M'sila University		Faculty of Mathematics and Computer Science
L 2 Mathematics	**Series n° : 5** **Correction of improper integrals**	Module: Analyse 3

Remark 1

The exercise marked with $(\star)$ or additional will not be corrected in the session of **TD**.

Correction of exercise 1

1 Study the convergence of $\int_0^{+\infty} {}^{-x}dx$.

The function $x \mapsto e^{-x}$ is continuous on $[0, +\infty[$, so we have a priori only a problem in $+\infty$. Let $x > 0$. Then,

$$\int_0^x e^{-t}dt = \Big[-e^{-t}\Big]_0^x = 1 - e^{-x} \underset{x\longrightarrow+\infty}{\longrightarrow} 1.$$

Therefore, the integral $\int_0^{+\infty} {}^{-x}dx$ is converged and $\int_0^{+\infty} {}^{-x}dx = 1$.

2 Convergence of $\int_0^1 \frac{1}{\sqrt{1-x}}dx$.

The function $x \mapsto \frac{1}{\sqrt{1-x}}$ is continuous on $[0, 1[$, so we have a priori only a problem in 1. Let $x \in [0, 1[$. Then,

$$\int_0^x \frac{1}{\sqrt{1-t}}dt = \Big[-\sqrt{1-t}\Big]_0^x = 2 - \sqrt{1-x} \underset{x\longrightarrow1}{\longrightarrow} 2.$$

Hence the convergence of $\int_0^1 \frac{1}{\sqrt{1-x}}dx$ and $\int_0^1 \frac{1}{\sqrt{1-x}}dx = 2$.

3 Study the convergence of $\int_0^{+\infty} \frac{x}{(1+x^2)^2}dx$.

The function $x \mapsto \frac{x}{(1+x^2)^2}$ is continuous on $[0, +\infty[$, so a priori we only have a problem in $+\infty$. Let $x > 0$. then, we have

$$\begin{aligned}\int_0^x \frac{t}{(1+t^2)^2}dt &= \frac{1}{2}\int_0^x \frac{2t}{(1+t^2)^2}dt = \frac{1}{2}\Big[-\frac{1}{1+t^2}\Big]_0^x \\ &= -\frac{1}{2(1+t^2)} + \frac{1}{4} \underset{x\longrightarrow+\infty}{\longrightarrow} \frac{1}{4}.\end{aligned}$$

Consequently, the integral $\int_0^{+\infty} \frac{x}{(1+x^2)^2}dx$ is converged and $\int_0^{+\infty} \frac{x}{(1+x^2)^2}dx = \frac{1}{4}$.

4 For the integral $\int\limits_{-\infty}^{0} xe^{-x^2}dx$. We have the function $x \mapsto xe^{-x^2}$ is continuous on $]-\infty, 0]$, so we have a priori only a problem in $-\infty$.
Let $x \in]-\infty, 0]$. Then, we have,

$$\begin{aligned} \int\limits_{x}^{0} te^{-t^2}dt &= -\frac{1}{2}\int\limits_{x}^{0}(-2t)e^{-t^2}dt = -\frac{1}{2}\left[e^{-t^2}\right]_x^0 \\ &= \frac{1}{2}e^{-t^2} - \frac{1}{2} \underset{x \longrightarrow -\infty}{\longrightarrow} -\frac{1}{2}. \end{aligned}$$

That is, $\int\limits_{-\infty}^{0} xe^{-x^2}dx$ converges to $-\frac{1}{2}$.

5 Study of the convergence of $\int\limits_{0}^{+\infty} xe^{-x}dx$. We know that the function $x \mapsto xe^{-x}$ is continuous on $[0, +\infty[$, so we have a priori only a problem in $+\infty$.
Let $x \in [0, +\infty[$. By integration by parts, we find,

$$\begin{cases} u(t) = t \\ v'(t) = e^{-t} \end{cases} \Rightarrow \begin{cases} u'(t) = 1 \\ v(t) = -e^{-t}. \end{cases}$$

Consequently,

$$\begin{aligned} \int\limits_{0}^{x} te^{-t}dt &= \left[-te^{-t}\right]_0^x + \int\limits_{0}^{x} e^{-t}dt \\ &= -xe^{-x} + \left[-e^{-t}\right]_0^x \\ &= -xe^{-x} - e^{-x} + 1 \underset{x \longrightarrow +\infty}{\longrightarrow} 1. \end{aligned}$$

This means $\int\limits_{0}^{+\infty} xe^{-xdx}$ converges to 1.

6 Convergence of $\int\limits_{0}^{+\infty} \frac{1}{1+x^2}dx$. It is clear that the function $x \mapsto \frac{1}{1+x^2}$ is continuous on $[0, +\infty[$, so we have, a priori, only one problem in $+\infty$.
Let $x \in [0, +\infty[$. As a result,

$$\int\limits_{0}^{x} \frac{1}{1+t^2}dt = \left[\arctan(t)\right]_0^x = \arctan(x) \underset{x \longrightarrow +\infty}{\longrightarrow} = \frac{\pi}{2}.$$

Hence the convergence of $\int\limits_{0}^{+\infty} \frac{1}{1+x^2}dx$ to $\frac{\pi}{2}$.

Correction of exercise 2

1 For all $x \geq 0$, we have,

$$\frac{1}{(1+e^x)(1+e^{-x})} = \frac{e^x}{(1+e^x)^2} = \left[-\frac{1}{1+e^x}\right]'.$$

So we get,

$$\begin{aligned}\int_0^{+\infty} \frac{1}{(1+e^x)(1+e^{-x})}dx &= \lim_{x\to+\infty}\int_0^x \frac{1}{(1+e^t)(1+e^{-t})}dt = \lim_{x\to+\infty}\int_0^x \frac{e^t}{(1+e^t)^2}dt \\ &= \lim_{x\to+\infty}\left[-\frac{1}{1+e^x}\right]_0^x = \frac{1}{2} - \lim_{x\to+\infty}\frac{1}{1+e^x} = \frac{1}{2}.\end{aligned}$$

2 Since the primitive of the functions $x \mapsto u'(x)e^{u(x)}$ is the function $x \mapsto e^{u(x)}$, such that u is a differentiable function, then we have,

$$\begin{aligned}\int_1^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx &= \lim_{x\to+\infty}\int_1^x \frac{e^{-\sqrt{t}}}{\sqrt{t}}dt = \lim_{x\to+\infty}\left[-2e^{-\sqrt{t}}\right]_1^x \\ &= \frac{2}{e} - \lim_{x\to+\infty} 2e^{-\sqrt{x}} = \frac{2}{e}.\end{aligned}$$

3 We have for all $x \in]0, \frac{\pi}{2}[$,

$$\left(\sqrt{\sin(2x)}\right)' = \frac{\cos 2x}{\sqrt{\sin 2x}}.$$

So,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\cos 2x}{\sqrt{\sin 2x}}dx &= \lim_{x\to 0}\int_x^{\frac{\pi}{4}} \frac{\cos 2t}{\sqrt{\sin 2t}}dt + \lim_{x\to\frac{\pi}{2}}\int_{\frac{\pi}{4}}^x \frac{\cos 2t}{\sqrt{\sin 2t}}dt \\ &= \lim_{x\to 0}\left[\sqrt{\sin 2t}\right]_x^{\frac{\pi}{4}} + \lim_{x\to\frac{\pi}{2}}\left[\sqrt{\sin 2t}\right]_{\frac{\pi}{4}}^x \\ &= \lim_{x\to 0}\left[1-\sqrt{\sin 2x}\right] + \lim_{x\to\frac{\pi}{2}}\left[\sqrt{\sin 2x}-1\right] = 0.\end{aligned}$$

4 For the integral $\int_a^{+\infty} \frac{1}{x(x+b)}dx,\ a>0,\ b>0$, by decomposing the fraction $\frac{1}{x(x+b)}$, we obtain

$$\frac{1}{x(x+b)} = \frac{1}{b} \times \left[\frac{1}{x} - \frac{1}{x+b}\right].$$

