M'sila University	**Series n° : 3**	Faculty of Mathematics and Computer Science
L 2 Mathematics	**Integers series (Correction)**	Module: Analysis 3

Remark 1

The exercise marked with $(\star)$ or additional will not be corrected in the session of **TD**.

Correction of exercise 1

1 Let $u_n(z) = \dfrac{n^2+1}{3^n} z^n$, $z \in \mathbb{C}$. So, $a_n = \dfrac{n^2+1}{3^n}$ and the radius R is given by

$$R = \lim_{n\to+\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to+\infty} \frac{3^{n+1}}{3^n} \times \frac{n^2+1}{(n+1)^2+1} = 3.$$

2 For $u_n(z) = \dfrac{(3n)!}{(n!)^3} z^n$, $z \in \mathbb{C}$. We have, $a_n = \dfrac{(3n)!}{(n!)^3}$ and

$$\begin{aligned} R &= \lim_{n\to+\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to+\infty} \frac{(3n)!}{(n!)^3} \times \frac{(3(n+1))!}{((n+1)!)^3} \\ &= \lim_{n\to+\infty} \left(\frac{(3(n+1))!}{(3n)!}\right) \times \left(\frac{n!}{(n+1)!}\right)^3 = \lim_{n\to+\infty} \frac{3(3n+2)(3n+1)}{(n+1)^2} = 27. \end{aligned}$$

3 For all $n \in \mathbb{N}^*$, we have,

$$a_n = (n+1)^{\frac{1}{n+1}} - n^{\frac{1}{n}} = e^{\frac{\ln(n+1)}{n+1}} - e^{\frac{\ln(n)}{n}}.$$

We know that $\displaystyle\lim_{n\to+\infty} \frac{\ln(n)}{n} = 0$, so we obtain,

$$\begin{aligned} a_n &= (n+1)^{\frac{1}{n+1}} - n^{\frac{1}{n}} \underset{+\infty}{\sim} \frac{\ln(n+1)}{n+1} - \frac{\ln(n)}{n} \\ &= -\frac{\ln(n)}{n(n+1)} + \frac{\ln(1+\frac{1}{n})}{n+1} \underset{+\infty}{\sim} \frac{\ln(n)}{n^2} \end{aligned}$$

Par conséquent,

$$\begin{aligned} R &= \lim_{n\to+\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to+\infty} \frac{\ln(n)}{n^2} \times \frac{(n+1)^2}{\ln(n+1)} \\ &= \lim_{n\to+\infty} \frac{\ln(n)}{\ln(n+1)} \times \frac{(n+1)^2}{n^2} = 1. \end{aligned}$$

4 Let $u_n(z) = \dfrac{\ln(n)}{n^2} z^{2n}$. Then, if $|z| < 1$ the series is convergent and if $|z| > 1$ it is divergent. Therefore $R = 1$. Because,

$$R = \lim_{n\to+\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to+\infty} \frac{\ln(n)}{n^2} \times \frac{(n+1)^2}{\ln(n+1)} = 1.$$

5. For the series $\sum_{n\geq 0} \frac{n^n}{n!} z^{3n}$, $z \in \mathbb{C}$, we have,

$$\frac{u_{n+1}(z)}{u_n(z)} = \frac{(n+1)^{n+1}}{n!} \times \frac{n!}{n^n}|z|^3.$$

So, as a consequence,

$$\lim_{n\to+\infty} \frac{u_{n+1}(z)}{u_n(z)} = |z|^3 \lim_{n\to+\infty} e^{n\ln(1+\frac{1}{n})} = e|z|^3.$$

Hence, the series converges if $|z|^3 < 1$, i.e., $|z| < \sqrt[3]{\frac{1}{e}}$, and diverges if $|z|^3 > 1$. Therefore the radius of convergence is $R = \sqrt[3]{\frac{1}{e}}$.

6. For $u_n(z) = e^{-n^2} z^n$, we have $a_n = e^{-n^2}$. According to Cauchy's rule we have

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = |z|^3 \lim_{n\to+\infty} e^{-n} = 0.$$

So we find $R = +\infty$.

