M'sila University	Series n° : 4 **Correction of Fourier series**	Faculty of Mathematics and Computer Science
L 2 Mathematics		Module: Analyse 3

Remark 1

The exercise marked with $(\star)$ or additional will not be corrected in the session of **TD**.

Correction of exercise 1

1 The function f is neither even nor odd, so the Fourier coefficients are,

a

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{0}^{\pi} xdx = \frac{\pi}{4}.$$

b For all $n \in \mathbb{N}^*$, and by integration by parts, we have,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)dx \\ &= \frac{1}{\pi}\Big[\frac{1}{n}x\sin(nx)\Big]_0^{\pi} - \frac{1}{n\pi}\int_0^{\pi}\sin(nx)dx = -\frac{1}{n\pi}\int_0^{\pi}\sin(nx)dx \\ &= \frac{1}{n^2\pi}\Big[\cos(nx)\Big]_0^{\pi} = \frac{1}{n^2\pi}\Big[(-1)^n - 1\Big]. \end{aligned}$$

So,

$$a_{2k} = 0, \text{ if } k \neq 0, \text{ and } a_{2k+1} = -\frac{2}{(2k+1)^2\pi}.$$

c

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)dx \\ &= \frac{1}{\pi}\Big[-\frac{1}{n}x\cos(nx)\Big]_0^{\pi} + \frac{1}{n\pi}\int_0^{\pi}\cos(nx)dx = -\frac{1}{n\pi}\Big[x\cos(nx)\Big]_0^{\pi} \\ &= \frac{(-1)^{n+1}}{n}. \end{aligned}$$

Hence the Fourier series associated with f is,

$$F(f)(x) = \frac{\pi}{4} + \sum_{n\geq 1}\left(\frac{(-1)^n - 1}{n^2\pi}\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)\right).$$

2 Since the function f is of class C^1 by parts (pieces) and $2\pi-$periodic, then according to Dirichlet's theorem, in particular at each point x where f is continuous, we have

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{+\infty}\left(\frac{(-1)^n - 1}{n^2\pi}\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)\right).$$

The function f is continuous at 0, so,

$$\begin{aligned} f(0) &= 0 = \frac{\pi}{4} + \sum_{n=1}^{+\infty} \frac{(-1)^n - 1}{n^2\pi} \\ &= \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}, \end{aligned}$$

Consequently,

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi}{8}.$$

Since,

$$\begin{aligned} \sum_{n=1}^{+\infty} \frac{1}{n^2} &= \sum_{n=1}^{+\infty} \frac{1}{(2n)^2} + \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} \\ &= \sum_{n=1}^{+\infty} \frac{1}{4n^2} + \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}, \end{aligned}$$

So, we find,

$$\frac{3}{4} \sum_{n=1}^{+\infty} \frac{1}{n^2} = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi}{8}.$$

Hence the results,

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi}{8}.$$

Correction of exercise 2

1 The function f is odd, so the Fourier coefficients are,

a

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x|dx = \frac{1}{\pi} \int_{0}^{\pi} xdx = \frac{\pi}{2}.$$

b For all $n \in \mathbb{N}^*$, and since the function $x \mapsto x\cos(nx)$ is odd, then, we have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x\cos(nx)dx = 0.$$

c

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi} \int_{0}^{\pi} x\sin(nx)dx \\ &= \frac{1}{\pi}\Big[-\frac{1}{n}x\cos(nx)\Big]_0^{\pi} + \frac{2}{n\pi} \int_{0}^{\pi} \cos(nx)dx = \frac{2(-1)^{n+1}}{n}. \end{aligned}$$

Hence the Fourier series associated with f is,

$$F(f)(x) = 2\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}\sin(nx).$$

2 For all $n \in \mathbb{N}$, we have

$$\sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & : \ n = 2k, \ (K \in \mathbb{N}) \\ (-1)^k & : \ n = 2k+1. \end{cases}$$