So, we find,

$$\begin{aligned}\int_a^{+\infty} \frac{1}{x(x+b)}dx &= \lim_{x\to+\infty}\int_a^x \frac{1}{b}\times\left[\frac{1}{t}-\frac{1}{t+b}\right]dt = \lim_{x\to+\infty}\frac{1}{b}\times\left[\ln\left(\frac{t}{t+b}\right)\right]_a^x \\ &= -\ln\left(\frac{a}{a+b}\right) + \frac{1}{b}\lim_{x\to+\infty}\times\ln\left(\frac{x}{x+b}\right) = -\ln\left(\frac{a}{a+b}\right).\end{aligned}$$

5 Since the function $x \mapsto \arctan x$ has as its derivative the function $x \mapsto \dfrac{1}{1+x^2}$, and the primitive of functions of the form $f.f'$ is $\dfrac{1}{2}f^2$. Then, we have,

$$\begin{aligned}\int_0^{+\infty} \frac{\arctan x}{1+x^2}dx &= \lim_{x\to+\infty}\int_0^x \frac{\arctan t}{1+t^2}dt = \lim_{x\to+\infty}\left[\frac{(\arctan t)^2}{2}\right]_0^x \\ &= \lim_{x\to+\infty}\frac{(\arctan x)^2}{2} = \frac{\pi^2}{8}.\end{aligned}$$

6 Integrating by parts, we find,

$$\int \frac{\ln x}{x}dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x}.$$

Par suite,

$$\begin{aligned}\int_1^{+\infty} \frac{\ln x}{x}dx &= \lim_{x\to+\infty}\int_1^x \frac{\ln t}{t}dt = -\lim_{x\to+\infty}\left[\frac{\ln t}{t}+\frac{1}{t}\right]_0^x \\ &= 1 - \lim_{x\to+\infty}\left(\frac{\ln x}{x}+\frac{1}{x}\right) = 1.\end{aligned}$$

Correction of exercise 3

We are studying the nature of the following integrals

1 For the integral $\int_0^{+\infty} \sin x dx$. we have

$$\int_0^{+\infty} \sin x dx = \lim_{x\to+\infty}\int_0^x \sin x dx = \lim_{x\to+\infty}\left(1-\cos x\right) \quad \text{n'existe pas.}$$

Because $\lim\limits_{x\to+\infty} \cos x$ takes several of the values, so the generalized integral doesn't exist.

2 Study convergence of $\int_1^{+\infty} \sin(x^2)dx.\}$. By changing the variable, let's assume $u = x^2$, so, $u' = 2xdx$, then

$$\int_1^{+\infty} \sin(x^2)dx = \int_1^{+\infty} \frac{\sin(u)}{2u}du.$$

Applying Abel's theorem to the improper integral $\int_1^{+\infty} \dfrac{\sin(u)}{2u}du$.

On the one hand, the function $u \mapsto \dfrac{1}{2u}$ is decreasing on $[1,+\infty[$, and $\lim\limits_{x\to+\infty}\dfrac{1}{2u} = 0$.

On the other hand, for all $x \in [1,+\infty[$, we have, $\left|\int_1^x \sin(t^2)dt\right| = \left|\sin(2x)-\sin(1)\right| \leq 2$.

Hence the improper integral $\int_1^{+\infty} \frac{\sin(u)}{2u} du$ converges, so the improper integral $\int_1^{+\infty} \sin(x^2) dx$ converges.

3 For the integral $\int_0^{+\infty} e^{-x^2} dx$. We have $x^2 \geq x \geq 1$ for all $x \geq 1$. From the growth of the exponential function, we find,

$$e^{-x^2} \leq e^{-x}.$$

As the improper integral $\int_1^{+\infty} e^{-x} dx$ converges, because,

$$\int_1^x e^{-t} dt = \Big[-e^{-t} \Big]_1^x = e^{-1} - e^{-x} \underset{x \longrightarrow +\infty}{\longrightarrow} e^{-1}.$$

Hence, according to the comparison theorem, the integral $\int_1^{+\infty} e^{-x^2} dx$ converges.

We know that the integral $\int_0^{e} e^{-x^2} dx$ also converges, since the function $x \mapsto e^{-x^2}$ is continuous on the closed interval $[0, 1]$.

4 We have the function $x \mapsto e^{-x^2}$ is even, therefore,

$$\int_0^{+\infty} e^{-x^2} dx = \int_{-\infty}^{0} e^{-x^2} dx.$$

From the previous question, the improper integral $\int_0^{+\infty} e^{-x^2} dx$ converges, hence the convergence of the integral $\int_{-\infty}^{0} e^{-x^2} dx$.

5 Study the integral $\int_0^{+\infty} \frac{2x}{\sqrt{x^5 + x + 1}} dx$.

It is known that $\frac{2x}{\sqrt{x^5 + x + 1}} \underset{+\infty}{\sim} \frac{2}{x^{\frac{3}{2}}}$. Since the improper Riemann integral $\int_1^{+\infty} \frac{2}{x^{\frac{3}{2}}} dx$ converge ($\alpha = \frac{3}{2} > 1$), then, by the function equivalence theorem, the improper integral $\int_1^{+\infty} \frac{2x}{\sqrt{x^5 + x + 1}} dx$ converges. Knowing that $\int_0^{1} \frac{2x}{\sqrt{x^5 + x + 1}} dx$ is finite, because the function $x \mapsto \frac{2x}{\sqrt{x^5 + x + 1}}$ is continuous on $[0, 1]$. Therefore the integral $\int_0^{+\infty} \frac{2x}{\sqrt{x^5 + x + 1}} dx$ converges.

6 We have,

$$\left| \frac{\cos(x^2)}{1 + x^2} \right| \leq \frac{1}{1 + x^2} \underset{+\infty}{\sim} \frac{1}{x^2}.$$

Since the Riemann integral $\int_1^{+\infty} \frac{1}{x^2}dx$ converges, then by the comparison theorem, the improper integral $\int_1^{+\infty} \frac{\cos(x^2)}{1+x^2}dx$ converges. Hence the improper integral $\int_0^{+\infty} \frac{\cos(x^2)}{1+x^2}dx$ converges, because the function $x \mapsto \frac{\cos(x^2)}{1+x^2}$ is continuous on $[0,1]$, i.e. integrable on $[0,1]$.

7 For all $x \geq e$, we have

$$\frac{\ln x}{x+2} \geq \frac{1}{x+2} \underset{+\infty}{\sim} \frac{1}{x}.$$

Since the improper Riemann integral $\int_e^{+\infty} \frac{1}{x}dx$ diverges, then according to the comparison and equivalence theorems the improper integral $\int_e^{+\infty} \frac{\ln x}{x+2}dx$ diverges. Knowing that the integral $\int_1^{e} \frac{\ln x}{x+2}dx$ is finite. Hence the divergence of the integral $\int_1^{+\infty} \frac{\ln x}{x+2}dx$.

8 We study the convergence of the integral $\int_0^{+\infty} \frac{\cos x}{x^2}dx$. Then, we have,

$$\int_0^{+\infty} \frac{\cos x}{x^2}dx = \int_0^{1} \frac{\cos x}{x^2}dx + \int_1^{+\infty} \frac{\cos x}{x^2}dx.$$

a For the integral $I_1 = \int_0^{1} \frac{\cos x}{x^2}dx$, we know that $x \mapsto \cos x$ is a positive function in the neighborhood of 0, moreover,

$$\cos x = 1 - \frac{x^2}{2} + o(x^2),$$

consequently, we have

$$\lim_{n\to 0} \frac{\frac{1}{x^2}}{\frac{\cos x}{x^2}} = \lim_{n\to 0} \frac{\frac{1}{x^2}}{\frac{1-\frac{x^2}{2}+o(x^2)}{x^2}} = \lim_{n\to 0} \frac{1}{1-\frac{x^2}{2}+o(x^2)} = 1.$$

Since the Riemann integral $\int_0^{1} \frac{1}{x^2}dx$. converges, then the integral $\int_1^{+\infty} \frac{\cos x}{x^2}dx$ is also converged. Hence the improper integral $\int_0^{1} \frac{\cos x}{x^2}dx$ converges.

b For the integral $I_2 = \int_1^{+\infty} \frac{\cos x}{x^2}dx$. We have

$$\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2}.$$

Since the Riemann integral $\int_0^{1} \frac{1}{x^2}dx$ converges, then by the Comparison theorem, the integral $\int_1^{+\infty} \frac{\cos x}{x^2}dx$ converges.

