).

Rules for Counting Significant Figures (digits)

Non-zero digits: All non-zero digits (1-9) are always significant.

Example: 1234 has four significant figures.

Leading zeros: Zeros that precede all non-zero digits are not significant.

Example: 0.0025 has two significant figures (2 and 5).

Captive zeros: Zeros between non-zero digits are significant.

Example: 1002 has four significant figures.

Trailing zeros: Zeros at the end of a number are significant if there is a decimal point.

Example: 1500.0 has five significant figures, while 1500 has only two.

Exact numbers: Numbers that are counted (like 25 apples) or defined quantities (like 100 centimeters in a meter) have an infinite number of significant figures.

Rounding an approximate number:

Rounding is a process of truncating numbers to the correct number of significant digits.

Rounding rules:

1) If the 1st digit to reject is < 5, the number is retained.
2) If the first digit to be rejected is ≥ 5, one unit is added to the last significant digit retained.

Exact Significant Figures of an Approximated Number

We say that the first n digits of an approximated number x^* are exact if the following condition is satisfied:

$$E_A \leq 0.5 \times 10^{m-n+1}$$

Exact significant figure

We say that the n significant digit after the decimal point of an approximated value of x is exact if:

$$E_A \leq 0.5 \times 10^{-n}$$

Example : Let $x = \frac{2}{3}$ and $x^* = 0.6671$. We have:

$$E_A = \left|\frac{2}{3} - 0.6671\right| = \frac{13}{3} \times 10^{-4} = 0.4333 \times 10^{-3} < 0.5 \times 10^{-3}$$

Thus, the significant digits 7, 6, and 6 of x^* are exact.

Proposition : If an approximated number x^* of x has n exact significant digits, then:

$$E_R \leq 0.5 \times 10^{-n+1}$$

Exercise

The bound for the absolute error on $X = 124.3675$ is 0.5×10^{-2} .

1) What is the bound for the relative error on X?
2) Determine the number of exact significant digits of X.
3) Round x to the Last Exact Significant Figure.

Solution:

The bound for the relative error on X:

$$E_R = \frac{0.5 \times 10^{-2}}{124.3675} = 0.4 \times 10^{-4}$$

The number of exact significant digits:

we use :$E_R \leq 0.5 \times 10^{-n+1}$

$$E_R = 0.4 \times 10^{-4} < 0.5 \times 10^{-n+1}$$

$-n + 1 = -4$ Hence $n = 5$

The last exact significant figure corresponds to the second decimal place. Rounding 124.3675

$X = 124.3675$ to two decimal places gives:

$$X \approx 124.37$$

Rounded Value $=$ 124.37.

Properties of absolute error and relative error

Let's consider

x^* as the approximate value of x

y^* as the approximate value of y

We have the following properties:

1. $E_A(x + y) = E_A(x) + E_A(y)$
2. $E_r(x + y) = \frac{1}{|x+y|}(xE_r(x) + yE_r(y))$
3. $E_A(x - y) = E_A(x) + E_A(y)$
4. $E_r(x - y) = \frac{1}{|x-y|}(xE_r(x) + yE_r(y)$
5. $E_A(xy) = x^*E_A(y) + y^*E_A(x)$
6. $E_r(xy) = E_r(x) + E_r(y)$
7. $E_A\left(\frac{x}{y}\right) = \frac{x}{y}(E_r(x) - E_r(y))$
8. $E_r\left(\frac{x}{y}\right) = E_r(x) - E_r(y)$

ERROR IN THE APPROXIMATION OF A FUNCTION

the error Δy in the function $y = f(x)$ corresponding to the error Δx is given by

$$\Delta y = |f'(x)|\Delta x$$

and the relative error in y is

$$E_r(y) = \frac{\Delta y}{|y|} = \left|\frac{f'(x)}{f(x)}\right|\Delta x$$

Example

Calculate the relative error and absolute error in calculating the value: $y = x^{\frac{2}{3}}$ where $x = 8000 \mp 3$.

$y = (8000)^{\frac{2}{3}} = 400$

$$E_r(y) = \frac{\Delta y}{|y|} = \frac{2}{3}\frac{\Delta x}{|x|} = \frac{2}{3} \times \frac{3}{8000} = \frac{1}{4000}$$

$$E_a(y) = yE_r(y) = 400 \times \frac{1}{4000} = \frac{1}{10} = 0.1$$