3 The function f is of class C^1 by pieces and continuous on $[-\pi, \pi[$, so according to Dirichlet's theorem, the Fourier series of f simply converges to f on $[-\pi, \pi[$, hence for all $x \in [-\pi, \pi[$, we have

$$f(x) = 2\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

In the particular case where $x = \dfrac{\pi}{2}$, we obtain,

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} = 2\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} \sin\left(n\frac{\pi}{2}\right).$$

According to question 2, we have

$$\frac{\pi}{4} = \sum_{k=0}^{+\infty} \frac{(-1)^{2k+1}}{2k+1}(-1)^k = \sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1}.$$

As a result, we will have

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

By using Parceval's equality,

$$|a_0|^2 + \frac{1}{2}\sum_{n=0}^{+\infty} \left[|a_n|^2 + |b_n|^2\right] = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx.$$

So we get,

$$\frac{1}{2}\sum_{n=0}^{+\infty} \frac{4}{n^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x) dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx.$$

Therefore, it follows that,

$$\sum_{n=0}^{+\infty} \frac{1}{n^2} = \frac{\pi}{6}.$$

Correction of exercise 3

1 The function f is even, so the Fourier coefficients are

a For all $n \in \mathbb{N}$, we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi} |x|\sin(nx)dx = 0.$$

b

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{0}^{\pi} x dx = \frac{\pi}{2}.$$

c Let $n \in \mathbb{N}^*$, and since the function $x \mapsto |c|\cos(nx)$ is even, and by integration by parts, we find,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos(nx)dx \\ &= \frac{2}{\pi}\left[\frac{1}{n}x\sin(nx)\right]_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi}\sin(nx)dx \\ &= -\frac{2}{n\pi}\int_0^{\pi}\sin(nx)dx = \frac{2((-1)^n-1)}{n^2\pi}. \end{aligned}$$

Thus, for all $n \in \mathbb{N}^*$, we have

$$a_n = \begin{cases} 0 & : \ n = 2k, \ (K \in \mathbb{N}) \\ -\dfrac{4}{(2k+1)^2\pi} & : \ n = 2k+1. \end{cases}$$

Hence the Fourier series associated with f is,

$$F(f)(x) = \frac{\pi}{2} - \frac{\pi}{4}\sum_{n\geq 0}\frac{1}{2n+1}\cos\left((2n+1)x\right).$$

2 For all $n \in \mathbb{N}$, we have

$$\sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & : \ n = 2k, \ (K \in \mathbb{N}) \\ (-1)^k & : \ n = 2k+1. \end{cases}$$

3 The function f is of class C^1 by pieces and continuous on $[-\pi, \pi[$, so according to Dirichlet's theorem, the Fourier series of f simply converges to f on $[-\pi, \pi[$, hence for all $x \in [-\pi, \pi[$, we have

$$f(x) = 2\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\sin(nx).$$

In the particular case where $x = \dfrac{\pi}{2}$, we obtain,

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} = 2\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\sin\left(n\frac{\pi}{2}\right).$$

According to question 2, we have

$$\frac{\pi}{4} = \sum_{k=0}^{+\infty}\frac{(-1)^{2k+1}}{2k+1}(-1)^k = \sum_{k=0}^{+\infty}\frac{(-1)^k}{2k+1}.$$

As a result, we have

$$\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

Using the following Parceval equality,

$$|a_0|^2 + \frac{1}{2}\sum_{n=0}^{+\infty}\left[|a_n|^2 + |b_n|^2\right] = \frac{1}{2\pi}\int_0^{2\pi}|f(x)|^2dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx.$$

So we get,

$$\frac{1}{2}\sum_{n=0}^{+\infty}\frac{4}{n^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}f^2(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}x^2dx.$$

Therefore, it follows that,

$$\sum_{n=0}^{+\infty}\frac{1}{n^2} = \frac{\pi}{6}.$$

Correction of exercise 4

1 The function f is odd, so the Fourier coefficients are,

a For all $n \in \mathbb{N}$, and since the function $x \mapsto x - x^3$ is odd, then, we have,