Hence, the improper integral $\int_0^{+\infty} \frac{\cos x}{x^2} dx$ converges.

9 For the integral $\int_0^{+\infty} \frac{\ln x}{x+e^x} dx$. So, we have

$$\int_0^{+\infty} \frac{\ln x}{x+e^x} dx = \int_0^1 \frac{\ln x}{x+e^x} dx + \int_1^{+\infty} \frac{\ln x}{x+e^x} dx = -\int_0^1 \frac{-\ln x}{x+e^x} dx + \int_1^{+\infty} \frac{\ln x}{x+e^x} dx.$$

a Let $I_1 = \int_0^1 \frac{-\ln x}{x+e^x} dx$, then we have,

$$\lim_{n\to 0} \frac{\frac{-\ln x}{x+e^x}}{-\ln x} = \lim_{x\to 0} \frac{1}{x+e^x} = 1.$$

Comme l'intégrale $-\int_0^1 \ln x dx$. converges, because,

$$-\int_0^1 \ln x dx = -\lim_{x\to 0} \int_x^1 \ln t dt = \lim_{x\to 0} \left(1 - t\ln t\right) = 1.$$

Then, according to the equivalence theorem, the integral $I_1 = \int_0^1 \frac{-\ln x}{x+e^x} dx$ converges.

b For the integral $I_2 = \int_{01}^{+\infty} \frac{\ln x}{x+e^x} dx$, it is clear that,

$$\lim_{n\to +\infty} \frac{\frac{\ln x}{x+e^x}}{\frac{1}{x^2}} = \lim_{x\to +\infty} \frac{\ln 2 + \ln(\frac{x}{2})}{e^{\frac{x}{2}}\left(\frac{x}{e^{\frac{x}{2}}} + 1\right)} \times \frac{x^2}{e^{\frac{x}{2}}} = 0.$$

As the Riemann integral $\int_1^{+\infty} \frac{1}{x^2} dx$ converges, then according to the comparison theorem, the integral $I_2 = \int_1^{+\infty} \frac{\ln x}{x+e^x} dx$ converges.

Therefore, the improper integral $\int_0^{+\infty} \frac{\ln x}{x+e^x} dx$ converges.

10 On a

$$\begin{aligned} I &= \int_0^{+\infty} \frac{\ln(x)}{x^2-1} dx \\ &= \int_0^{\frac{1}{2}} \frac{\ln(x)}{x^2-1} dx + \int_{\frac{1}{2}}^1 \frac{\ln(x)}{x^2-1} dx + \int_1^2 \frac{\ln(x)}{x^2-1} dx + \int_2^{+\infty} \frac{\ln(x)}{x^2-1} dx \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Since,

$$\int_0^{\frac{1}{2}} \frac{\ln(x)}{x^2-1}dx \underset{0}{\sim} \frac{1}{\sqrt{x}} \text{ (converges)}.$$

Then, $I_1 = \int_0^{\frac{1}{2}} \frac{\ln(x)}{x^2-1}dx$ converges.

For the integrals, I_2 and I_3, we have,

$$\lim_{x\to 1} \frac{\frac{\ln(x)}{x^2-1}}{\frac{1}{x+1}} = \lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1}{x} = 1.$$

Since the two integrals $\int_{\frac{1}{2}}^{1} \frac{\ln x}{x+1}$ and $\int_1^2 \frac{\ln x}{x+1}$ exist (finite), then by the equivalence theorem, the two improper integrals I_2 and I_3 are convergent

For the integral $I_4 = \int_2^{+\infty} \frac{\ln(x)}{x^2-1}dx$. We note that,

$$\int_2^{+\infty} \frac{\ln(x)}{x^2-1}dx \underset{+\infty}{\sim} \frac{1}{x^{\frac{3}{2}}} \text{ (intégrale de Riemann, donc converg)}.$$

Therefore the integral I_4 converges. We result that I the sum of I_1, I_2, I_3 and I_4 is converged.

Correction of exercise 4

Study of the convergence of the integral $\int_1^{+\infty} \frac{\sin(x)}{x^\alpha}dx$.

1 Simple convergence. Using Abel's theorem, it is easy to see that for all $\alpha > 0$, the integral $\int_1^{+\infty} \frac{\sin(x)}{x^\alpha}dx$ is convergent. Indeed, the function $x \mapsto \frac{1}{x^\alpha}$ decreasing on $[1, +\infty[$ and $\lim_{x\to 1=\infty} \frac{1}{x^\alpha} = 0$. On the other hand for all $x \in [1, +\infty[$, we have, $\left| \int_1^x \sin(t)dt \right| = \left| \sin x - \sin(1) \right| \leq 2$.

Hence the improper integral $\int_1^{+\infty} \frac{\sin(x)}{x^\alpha}dx$ is converged.

2 Study absolute convergence. We have for all $\alpha > 0$

$$\left| \int_1^{+\infty} \frac{\sin(x)}{x^\alpha}dx \right| \leq \int_1^{+\infty} \frac{1}{x^\alpha}dx.$$

So, we distinguish two cases

a If $\alpha > 1$, the Riemann integral $\int_1^{+\infty} \frac{1}{x^\alpha} dx$ converges, therefore the improper integral $\int_1^{+\infty} \frac{\sin(x)}{x^\alpha} dx$ est converge.

b If $0 < \alpha \leq 1$. We must study the convergence of the integral $\int_1^{+\infty} \frac{|\sin(x)|}{x^\alpha} dx$.
We know that for $x \in [0, 1]$ we have $x \geq x^2$ and therefore, for all $t \in [1, +\infty[$

$$|\sin(x)| \geq \sin^2(x).$$

Hence, for any $t \geq 1$, we find,

$$0 \leq \frac{\sin^2 t}{t^\alpha} \leq \frac{|\sin t|}{t^\alpha}.$$

Since,

$$\frac{\sin^2 t}{t^\alpha} = \frac{1}{2t^\alpha} - \frac{\cos(2t)}{2t^\alpha},$$

and the generalized integral $\int_1^{+\infty} \frac{1}{t^\alpha} dt$ diverges, so the integral $\int_1^{+\infty} \frac{|\sin(x)|}{x^\alpha} dx$ is divergent.

3 $\int_0^1 \frac{x^\alpha}{\ln^\beta(x+1)} dx$ is an improper integral in 0.

$$\lim_{x \to 0} \frac{\frac{x^\alpha}{\ln(x+1)^\beta}}{\frac{1}{x^{\beta-\alpha}}} = 1, \quad \text{i.e,} \quad \frac{x^\alpha}{\ln^\beta(x+1)} \underset{0}{\sim} \frac{1}{x^{\beta-\alpha}}.$$

Then, the two integrals of the same nature.
Now, the Riemann integral $\int_0^1 \frac{1}{x^{\beta-\alpha}} dx$ is convergent if and only if $\beta - \alpha < 1$.

Correction of exercise 5

We study the nature of Bertrand's integral $\int_2^{+\infty} \frac{1}{x^\alpha \ln^\beta(x)} dx$, according to the values of the real parameters α and β. Four cases can be distinguished.