7. If $u_n(z) = \sin\left(\frac{1}{\sqrt{n}}\right) z^n$, we have $a_n = \sin\left(\frac{1}{\sqrt{n}}\right)$. Let R be the radius of convergence of the series $\sum_{n\geq 1} u_n(z)$, we distinguish two cases,

   a. For $|z| < 1$, we know that for all $n \in \mathbb{N}^*$,

$$|\sin\left(\frac{1}{\sqrt{n}}\right) z^n| < |z|^n,$$

   Since the series $\sum_{n\geq 1} |z|^n$ converges, then the series $\sum_{n\geq 1} \sin\left(\frac{1}{\sqrt{n}}\right) z^n$ converges. Consequently the radius of convergence $R = 1$.

   b. For $|z| > 1$, we know that for all $n \in \mathbb{N}^*$,

$$|\sin\left(\frac{1}{\sqrt{n}}\right) z^n| \underset{+\infty}{\sim} \frac{1}{\sqrt{n}}|z|^n = \frac{1}{\sqrt{n}} e^{n\ln|z|} \underset{n\longrightarrow\infty}{\longrightarrow} +\infty.$$

   In other words, the $\sum_{n\geq 1} |z|^n$ series is divergent in this case.

Correction of exercise 2

According to Cauchy's rule we have,

$$\lim_{n\to+\infty} \sqrt[n]{|a_n||z|^2} = |z|^{2n} \lim_{n\to+\infty} \sqrt[n]{|a_n|} = \frac{|z|^2}{R}.$$

The series converges if $\frac{|z|^2}{R} < 1$ and diverges if $\frac{|z|^2}{R} > 1$, i.e., converges if $|z| < \sqrt{R}$ and diverges if $|z| > \sqrt{R}$. So the radius of convergence is $R' = \sqrt{R}$.

Correction of exercise 3

1 Radius and domain of convergence and convergence at boundary points

a If $u_n(z) = z^n$, ici $a_n = 1$, so, $R = \lim\limits_{n\to+\infty} \frac{a_n}{a_{n+1}} = \lim\limits_{n\to+\infty} 1 = 1$. The series converges for $|z| < 1$ and diverges on the boundary (circle of center 0 and radius 1) $|z| = 1$, because, $\lim\limits_{\substack{n\to+\infty \\ |z|=1}} |z|^n = 1 \neq 0$.

b $u_n(z) = nz^n$, $a_n = n$, so, $R = \lim\limits_{n\to+\infty} \frac{a_n}{a_{n+1}} = \lim\limits_{n\to+\infty} \frac{n}{n+1} = 1$. As a result, the series converges for $|z| < 1$ and diverges on the boundary (circle of center 0 and radius 1) $|z| = 1$, because, $\lim\limits_{\substack{n\to+\infty \\ |z|=1}} n|z|^n = +\infty \neq 0$.

c For the series of the general term $u_n(z) = \frac{1}{n}z^n$, $a_n = \frac{1}{n}$, therefore, $R = 1$. Then, the series converges for $|z| < 1$.

For $z = 1$, the harmonic series $\sum\limits_{n\geq 1} \frac{1}{n}$ diverges.

If $z = -1$, the alternating series $\sum\limits_{n\geq 1} \frac{(-1)^n}{n}$ converges.

It follows that the series $\sum\limits_{n\geq 1} \frac{1}{n}z^n$ converges if $z \in [-1, 1[$.

d Let $u_n(z) = \frac{5n}{n+2}z^n$, $a_n = \frac{5n}{n+2}$, so, $R = 1$. Then, the series converges for $|z| < 1$, and diverges if $|z| = 1$ because, $\lim\limits_{\substack{n\to+\infty \\ |z|=1}} \frac{5n}{n+2}|z|^n = 5 \neq 0$.

e $u_n(z) = \frac{n^2+n-1}{n!}z^n$. Therefore, $a_n = \frac{n^2+n-1}{n!}$, so, we find,

$$\frac{1}{R} = \lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{n^2+3n+1}{n^2+n-1} \times \frac{1}{n+1} = 0.$$

Consequently, $R = +\infty$.

f $u_n(z) = (-1)^n(4z+1)^n$. So we obtain

$$\lim_{n\to+\infty} \sqrt[n]{|u_n(z)|} = \lim_{n\to+\infty} \sqrt[n]{|(4z+1)^n|} = |4z+1|.$$

The series converges for $|4z+1| < 1$ and diverges for $|4z+1| > 1$. That is, the series converges for $|z+\frac{1}{4}| < \frac{1}{4}$ and diverges for $|z+\frac{1}{4}| > \frac{1}{4}$. Hence, the domain of convergence is the disk of center $A\left(\frac{1}{4}, 0\right)$ and radius $\frac{1}{4}$.