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)dx = \int_{-1}^{1}(x - x^3)\cos(n\pi x)dx = 0.$$

b

$$\begin{aligned}
b_n &= \int_{-1}^{1} f(x)\sin(n\pi x)dx = 2\int_0^1 (x - x^3)\sin(n\pi x)dx \\
&= -\frac{2}{n\pi}\left[(x - x^3)\cos(n\pi x)\right]_0^1 - \frac{2}{n\pi}\int_0^1(3x^2 - 1)\cos(n\pi x)dx \\
&= \frac{12}{n^2\pi^2}\int_0^1 x\sin(n\pi x)dx + \frac{12}{n^2\pi^2}\left[-\frac{x\cos(n\pi x)}{n\pi}\right]_0^1 + \frac{12}{n^2\pi^3}\int_0^1\cos(n\pi x)dx \\
&= \frac{(-1)^{n+1}12}{n^3\pi^3}.
\end{aligned}$$

Hence the Fourier series associated with f is,

$$F(f)(x) = \sum_{n=0}^{+\infty}\frac{(-1)^{n+1}12}{n^3\pi^3}\sin(n\pi x).$$

2 The function f is of class C^1 by pieces (parts) and continuous on $[-1, 1[$, so according to Dirichlet's theorem, the Fourier series of f simply converges to f on $[-1, 1[$, hence for all $x \in [-1, 1[$, we have

$$f(x) = \sum_{n=0}^{+\infty}\frac{(-1)^{n+1}12}{n^3\pi^3}\sin(n\pi x).$$

In the special case if $x = dfrac12$, we obtain,

$$f\left(\frac{1}{2}\right) = \frac{3}{8} = 2\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\sin\left(n\frac{\pi}{2}\right).$$

From the above, we have

$$\sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & : n = 2k, \ (K \in \mathbb{N}) \\ (-1)^k & : n = 2k+1. \end{cases}$$

Hence,

$$\frac{3}{8} = \sum_{k=0}^{+\infty} \frac{(-1)^{2k+1}}{2k+1}(-1)^k = \sum_{k=0}^{+\infty} \frac{(-1)^k}{\pi^3(2k+1)^3}.$$

As a result, we have

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

Correction of exercise 5

1 Indeed, for $n \neq 0$, the coefficients of the Fourier series are,

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx} = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)e^{-inx}dx + \frac{1}{2i}\int_{0}^{\pi} f(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-inx}dx + \frac{1}{\pi}\int_{0}^{2\pi} f(-x)e^{inx}dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx dx, \ \text{(because f is even)} \\
&= \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} \cos nx dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{2(\pi - x)}{\pi}\cos nx dx \\
&= \left[\frac{\sin(nx)}{n\pi}\right]_0^{\frac{\pi}{2}} + \left[\frac{2(\pi - x)\sin(nx)}{n\pi^2}\right]_{\frac{\pi}{2}}^{\pi} + \frac{2}{n\pi^2}\int_{\frac{\pi}{2}}^{\pi} \frac{2(\pi - x)}{\pi}\sin nx dx \\
&= -\left[\frac{2\cos(nx)}{n^2\pi^2}\right]_{\frac{\pi}{2}}^{\pi} = \frac{2\Big(\cos(\frac{n\pi}{2} - \cos(n\pi))\Big)}{n^2\pi^2}.
\end{aligned}$$

On the other hand, an immediate calculation shows that $c_0 = \frac{3}{4}$.
We know that for all $n \in \mathbb{N}^*$, we have $c_n = c_{-n}$, so, we find,

$$F(f)(x) = \frac{3}{4} + \sum_{n=1}^{+\infty} 2c_n \cos(nx).$$

Which means,

$$F(f)(x) = \frac{3}{4} + \frac{4}{\pi^2}\sum_{n=0}^{+\infty} \frac{\cos(2n+1)x}{(2n+1)^2} - \frac{2}{\pi^2}\sum_{n=0}^{+\infty} \frac{\cos(4n+2)x}{(2n+1)^2}.$$