1 If $\alpha > 1$ we choose a real γ so that $1 < \gamma < \alpha$, we find,

$$\lim_{x \to +\infty} \frac{\frac{1}{x^\alpha \ln^\beta(x)}}{\frac{1}{x^\gamma}} = \lim_{x \to +\infty} \frac{1}{x^{\gamma-\alpha} \ln^\beta(x)} = 0,$$

for all $\beta \in \mathbb{R}$, that is $\frac{1}{x^\alpha \ln^\beta(x)} = o\left(\frac{1}{x^\gamma}\right)$. Since the Riemann integral $\int_2^{+\infty} \frac{1}{x^\gamma} dx$ is convergent because $\gamma > 1$, we deduce that the integral $\int_2^{+\infty} \frac{1}{x^\alpha \ln^\beta(x)} dx$ is convergent.

2 If $\alpha < 1$, then if we take a real number so that $0 < \gamma < \alpha$, we find,

$$\lim_{x\to+\infty} \frac{\frac{1}{x^\gamma}}{\frac{1}{x^\alpha \ln^\beta(x)}} = \lim_{x\to+\infty} \frac{1}{x^{\gamma-\alpha}\ln^{-\beta}(x)} = 0,$$

for all $\beta \in \mathbb{R}$, i.e. $\frac{1}{x^\gamma} = o\Big(\frac{1}{x^\alpha \ln^\beta(x)}\Big)$. Since the Riemann integral $\int_2^{+\infty} \frac{1}{x^\gamma}dx$ is divergent because $\gamma < 1$, we deduce that the integral $\int_2^{+\infty} \frac{1}{x^\alpha \ln^\beta(x)}dx$ is divergent.

3 If $\alpha = 1$ and $\beta = 1$. We know that for all $x > 2$, we have

$$\Big(\ln(\ln x)\Big)' = \frac{1}{x\ln(x)}.$$

As,

$$\begin{aligned}\int_2^{+\infty} \frac{1}{x\ln(x)}dx &= \lim_{x\to+\infty}\int_2^x \frac{1}{t\ln(t)}dt = \lim_{x\to+\infty}\Big[\ln(\ln t)\Big]_2^x \\ &= -\ln(\ln 2) + \lim_{x\to+\infty}\ln(\ln x) = +\infty.\end{aligned}$$

Hence the divergence of the Bertrand integral in this case.

4 If $\alpha = 1$ and $\beta \neq 1$. We have that for all $x > 2$,

$$\frac{1}{1-\beta}\Big(\ln^{1-\beta}(x)\Big)' = \frac{1}{x\ln^\beta(x)}.$$

Since,

$$\begin{aligned}\int_2^{+\infty} \frac{1}{x\ln^\beta(x)}dx &= \frac{1}{1-\beta}\lim_{x\to+\infty}\int_2^x \ln^{1-\beta}(t)dt = \frac{1}{1-\beta}\lim_{x\to+\infty}\Big[\ln^{1-\beta}(t)\Big]_2^x \\ &= -\frac{\ln^{1-\beta}(2)}{1-\beta} + \frac{1}{1-\beta}\lim_{x\to+\infty}\ln^{1-\beta}(x) = \begin{cases} +\infty & : \ \beta < 1, \\ 0 & : \ \beta > 1. \end{cases}\end{aligned}$$

So, we conclude from this study that,

$$\int_2^{+\infty} \frac{1}{x^\alpha \ln^\beta(x)}dx \text{ converges} \Leftrightarrow \begin{cases} \alpha > 1 \\ or \\ \alpha = 1 \text{ and } \beta < 1. \end{cases}$$

Correction of exercise 6

1 The integral I_n is a Bertrand integral $\int_1^{+\infty} \frac{1}{x^n \ln^{-1} x}dx$ which converges if and only if $n \geq 2$.

2. To calculate it in this case, let's integrate by parts $\int_1^{+\infty}\frac{\ln x}{x^n}dx$. We thus find,

$$\int \frac{\ln x}{x^n}dx = \frac{x^{-n+1}}{-n+1}\ln x - \int \frac{x^{-n}}{-n+1}dx = \frac{x^{-n+1}}{-n+1}\ln x - \frac{x^{-n}}{(-n+1)^2}.$$

Then, it follows that,

$$I_n = \int_1^{+\infty}\frac{\ln x}{x^n}dx = \left[\frac{x^{-n+1}}{-n+1}\ln x - \frac{x^{-n}}{(-n+1)^2}\right]_1^{+\infty} = \frac{1}{(-n+1)^2}.$$

Correction of exercise 7

Let be the integral $F(x) = \int_1^x \frac{\ln(1+t^2)}{t^2}dt$.

1. By integration by parts, and so let's pose,

$$\begin{cases} u'(t) = \frac{1}{t^2} \\ v(t) = \ln(1+t^2) \end{cases} \Rightarrow \begin{cases} u(t) = -\frac{1}{t} \\ v(t) = \frac{2t}{1+t^2}. \end{cases}$$

So we obtain,

$$\begin{aligned} F(x) &= \left[-\frac{1}{t}\ln(1+t^2)\right]_1^x + 2\int_1^x \frac{1}{1+t^2}dt \\ &= -\frac{1}{x}\ln(1+x^2) + \ln 2 + 2\arctan x - \arctan 1 \\ &= -\frac{1}{x}\ln(1+x^2) + 2\arctan x + \ln 2 - \frac{\pi}{2}. \end{aligned}$$

2. Since,

$$\lim_{x\to+\infty}\frac{1}{x}\ln(1+x^2) = 0 \text{ and } \lim_{x\to+\infty}\arctan x = \frac{\pi}{2}.$$

Therefore, $F(x)$ has a finite limit, which means that the integral I is convergent and

$$I = \lim_{x\to+\infty}F(x) = \lim_{x\to+\infty}\left[-\frac{1}{x}\ln(1+x^2) + 2\arctan x + \ln 2 - \frac{\pi}{2}\right] = \ln 2 + \frac{\pi}{2}.$$

Correction of exercise 8

1. We make the change of variable, so let us pose, $u = \sqrt{1+t^2}$ so, $t = \sqrt{u^2-1}$ and

$$dt = \frac{2udu}{2\sqrt{u^2-1}} = \frac{udu}{\sqrt{u^2-1}},$$

As a consequence,

$$\begin{cases} t = 1 \Rightarrow u = \sqrt{2} \\ t = x \Rightarrow u = \sqrt{x^2+1}. \end{cases}$$

Therefore, we have,

$$F(x)=\int_1^x \frac{1}{t\sqrt{t^2+1}}dt=\int_{\sqrt{2}}^{\sqrt{x^2+1}} \frac{1}{\sqrt{u^2-1}}\times\frac{udu}{\sqrt{u^2-1}}=\int_{\sqrt{2}}^{\sqrt{x^2+1}} \frac{du}{\sqrt{u^2-1}}.$$

We know that we can develop the fraction $\dfrac{1}{\sqrt{u^2-1}}$ as follows,

$$\frac{1}{\sqrt{u^2-1}}=\frac{1}{2}\times\frac{1}{u-1}-\frac{1}{2}\frac{1}{u+1}.$$

So we have

2 .

$$\begin{aligned} F(x) &= \int_{\sqrt{2}}^{\sqrt{x^2+1}} \frac{du}{\sqrt{u^2-1}}=\frac{1}{2}\int_{\sqrt{2}}^{\sqrt{x^2+1}} \frac{du}{\sqrt{u-1}}-\frac{1}{2}\int_{\sqrt{2}}^{\sqrt{x^2+1}} \frac{du}{\sqrt{u+1}} \\ &= \frac{1}{2}\Big[\ln|u-1|-\ln|u+1|\Big]_{\sqrt{2}}^{\sqrt{x^2+1}} \\ &= \frac{1}{2}\left[\ln\left|\frac{u-1}{u+1}\right|\right]_{\sqrt{2}}^{\sqrt{x^2+1}}=\frac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right|-\frac{1}{2}\ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|. \end{aligned}$$

3 **a** For $\alpha=\dfrac{3}{2}>1$, we have

$$\lim_{x\to+\infty} t^{\frac{3}{2}}\times\frac{1}{t\sqrt{1+t^2}}=0.$$

Therefore, the integral $I=\int_1^{+\infty}\dfrac{1}{t\sqrt{t^2+1}}dt$ converges.

b Note that $t=\sqrt{1+x^2}$ and tends to $+\infty$ when x tends to $+\infty$, hence

$$\lim_{x\to+\infty} F(x)=-\frac{1}{2}\ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|+\lim_{t\to+\infty}\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|=-\frac{1}{2}\ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|.$$

Correction of exercise 9

Let's show the semi-convergence of the following integrals.