2 Note that the Fourier series of f is normally convergent, so the continuity of f shows that for all $x \in]-\pi, \pi]$,

$$f(x) = \frac{3}{4} + \frac{4}{\pi^2}\sum_{n=0}^{+\infty} \frac{\cos(2n+1)x}{(2n+1)^2} - \frac{2}{\pi^2}\sum_{n=0}^{+\infty} \frac{\cos(4n+2)x}{(2n+1)^2}.$$

In the special case, taking $x = 0$, we obtain

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi}{8}.$$

3 Note also that $g(x) = 1 - f(x-\pi)$ for all $x \in]-\pi, \pi]$. Then, $\forall x \in]-\pi, \pi]$, we have,

$$\begin{aligned} g(x) &= 1 - \left(\frac{3}{4} + \frac{4}{\pi^2} \sum_{n=0}^{+\infty} \frac{\cos(2n+1)(\pi-x)}{(2n+1)^2} - \frac{2}{\pi^2} \sum_{n=0}^{+\infty} \frac{\cos(4n+2)(\pi-x)}{(2n+1)^2} \right) \\ &= \frac{1}{4} + \frac{4}{\pi^2} \sum_{n=0}^{+\infty} \frac{\cos(2n+1)x}{(2n+1)^2} - \frac{2}{\pi^2} \sum_{n=0}^{+\infty} \frac{\cos(4n+2)x}{(2n+1)^2}. \end{aligned}$$

Correction of exercise 6

1 Since the function f is odd, $a_n = 0$ for all $n \in \mathbb{N}$. Then for $n \geq 1$, we have,

$$\begin{aligned} b_n &= \frac{1}{2\pi} \int_0^{\pi} \sin(nx)dx = \left[-\frac{\cos(nx)}{n} \right]_0^{\pi} = \frac{2(1-(-1)^n)}{n\pi} \\ &= \begin{cases} \frac{4}{n\pi} & : n \text{ is odd} \\ 0 & : n \text{ is even} \end{cases} \end{aligned}$$

The trigonometric Fourier series of f is therefore given by

$$F(f)(x) = \sum_{k=0}^{+\infty} \frac{4}{(2n+1)\pi} \sin(2k+1)x.$$

2 The function f satisfies the assumptions of Dirichlet's theorem, and the series $F(f)$ converges in all x to,

$$\frac{f(x+) + f(x-)}{2} = f(x).$$

The convergence cannot be uniform because the limit f is not continuous.

3 For $x = \frac{\pi}{2}$, we have,

$$\sin(2k+1)\frac{\pi}{2} = \sin\left(\frac{\pi}{2} + k\pi\right) = (-1)^k,$$

We deduce

$$\sum_{k=0}^{+\infty} \frac{4(-1)^k}{(2n+1)\pi} = f\left(\frac{\pi}{2}\right) = 1 \Rightarrow \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2n+1)} = \frac{\pi}{4}.$$

Since f is odd, Parseval's equality gives,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \left|f(x)\right|^2 dx = \frac{1}{2} \sum_{K=0}^{+\infty} \left|b_k\right|^2 = \frac{8}{\pi^2} \sum_{k=0}^{+\infty} \frac{1}{(2n+1)^2}, \quad \text{so} \quad \sum_{k=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi}{8}.$$

Next, we have,

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \sum_{k=0}^{+\infty} \frac{1}{(2k+1)^2} + \sum_{k=1}^{+\infty} \frac{1}{(2k)^2} = \frac{\pi}{8} + \frac{1}{4} \sum_{n=1}^{+\infty} \frac{1}{n^2}.$$

So, we find,

$$\frac{3}{4}\sum_{n=1}^{+\infty}\frac{1}{n^2}=\frac{\pi}{8}\Rightarrow\sum_{n=1}^{+\infty}\frac{1}{n^2}=\frac{4}{3}\times\frac{\pi}{8}=\frac{\pi}{6}.$$

Finally, from the above, we have,

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^2}+\sum_{k=1}^{+\infty}\frac{1}{(2k)^2}=\sum_{k=1}^{+\infty}\frac{1}{(2n+1)^2}=\frac{\pi}{8},\text{ and }\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\pi}{8}-\frac{1}{4}\frac{\pi}{6}=\frac{\pi}{12}.$$

Correction of exercise 7

1 Note that f is even, so that $b_n=0$ for all n. Furthermore,

$$a_0=\frac{1}{\pi}\int_0^{2\pi}(x-\pi)^2dx=\frac{2\pi^2}{3}.$$

For all $n\geq 1$, and integrations by parts, we have

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi}(x-\pi)^2\cos(nx)dx\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}t^2\cos(nt+n\pi)dt=\frac{(-1)^n}{\pi}\int_{-\pi}^{\pi}t^2\cos(nt)dt\\
&= \frac{(-1)^n}{\pi}\times\frac{4}{n^2}(-1)^n=\frac{4}{n^2}.
\end{aligned}$$

Hence, the Fourier series of f is written as,

$$F(f)(x)=\frac{\pi}{3}+4\sum_{n=1}^{+\infty}\frac{1}{n^2}\cos(nx).$$

2 The function f is of class $C^1(\mathbb{R})$, Dirichlet's theorem leads to the conclusion that for all $x\in\mathbb{R}$, we have

$$F(f)(x)=f(x)=\frac{\pi}{3}+4\sum_{n=1}^{+\infty}\frac{1}{n^2}\cos(nx).$$

3 From the previous question,

$$\pi^2=f(0)=\frac{\pi}{3}+4\sum_{n=1}^{+\infty}\frac{1}{n^2}\quad\text{and}\quad 0=f(\pi)=\frac{\pi}{3}+4\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}$$

We deduce that,

$$\sum_{n=1}^{+\infty}\frac{1}{n^2}\frac{\pi^2}{6}=\text{ and }\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}=-\frac{\pi^2}{12}.$$

Correction of exercise 8

1 Posons,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{+\infty}(a_n\cos(nx)+b_n\sin(nx)),\ \text{and}\ y(x)=\frac{\alpha_0}{2}+\sum_{n=1}^{+\infty}(\alpha_n\cos(nx)+\beta_n\sin(nx)),$$

and, deriving term by term,

$$y'(x)=\sum_{n=1}^{+\infty}(n\beta_n\cos(nx)-n\alpha_n\sin(nx)).$$

So we have,

$$y'(x)+\lambda y(x)=\frac{\lambda\alpha_0}{2}+\sum_{n=1}^{+\infty}\Big([n\beta_n+\lambda\alpha_n]\cos(nx)+[\lambda\beta_n-n\alpha_n]\sin(nx)\Big).$$

If we identify the coefficients, we obtain

$$\begin{cases} a_0=\lambda\alpha_0,\\ a_n=[n\beta_n+\lambda\alpha_n],\\ b_n=[\lambda\beta_n-n\alpha_n], \end{cases}\quad\Rightarrow\begin{cases} \alpha_0=\dfrac{a_0}{\lambda},\\ \alpha_n=\dfrac{\lambda a_n-nb_n}{n^2+\lambda^2},\\ \beta_n=\dfrac{na_n+\lambda b_n}{n^2+\lambda^2}. \end{cases}$$

2 The proposed function f is continuously differentiable on $\mathbb{R}$. Let's compute its Fourier coefficients. We have

$$a_0=\frac{1}{\pi}\int_0^{\pi}(x-\frac{\pi}{2})^2dx+\frac{1}{\pi}\int_{\pi}^{2\pi}\left[(x-\frac{\pi}{2})^2+\frac{\pi^2}{2}\right]dx=\frac{\pi^2}{2}.$$