1 For $\int_\pi^{+\infty}\dfrac{\cos x}{\sqrt{x}}dx$. Applying Abel's criterion. The function $x\mapsto\dfrac{1}{\sqrt{x}}$ is decreasing and tends to 0 at infinity, moreover for all $x,\ x'\geq\pi$, we have,

$$\left|\int_x^{x'}\cos t dt\right|=\left|\sin x'-\sin x\right|\leq 2.$$

So the integral converges.
To show that it does not converge absolutely, we can use the inequality

$$\forall x\in[\pi,+\infty[:\ \ |\cos x|\geq\cos^2 x=\frac{1-\cos 2x}{2}.$$

So,
$$\int_{\pi}^{x} \frac{|\cos t|}{\sqrt{t}} \geq \int_{\pi}^{x} \frac{1-\cos 2t}{2\sqrt{t}}dt = \int_{\pi}^{x} \frac{1}{2\sqrt{t}}dt + \int_{\pi}^{x} \frac{\cos 2t}{2\sqrt{t}}dt. \tag{1}$$
But,
$$\lim_{x\to+\infty} \int_{\pi}^{x} \frac{1}{2\sqrt{t}}dt = \lim_{x\to+\infty} [\sqrt{x} - \sqrt{\pi}] = +\infty.$$
On the other hand, by setting $u = 2t$, we obtain
$$\int_{\pi}^{x} \frac{\cos 2t}{2\sqrt{t}}dt = \frac{1}{2\sqrt{2}} \int_{2\pi}^{2x} \frac{\cos u}{\sqrt{u}}du.$$
According to the above, this expression has a finite limit when x tends to $+\infty$, so the member of right of the inequality (1) tends to $+\infty$, and it follows that the left one has the same limit. Consequently, the integral $\int_{\pi}^{+\infty} \frac{\cos x}{\sqrt{x}}dx$ diverges.

2 Since the function that at $x \mapsto \cos(x^2)$ is continuous on $[-1,+\infty[$, the integral $\int_{-1}^{+\infty} \cos(x^2)dx$ converges if and only if the integral $\int_{\sqrt{\pi}}^{+\infty} \cos(x^2)dx$ converges.

Similarly the integral $\int_{-1}^{+\infty} |\cos(x^2)|dx$ converges if and only if the integral $\int_{\sqrt{\pi}}^{+\infty} |\cos(x^2)|dx$ converges. By making the change of variable $x^2 = u$, we obtain,
$$\int_{\sqrt{\pi}}^{+\infty} \cos(x^2)dx = \int_{\pi}^{+\infty} \frac{\cos(u)}{2\sqrt{u}}du,$$
and also,
$$\int_{\sqrt{\pi}}^{+\infty} |\cos(x^2)|dx = \int_{\pi}^{+\infty} \frac{|\cos(u)|}{2\sqrt{u}}du.$$
According to the result of question (a), the integral $\int_{\pi}^{+\infty} \frac{\cos(u)}{2\sqrt{u}}du$ is semi-convergent, and therefore the integral $\int_{\sqrt{\pi}}^{+\infty} \cos(x^2)dx$ is also semi-convergent. Therefore $\int_{-1}^{+\infty} \cos(x^2)dx$ is semi-convergent.

3 We make the change of variable $u = x^4$. The integral becomes,
$$\int_{\sqrt{\pi}}^{+\infty} x^2 \sin(x^4)dx = \frac{1}{4} \int_{\pi^4}^{+\infty} \frac{\sin(u)}{u^{\frac{1}{4}}}du.$$
The situation is identical to that in question (a). Such that the function $u \mapsto \frac{1}{\sqrt[4]{u}}$ is decreasing and tends to 0 at infinity. Moreover, for all $x,\ x' \geq \pi^4$, we have
$$\left| \int_{x}^{x'} \sin u du \right| = \left| \cos u' - \cos u \right| \leq 2.$$

Therefore, Abel's criterion allows us to conclude that the integral $\int\limits_{\pi^4}^{+\infty} \frac{\sin(u)}{u^{\frac{1}{4}}} du$ converges. To show that it does not converge absolutely, we can use the inequality,

$$\forall x \in [\pi^4, +\infty[: \ |\sin x| \geq \sin^2 x = \frac{1 - \cos 2x}{2},$$

and conclude as in question (a).

4 Making the change of variable $u = \sqrt{x}$. The integral becomes,

$$\int\limits_{\pi}^{+\infty} \frac{e^{i\sqrt{x}}}{x} dx = 2 \int\limits_{\sqrt{\pi}}^{+\infty} \frac{e^{iu}}{u} du.$$

We then apply the same method as in question (a) to the real and imaginary parts

$$\int\limits_{\sqrt{\pi}}^{+\infty} \frac{2\cos u}{u} du \text{ and } \int\limits_{\sqrt{\pi}}^{+\infty} \frac{2\sin u}{u} du.$$

Which shows that these two integrals converge, so $\int\limits_{\pi}^{+\infty} \frac{e^{i\sqrt{x}}}{x} dx$ converges. On the other hand, since $|e^{i\sqrt{x}}| = 1$, and the integral $\int\limits_{\pi}^{+\infty} \frac{1}{x} dx$ diverges, the proposed integral is not absolutely convergent.

Correction of exercise 10

1 We write $\tan x = \frac{sinx}{\cos x}$, so we have immediately,

$$\int \tan x dx = -\int \frac{(\cos x)'}{\cos x} dx = -\ln(\cos x) + c, \quad c \in \mathbb{R}.$$

Consequently,

$$\begin{aligned} \int\limits_0^{\frac{\pi}{2}} \tan x dx &= \lim_{x \to \frac{\pi}{2}} \int\limits_0^x \tan t dt = \lim_{x \to \frac{\pi}{2}} \Big[-\ln(\cos t) \Big]_0^x \\ &= \lim_{x \to \frac{\pi}{2}} \Big[-\ln(\cos x) \Big] = +\infty. \end{aligned}$$

So, the integral $\int\limits_0^{\frac{\pi}{2}} \tan x dx$ diverges.

2 The tangent function has a discontinuity at $\frac{\pi}{2}$. We return to 0 by setting $u = \frac{\pi}{2} - x$. Then,

$$\tan x = \tan\left(\frac{\pi}{2} - u\right) = cotanu = \frac{1}{\tan u}.$$

But in the neighborhood of 0, we have, $\tan(u) \sim u$, therefore,

$$\tan x \sim \frac{1}{u}.$$

Since the integral $-\int_0^{\frac{\pi}{2}} \frac{du}{u}$ diverges, it follows that the integral $\int_0^{\frac{\pi}{2}} \tan x dx$ also diverges.

Correction of exercise 11

1 For all $x \in]0,+\infty[$, if the integral $\int_0^{+\infty} \frac{\sin(5x)-\sin(3x)}{x^{\frac{5}{3}}}$ exists, we can write as follows

$$\int_0^{+\infty} \frac{\sin(5x)-\sin(3x)}{x^{\frac{5}{3}}} dx = \int_0^{1} \frac{\sin(5x)-\sin(3x)}{x^{\frac{5}{3}}} dx + \int_1^{+\infty} \frac{\sin(5x)-\sin(3x)}{x^{\frac{5}{3}}} dx.$$

We'll use the equivalence criterion. The function to be integrated is continuous on $]0,+\infty[$ and therefore locally integrable on this interval. The integral is automatically improper in $+\infty$. The study must be done in 0.

a At point 0. we have,

$$\sin(3x) = 3x + o(x) \text{ and } \sin(5x) = 5x + o(x).$$

So,

$$\sin(5x) - \sin(3x) \underset{0}{\sim} 2x.$$

Then we find,

$$\frac{\sin(5x)-\sin(3x)}{x^{\frac{5}{3}}} \underset{0}{\sim} \frac{2}{x^{\frac{2}{3}}}.$$

We therefore have that the functions are of constant sign in the neighbourhood of 0. We can therefore use the criterion on equivalents,

$$\frac{\sin(5x)-\sin(3x)}{x^{\frac{5}{3}}} \underset{0}{\sim} \frac{2}{x^{\frac{2}{3}}}$$

So, $\int_0^{1} \frac{\sin(5x)-\sin(3x)}{x^{\frac{5}{3}}} dx$ and $\int_0^1 \frac{2}{x^{\frac{2}{3}}} dx$ are of the same nature.