Furthermore, we note that the function $f-\frac{a_0}{2}$ is odd, so that $a_n=0$ for $n\geq 1$. Finally,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{\pi}(x-\frac{\pi}{2})^2\cos(nx)dx+\frac{1}{\pi}\int_{\pi}^{2\pi}\left[(x-\frac{\pi}{2})^2+\frac{\pi^2}{2}\right]\sin(nx)dx\\ &= \frac{1-(-1)^n}{\pi}\int_0^{\pi}(x-\frac{\pi}{2})^2\cos(nx)dx+\frac{\pi}{2}\int_{\pi}^{2\pi}\sin(nx)dx\\ &= \frac{1-(-1)^n}{\pi}I_1+\frac{\pi}{2}I_2. \end{aligned}$$

By integrating by parts twice, we calculate the integral I_1, which we find,

$$I_1=\int_0^{\pi}(x-\frac{\pi}{2})^2\cos(nx)dx=(1-(-1)^n)\Big[\frac{\pi^2}{4n}-\frac{2}{n^3}\Big].$$

Thus,

$$I_2=\int_{\pi}^{2\pi}\sin(nx)dx=\frac{(-1)^n-1}{n}.$$

Hence,

$$b_n = \frac{4((-1)^n - 1)}{\pi n^3}.$$

We obtain $\alpha_0 = \dfrac{\pi^2}{2}$ and, for all $k \in \mathbb{N}^*$, $\alpha_{2k} = \beta_{2k} = 0$,

$$\alpha_{2k-1} = \frac{8}{\pi(2k-1)^2[(2k-1)^2+1]},$$

$$\text{and} \quad \beta_{2k-1} = \frac{-8}{\pi(2k-1)^3[(2k-1)^2+1]}.$$

Finally, it is easy to see that the series

$$\sum_{n=1}^{+\infty}(n\beta_n \cos(nx) - n\alpha_n \sin(nx)).$$

is uniformly convergent, and therefore the series,

$$\frac{\alpha_0}{2} + \sum_{n=1}^{+\infty}(\alpha_n \cos(nx) + \beta_n \sin(nx)),$$

which is also uniformly convergent, has as it's sum a function $y(x)$ continuously derivable on $\mathbb{R}$, which is a solution of the given differential equation (**??**).

Remark 2

Let f and g be two $2\pi-$ periodic functions defined on $\mathbb{R}$. Then, the convolution product of f and g is given by

$$\forall x \in \mathbb{R}: \ (f * g)(x) = \sum_{n \in \mathbb{Z}} c_n(f).c_n(g)e^{-inx}.$$

Correction of supplementary exercise 1

1 **a** We assume that $a \in \mathbb{C} \setminus i\mathbb{Z}$, and we write f_a to designate the $2\pi-$periodic function that verifies $f_a(x) = e^{ax}$ on $]-\pi, \pi]$. Note that,

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}dx \\ &= \left[\frac{e^{(a-in)x}}{2\pi(a-in)}\right]_{-\pi}^{\pi} = \frac{(-1)^n}{a-in} \times \frac{sh(a\pi)}{\pi} \\ &= \frac{(-1)^n(a+in)}{a^2+n^2} \times \frac{sh(a\pi)}{\pi}. \end{aligned}$$

Then, the Fourier series of f_a is,

$$F(f_a)(x) = \frac{sh(a\pi)}{a\pi} + 2\frac{sh(a\pi)}{\pi}\left(\sum_{n=1}^{+\infty}\frac{(-1)^n a}{n^2+a^2}\cos(nx) - \frac{2}{\pi^2}\sum_{n=0}^{+\infty}\frac{(-1)^n a}{n^2+a^2}\sin(nx)\right).$$

According to Dirichlet's theorem, the series $F(f_a)$ converges for all x and in particular we have,

$$\forall x \in]-\pi,\pi[:\ e^{ax} = \frac{sh(a\pi)}{a\pi} + 2\frac{sh(a\pi)}{\pi}\left(\sum_{n=1}^{+\infty}\frac{(-1)^n a}{n^2+a^2}\cos(nx) - \frac{2}{\pi^2}\sum_{n=0}^{+\infty}\frac{(-1)^n a}{n^2+a^2}\sin(nx)\right).$$