Since the integral $\int_0^1 \frac{2}{x^{\frac{2}{3}}} dx$ converges, then the improper integral $\int_0^{1} \frac{\sin(5x)-\sin(3x)}{x^{\frac{5}{3}}} dx$ converges.

b In $+\infty$. For all $x \in [1,+\infty[$, we have,

$$\left| \frac{\sin(5x)-\sin(3x)}{x^{\frac{5}{3}}} \right| \leq \frac{2}{x^{\frac{5}{3}}}.$$

As the Riemann improper integral $\int_1^{+\infty} \frac{2}{x^{\frac{5}{3}}} dx$ converges, so the improper integral $\int_1^{+\infty} \frac{\sin(5x)-\sin(3x)}{x^{\frac{5}{3}}} dx$ absolutely converges.

We conclude from (a) and (b) that the integral $\int_0^{+\infty} \frac{\sin(5x)-\sin(3x)}{x^{\frac{5}{3}}} dx$ converges.

2 The function to be integrated is continuous on $]0,+\infty[$ and therefore locally integrable on this interval. We also note according to Chasle's relation (if the integral exists)

$$\int_0^{+\infty} \frac{x}{e^x-1}dx = \int_0^{1} \frac{x}{e^x-1}dx + \int_1^{+\infty} \frac{x}{e^x-1}dx.$$

a At point 0. we have

$$\frac{x}{e^x-1} \underset{0}{\sim} 1.$$

Therefore, the function $x \mapsto \frac{x}{e^x-1}$ is extendable by continuity at 0. So the integral at 0 is a Riemann integral, i.e. $\int_0^1 \frac{x}{e^x-1}dx$ is finite.

b In $+\infty$. the Riemann criterion should work well. So we have,

$$x^2 \times \frac{x}{e^x-1} = \frac{x^3}{e^x} \times \frac{1}{1-e^{-x}} = x^3e^{-x}\frac{1}{1-e^{-x}}.$$

We know that $\lim\limits_{x\to 0} x^3e^{-x} = 0$, so, $x^2 \times \frac{x}{e^x-1} = x^3e^{-x}\frac{1}{1-e^{-x}} \underset{+\infty}{\sim} 0$. The Riemann criterion allows us to conclude that $\int_1^{+\infty} \frac{x}{e^x-1}dx$ converges.

Hence, according to (a) and (b) the improper integral $\int_0^{+\infty} \frac{x}{e^x-1}dx$ is converged.

Correction of exercise 12

1 f is none other than the square of a primitive function of $t \mapsto e^{-t^2}$ which is continuous on $\mathbb{R}$, therefore f is of class C^1 on $\mathbb{R}$, and $f'(x) = 2\left(\int_0^x e^{-t^2}dt\right)e^{-x^2}$. We have,

a the application $(x,t) \mapsto \frac{e^{-x^2(1+t^2)}}{1+t^2}$ is defined and continuous on $\mathbb{R} \times [0,1]$.

b

$$\frac{\partial}{\partial x}\frac{e^{-x^2(1+t^2)}}{1+t^2} = -2xe^{-x^2(1+t^2)}.$$

exists and is continuous on $\mathbb{R} \times [0,1]$, so according to Leibniz's theorem g is of class C^1 on $\mathbb{R}$, and we have,

$$g'(x) = -2\int_0^1 xe^{-x^2(1+t^2)}dt.$$

c By changing the variable $u = xt$, we get

$$g'(x) = -2\int_0^1 e^{-x^2-u^2}du = -f'(x).$$

Thus, for all $x \in \mathbb{R}$, we have

$$(f+g)'(x) = 0.$$

2 We have for everything $x \in \mathbb{R}$,

$$\begin{aligned} 0 \leq g(x) &\leq e^{-x^2}\int_0^1 \frac{e^{-x^2t^2}}{1+t^2}dt \leq e^{-x^2}\int_0^1 \frac{1}{1+t^2}dt \\ &= e^{-x^2}\Big[\arctan t\Big]_0^1 = \frac{\pi}{4}e^{-x^2}. \end{aligned}$$

Consequently,

$$0 \leq \lim_{x\to} g(x) \leq \frac{\pi}{4} \leq \lim_{x\to} e^{-x^2} = 0.$$

3 The application $h(x) = f(x) + g(x)$ is constant on $\mathbb{R}$ and $h(0) = 0 + \int_0^1 \frac{1}{1+t^2}dt = \frac{\pi}{4}$. Therefore, for all $x \in \mathbb{R}$, we find,

$$f(x) + g(x) = \frac{\pi}{4}.$$

Since,

$$\left|\int_0^{+\infty} e^{-t^2}dt\right| = \lim_{x\to+\infty} \sqrt{f(x)} = \lim_{x\to+\infty} \sqrt{f(x)+g(x)} = \sqrt{\frac{\pi}{4}}.$$

Or $\int_0^{+\infty} e^{-t^2}dt \geq 0$. Hence, $\int_0^{+\infty} e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$.

Correction of supplementary exercise 1

1 We have, $I_n = \int_0^{+\infty} \frac{1}{(x^3+1)^n}dx \underset{+\infty}{\sim} \frac{1}{x^{3n}}$. Therefore, the integral I_n converges if and only if $3n > 1$, i.e., $n \geq 1$.

2 Using factorization, we find,

$$x^3 + 1 = (x+1)(x^2 - x + 1).$$

Then, the rational fraction $\frac{1}{x^3+1}$ can be can be decomposed into simple elements in the form,

$$\frac{1}{x^3+1} = \frac{1}{3} \times \frac{1}{x+1} + \frac{1}{3} \times \frac{-x+2}{x^2-x+1},$$

which can still be written

$$\frac{1}{x^3+1} = \frac{1}{3} \times \frac{1}{x+1} - \frac{1}{6} \times \frac{2x-1}{x^2-x+1} + \frac{1}{3} \times \frac{1}{x^2-x+1}.$$

For $x \geq 0$, we then obtain a primitive,

$$\begin{aligned} \int \frac{1}{x^3+1}dx &= \frac{1}{3}\ln(x+1) - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) \\ &= \frac{1}{3}\ln\left(\frac{x+1}{\sqrt{x^2-x+1}}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right). \end{aligned}$$

Then,

$$\begin{aligned}\int_0^{+\infty} \frac{1}{x^3+1}dx &= \lim_{x\to+\infty}\left[\frac{1}{3}\ln\left(\frac{t+1}{\sqrt{t^2-t+1}}\right)+\frac{1}{\sqrt{3}}\arctan\left(\frac{2t-1}{\sqrt{3}}\right)\right]_0^x \\ &= \frac{1}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right)+\lim_{x\to+\infty}\left[\frac{1}{3}\ln\left(\frac{x+1}{\sqrt{x^2-x+1}}\right)+\frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)\right] \\ &= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}+\frac{\pi}{6}\right)=\frac{2\pi\sqrt{3}}{9}.\end{aligned}$$

Because, $\lim\limits_{u\to+\infty}\arctan(u)=\frac{\pi}{2}$, and $\arctan\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$. Hence the convergence of the improper integral $\int_0^{+\infty}\frac{1}{x^3+1}dx$.

3 If $n \geq 1$, the integral I_n is convergent, and we can integrate by parts. We find

$$I_n = \lim_{x\to+\infty}\left[\frac{t}{(t^3+1)^n}\right]_0^x + \int_0^{+\infty}\frac{3nx^3}{(x^3+1)^{n+1}}dx = \int_0^{+\infty}\frac{3nx^3}{(x^3+1)^{n+1}}dx.$$

But,

$$\int_0^{+\infty}\frac{3nx^3}{(x^3+1)^{n+1}}dx = \int_0^{+\infty}\frac{x^3+1}{(x^3+1)^{n+1}}dx - \int_0^{+\infty}\frac{1}{(x^3+1)^{n+1}}dx = I_n - I_{n+1}.$$

So,

$$I_n = 3n(I_n - I_{n+1}),$$

from which we deduce

$$I_{n+1} = \frac{3n-1}{3n}I_n.$$

Consequently, we obtain,

$$I_n = \frac{3n-4}{3n-3}\frac{3n-7}{3n-6}\cdots\frac{2}{3}I_1 = \frac{3n-4}{3n-3}\frac{3n-7}{3n-6}\cdots\frac{2}{3}\frac{2\pi\sqrt{3}}{9}.$$

Correction of supplementary exercise 2

1 Since f is decreasing on $[a,+\infty[$, then for all $t\in[x,2x]\subset[a,+\infty[$, we have,

$$f(t)\geq f(2x).$$

So,

$$\int_x^{2x} f(t)dt \geq \int_x^{2x} f(2x)dt = f(2x)\int_x^{2x} dt = xf(2x) \tag{2}$$

Replacing in the inequality (2), x by $\frac{x}{2}$ if $x\geq 2a$, we therefore have,

$$0\leq xf(x)\leq 2\int_{\frac{x}{2}}^{x} f(t)dt.$$

As the integral converges, there exists, according to the Cauchy criterion, a number A, such that the inequalities $A \leq u \leq v$ imply

$$\left| \int_u^v f(t)dt \right| < \frac{\varepsilon}{2}.$$

Then if $x \geq 2A$, we have $A \leq \frac{x}{2} \leq x$, and

$$0 \leq xf(x) \leq 2 \int_{\frac{x}{2}}^{x} f(t)dt < \varepsilon.$$

We deduce that $\lim\limits_{x \to +\infty} xf(x) = 0$.

2 For the function f defined on the interval $[2, +\infty[$ by $f(x) = \dfrac{1}{x \ln x}$. since $\lim\limits_{x \to +\infty} xf(x) = 0$. we can easily check that it is decreasing. On the other hand, the integral of f diverges, because,

$$\begin{aligned} \int_2^{+\infty} \frac{1}{x \ln x} dx &= \lim_{x \to +\infty} \int_2^x \frac{1}{t \ln t} dt = \lim_{x \to +\infty} \Big[\ln(\ln t) \Big]_2^x \\ &= -\ln(\ln 2) + \lim_{x \to +\infty} \ln(\ln x) = +\infty. \end{aligned}$$

Hence the convergence of the improper integral $\int_2^{+\infty} \dfrac{1}{x \ln x} dx$, i.e. the inverse of the implication is false.

Correction of supplementary exercise 3

1 If x is in the neighbourhood of 0, we have

$$\sin(xt) \underset{0}{\sim} xt.$$

So

$$_{t \to +\infty} \frac{\sin(xt)}{t} e^{-t} = 0,$$

then the function extends by continuity into 0. Thus the function $x \mapsto \dfrac{\sin(xt)}{t} e^{-t}$ is continuous on $[0, +\infty[$ and we have for $t \geq 1$,

$$\left| \frac{\sin(xt)}{t} e^{-t} \right| \leq e^{-t},$$

$\int_1^{+\infty} e^{-t} dt$ is convergent, then according to the comparison criterion $\int_1^{+\infty} \dfrac{\sin(xt)}{t} e^{-t} dt$ is convergent.

2 Let's assume, for $x \in \mathbb{R}$ and $t > 0$, that $f(x, t) = \dfrac{\sin(xt)}{t} e^{-t}$. Then, $\dfrac{\partial f}{\partial x}$ is defined on

$\mathbb{R}\times]0,+\infty[$ and
$$\frac{\partial}{\partial x}f(x,t)=\cos(xt)e^{-t}.$$
It is therefore a continuous function on $\mathbb{R}\times]0,+\infty[$. Moreover, for all $x\in\mathbb{R}$ and all $t\in\,]0,+\infty[$,
$$\left|\frac{\partial}{\partial x}f(x,t)\right|\leq e^{-t}.$$

Since, $\int\limits_1^{+\infty} e^{-t}dt$ is convergent. We deduce, by the theorem of derivation of integrals with parameters, that F is of class C^1 on $\mathbb{R}$ and
$$F'(x)=\int\limits_1^{+\infty}\cos(xt)e^{-t}dt.$$

3. We can calculate this last integral by writing $\cos(xt)$ as $Re(e^{ixt})$, we therefore have,
$$\begin{aligned}F'(x)&=Re\left[\int\limits_0^{+\infty}e^{(ix-1)t}dx\right]\\&=Re\left[\frac{1}{1-ix}\right]=\frac{1}{1+x^2}.\end{aligned}$$

4. By integrating F' between 0 and x, we find that
$$\int\limits_0^x F'(t)dt=\Big[F(t)\Big]_0^x=\arctan(x)-\arctan(0).$$

Therefore,
$$F(x)=\arctan(x).$$

Correction of supplementary exercise 4

1. For all $t>0$ and $x\in\mathbb{R}$, let $f(t,x)=\frac{1}{t^x(t+1)}$. It is immediate that the integral $\int\limits_0^1\frac{1}{t^x(t+1)}dt$ converges if $x<1$ and $\int\limits_1^{+\infty}\frac{1}{t^x(t+1)}dt$ converges if $x>0$. Therefore, the integral $\int\limits_0^{+\infty}f(t,x)dt$ converges if and only if $x\in]0,1[$. Then, the definition set of F is $]0,1[$.

2. Note that $\int\limits_1^{+\infty}\frac{1}{t^{x+1}}dt=\lim\limits_{x\to+\infty}\left[-\frac{1}{xt^{x+1}}\right]_1^{+\infty}=\frac{1}{x}$,

and by changing the variable $t = \dfrac{1}{u}$, we get

$$\int_1^{+\infty} \frac{1}{t^{x+1}(t+1)}dt = \int_0^1 \frac{u^x}{u+1}du = \int_0^1 \frac{t^x}{t+1}dt.$$

So, we obtain,

$$\begin{aligned} F(x) - \frac{1}{x} &= \int_0^{+\infty} \frac{1}{t^x(t+1)}dt - \int_1^{+\infty} \frac{1}{t^{x+1}}dt \\ &= \int_0^1 \frac{1}{t^x(t+1)}dt + \int_1^{+\infty} \frac{1}{t^x(t+1)}dt - \int_1^{+\infty} \frac{1}{t^{x+1}}dt \\ &= \int_0^1 \frac{1}{t^x(t+1)}dt - \int_1^{+\infty} \frac{1}{t^{x+1}(t+1)}dt \\ &= \int_0^1 \frac{1}{t^x(t+1)}dt - \int_0^1 \frac{t^x}{t+1}dt \end{aligned}$$

We conclude that for all $x \in]0,1[$,

$$0 \leq F(x) - \frac{1}{x} \leq \int_0^1 \frac{1}{t^x(t+1)}dt \leq \int_0^1 \frac{1}{t^x}dt = \frac{1}{1-x}.$$

So, $\lim\limits_{\substack{x \to 0 \\ >}} xF(x) = 1$, namely $F(x) \underset{0}{\sim} \dfrac{1}{x}$.

3 The change of variable $t = \frac{1}{or}$, gives us that $F(x) = F(1-x)$ for $x \in]0,1[$. Then the graph of f is symmetrical with respect to the line with equation $x = \dfrac{1}{2}$.