(b) On the other hand, the $2\pi-$periodic function g_a, which coincides with $x \mapsto chax$ on $]-\pi,\pi]$, verifies that, $g_a(x) = \dfrac{f_a(x^+)+f_a(x^-)}{2}$. Then we have,

$$c_n = \frac{(-1)^n a}{a^2+n^2} \times \frac{sh(a\pi)}{\pi},$$

and the Fourier series of g_a is,

$$F(g_a)(x) = \frac{sh(a\pi)}{a\pi} + \frac{sh(a\pi)}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^n 2a}{n^2+a^2}\cos(nx).$$

The Fourier series of g_a is normally convergent and g_a is continuous on $\mathbb{R}$, then,

$$\forall x \in [-\pi,\pi]:\ g_a(x) = ch(ax) = \frac{sh(a\pi)}{a\pi}\left(1 - \frac{sh(a\pi)}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^n 2a}{n^2+a^2}\cos(nx)\right).$$

In the special case, we obtain for any $x \in [-\pi,\pi]$,

$$\begin{aligned} ch(ax) &= \frac{sh(a\pi)}{a\pi} + \frac{sh(a\pi)}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^n 2a}{n^2+a^2}\cos(nx) \\ &= \frac{sh(a\pi)}{a\pi} - 2\frac{sh(a\pi)}{a\pi} + \frac{sh(a\pi)}{\pi}\sum_{n=0}^{+\infty}\frac{(-1)^n 2a}{n^2+a^2}\cos(nx) \\ &= \frac{sh(a\pi)}{a\pi}\left(-1 + \sum_{n=0}^{+\infty}\frac{(-1)^n 2a^2}{n^2+a^2}\cos(nx)\right). \end{aligned} \tag{1}$$

(2) Substituting x for 0 in equation (1) we get,

$$\frac{\pi}{2sh(a\pi)} + \frac{1}{2a} = \sum_{n=0}^{+\infty}\frac{(-1)^n a}{n^2+a^2}.$$

In particular, if $a=1$, we have

$$\frac{\pi}{2sh(\pi)} + \frac{1}{2} = \sum_{n=0}^{+\infty}\frac{(-1)^n}{n^2+1}.$$

On the other hand, if we replace x with π in the equation (1), we get

$$\frac{\pi}{2th(a\pi)} + \frac{1}{2a} = \sum_{n=0}^{+\infty}\frac{a}{n^2+a^2}.$$

Since the Fourier series of $g_a * g_a$ is normally convergent and its coefficients $c_{g_a*g_a}$ defined by

$$c_{g_a*g_a} = (c_{g_a})^2 = \frac{a^2}{n^2+a^2} \times \frac{sh^2 a\pi}{\pi^2}.$$

Then the sum of $g_a * g_a$ is given by,

$$g_a * g_a(x) = \frac{sh^2(a\pi)}{a^2\pi^2}\left(1 + 2\sum_{n=1}^{+\infty} \frac{a^4}{(n^2+a^2)^2}\cos(nx)\right).$$

On en déduit en prenant $x = 0$,

$$g_a * g_a(0) = \frac{sh^2(a\pi)}{a^2\pi^2}\left(-1 + 2\sum_{n=0}^{+\infty} \frac{a^4}{(n^2+a^2)^2}\right).$$

And also, $g_a * g_a(0) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} ch^2(ax)dx = \frac{1}{2} + \frac{sh(2a\pi)}{4a\pi}$, which gives,

$$\frac{\pi^2}{4sh^2(a\pi)} + \frac{\pi}{4ath(a\pi)} + \frac{1}{2a^2} = \sum_{n=0}^{+\infty} \frac{a^2}{(n^2+a^2)^2}$$