4 Let's put $F_n(x) = \int_{\frac{1}{n}}^{n} \dfrac{1}{t^x(t+1)}dt$, for all $n \in \mathbb{N}^*$ and $x \in]0,1[$.

It is immediate that $(t,x) \mapsto \dfrac{1}{t^x(t+1)}$ is continuous on $]\dfrac{1}{n}, n[\times]0,1[$. Then, for $n \in \mathbb{N}^*$ the function F_n is continuous on $]0,1[$. On the other hand, let $\varepsilon \in]0, \dfrac{1}{2}[$. For all $x \in [\varepsilon, 1-\varepsilon]$, we have

$$\begin{aligned} F(x) - F_n(x) &= \int_0^{\frac{1}{n}} \frac{1}{t^x(t+1)}dt + \int_n^{+\infty} \frac{1}{t^x(t+1)}dt \\ &\leq \int_0^{\frac{1}{n}} \frac{1}{t^x}dt + \int_n^{+\infty} \frac{1}{t^{x+1}}dt \\ &\leq \frac{1}{1-x}\frac{1}{n^{1-x}} + \frac{1}{x}\frac{1}{n^x} \leq \frac{2}{\varepsilon n^{\varepsilon}}. \end{aligned}$$

It follows that the sequence F_n converges uniformly to F on $[\varepsilon, 1-\varepsilon]$ the function F is then continuous on $[\varepsilon, 1-\varepsilon]$ for all $\varepsilon \in]0, \dfrac{1}{2}[$. We conclude that F is continuous on $]0,1[$.

5 Note that

$$F(x)=\int_0^1\left(\frac{1}{t^x}+\frac{1}{t^{1-x}}\right)\frac{dt}{t+1}=\int_0^{+\infty}\left(e^{xu}+e^{(1-x)u}\right)\frac{du}{1+e^u}.$$

By a simple study shows that the function $x\mapsto e^{xu}+e^{(1-x)u}$ is decreasing on $]0,\frac{1}{2}[$ for all $u>0$. Then F is decreasing on $]0,\frac{1}{2}[$, and using the symmetry of F and that according to question (3). So, the function F is increasing on $]\frac{1}{2},1[$. We conclude that

$$\inf_{x\in]0,1[}F(x)=F\left(\frac{1}{2}\right)=\pi.$$

Correction of supplementary exercise 5

1 For all $x\in\mathbb{R}$, we define the function

$$\begin{array}{rccl}\varphi_x: &]0,+\infty[& \longrightarrow & \mathbb{R}\\ & t & \mapsto & \varphi_x(t)=\dfrac{\ln(x^2+t^2)}{1+t^2}.\end{array}$$

So it is continuous and therefore locally integrable. If $x\neq 0$, then $\lim\limits_{\substack{t\to 0\\ >}}\varphi_x(t)=\ln(x^2)$, on the other hand, if $x=0$ then in the neighbourhood of 0 we have $\varphi_x(t)\underset{0}{\sim}2\ln(t)$. On the other hand, in the neighbourhood of $+\infty$ and $\varphi_x(t)\underset{+\infty}{\sim}\dfrac{2\ln(t)}{t^2}$. This boundary study gives the existence of the integral,

$$F(x)=\int_0^{+\infty}\frac{\ln(x^2+t^2)}{1+t^2}dt.$$

F is obviously even.
Let's introduce the functions sequence as follows,

$$\begin{array}{rccl}F_n: &]0,+\infty[& \longrightarrow & \mathbb{R}\\ & x & \mapsto & F_n(x)=\displaystyle\int_0^n\frac{\ln(x^2+t^2)}{1+t^2}dt,\end{array}$$

and

$$\begin{array}{rccl}h_n: & [0,n]\times]0,+\infty[& \longrightarrow & \mathbb{R}\\ & (t,x) & \mapsto & h_n(t,x)=\dfrac{\ln(x^2+t^2)}{1+t^2}dt.\end{array}$$

It is immediate that,

$$\frac{\partial h_n}{\partial x}(t,x)=\frac{2x}{(t^2+1)(x^2+t^2)},$$

so, h_n and $\dfrac{\partial h_n}{\partial x}$ are continuous on $[0,n]\times]0,+\infty[$, which proves that F_n is of class C^1 on $]0+\infty[$, with

$$F_n'(x)=\int_0^n\frac{2x}{(t^2+1)(x^2+t^2)}dt.$$

Let us introduce the convergent integral $g(x) = \int_0^{+\infty} \frac{2x}{(t^2+1)(x^2+t^2)}dt$. It is immediate that,

$$g(x) - F_n'(x) = \int_n^{+\infty} \frac{2x}{(t^2+1)(x^2+t^2)}dt,$$

hence, using $2xt \leq x^2+t^2$, we have,

$$g(x) - F_n'(x) \leq \int_n^{+\infty} \frac{dt}{t(t^2+1)} \leq \int_n^{+\infty} \frac{dt}{t^3} = \frac{1}{2n^2}.$$

This shows that the sequence $(F_n')_{n\geq 1}$ converges uniformly on $]0+\infty[$ to g and that F is of class C^1 on $]0+\infty[$ with $F' = g$.

2 For $x \neq 1$, we have

$$\frac{2x}{(t^2+1)(x^2+t^2)} = \frac{2x}{x^2-1}\left(\frac{1}{t^2+1} - \frac{1}{x^2+t^2}\right),$$

hence,

$$F'(x) = \frac{2x}{x^2-1}\int_0^{+\infty}\left(\frac{1}{t^2+1} - \frac{1}{x^2+t^2}\right)dt = \frac{2x}{x^2-1}\left(\frac{\pi}{2} - \frac{\pi}{2x}\right) = \frac{\pi}{x+1}.$$

The continuity of F' at 1 allows us to write, for all $x > 0$,

$$F'(x) = \frac{\pi}{x+1} \text{ and } F(x) = \pi\ln(x+1) + c,$$

with c a constant.

We have seen the existence of $F(0) = \int_0^{+\infty} \frac{\ln(t^2)}{1+t^2}dt$. Then, for all $x > 0$, we have,

$$\begin{aligned}
0 \leq F(x) - F(0) &= \int_0^{+\infty} \frac{\ln(x^2+t^2) - \ln(t^2)}{1+t^2}dt \\
&= \int_0^{1} \frac{\ln(x^2+t^2) - \ln(t^2)}{1+t^2}dt + \int_1^{+\infty} \frac{\ln(1+\frac{x^2}{t^2})}{1+t^2}dt \\
&\leq \int_0^{1} \Big(\ln(x^2+t^2) - \ln(t^2)\Big)dt + x^2\int_1^{+\infty} \frac{dt}{t^2(1+t^2)} \\
&\leq \frac{x^2}{3} - 2 + \int_0^1 \ln(x^2+t^2)dt.
\end{aligned}$$

An integration by parts shows that,

$$\int_0^1 \ln(x^2+t^2)dt = \ln(x^2+1) - 2 + 2x\arctan\left(\frac{1}{x}\right),$$

so,
$$\forall x > 0:\ 0 \leq F(x) - F(0) \leq \frac{x^2}{3} + \ln(x^2+1) + 2x\arctan\left(\frac{1}{x}\right) \leq \pi x + \frac{4}{3}x^2.$$
As a result, $\lim\limits_{\substack{x \to 0 \\ >}} F(x) = F(0)$ and F is continuous on $\mathbb{R}$, (due to the parity of F). Finally, the change of variable $t = \dfrac{1}{u}$ in $F(0)$ shows us that,
$$F(0) = 2\int_0^{+\infty} \frac{\ln(t)}{1+t^2}dt = -\int_0^{+\infty} \frac{\ln(u)}{1+u^2}du = -F(0).$$
Consequently, $\lim\limits_{\substack{x \to 0 \\ >}} F(x) = F(0) = 0$.. This shows that $c = 0$ and that $F(x) = \pi\ln(x+1)$, for all $x \geq 0$, then, $F(x) = \pi\ln(|x|+1)$, for all $x \in \mathbb{R}$.