For convergence on the boundary $|z+\frac{1}{4}| = \frac{1}{4}$. We have,

$$\lim_{\substack{n\to+\infty \\ |z+\frac{1}{4}|=\frac{1}{4}}} (-1)^n \neq 0.$$

Since the general term does not tend towards 0, then, the series does not converge.

(g) $u_n(z) = \dfrac{(z-1)^{2n}}{4^n}$. The series converges for

$$\lim_{n\to+\infty} \sqrt[n]{|u_n(z)|} = \lim_{n\to+\infty} \sqrt[n]{\left|\frac{(z-1)^{2n}}{4^n}\right|} = \frac{|z-1|^2}{4} < 1,$$

and diverges for $\dfrac{|z-1|^2}{4} > 1$, so the series converges for all $z \in]-1, 4[$ and diverges otherwise. Study convergence at boundary points. Then, if $z = -1$ the general term is

$$u_n(-1) = \frac{(-2)^{2n}}{4^n} = 1,$$

which does not tend to 0, so the series does not converge.
For $z = 4$ the general term is

$$u_n(4) = \frac{(3)^{2n}}{4^n} = \left(\frac{9}{4}\right)^n,$$

which also diverges, as a geometric series, and its reason $q = \dfrac{9}{4} > 1$.

(2) Calculate sums of integer series in the convergence interval.

(a) The sum of the series $\sum\limits_{n\geq 0} z^n$ on the set $\{z \in \mathbb{C} : |z| < 1\}$ is $\sum\limits_{n\geq 0}^{+\infty} z^n = \dfrac{1}{1-z}$.

(b) The sum of the series $\sum\limits_{n\geq 0} nz^n$ on the set $\{z \in \mathbb{R} : |z| < 1\}$.
For all $|z| < 1$, we have,

$$\sum_{n\geq 0}^{+\infty} z^n = z\sum_{n=1}^{+\infty} nz^{n-1} = z\Big(\sum_{n=0}^{+\infty} z^n\Big)' = z\Big(\frac{1}{1-z}\Big)' = \frac{z}{(1-z)^2}.$$

(c) The sum of the series $\sum\limits_{n\geq 1} \dfrac{1}{n} z^n$ on the set $\{z \in \mathbb{R} : |z| < 1\}$ is

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{1}{n} z^n &= \sum_{n=1}^{+\infty} \int_{z=0}^{z} t^{n-1} dt \\
&= \int_{z=0}^{z} \Big(\sum_{n=1}^{+\infty} t^n\Big) dt = \int_{z=0}^{z} \frac{1}{1-z} dt \\
&= -\Big[\ln(1-t)\Big]_{z=0}^{z} = -\ln(1-z).
\end{aligned}$$

(d) Calculate the sum of the series $\sum\limits_{n\geq 1} \dfrac{5n}{n+2} z^n$ on the set $\{z \in \mathbb{R} : |z| < 1\}$.
We have for all $z \in]-1, 1[$ and for all $n \geq 0$,

$$\frac{5n}{n+2} z^n = 5z^n - \frac{10}{n+2} z^n.$$

Consequently, we will have,

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{5n}{n+2}z^n &= 5\sum_{n=1}^{+\infty}z^n-\frac{10}{z^2}\sum_{n=1}^{+\infty}\frac{1}{n+2}z^{n+2}=\frac{5z}{1-z}-\frac{10}{z^2}\sum_{n=3}^{+\infty}\frac{1}{n}z^n\\ &= \frac{5z}{1-z}-\frac{10}{z^2}\Big(\sum_{n=1}^{+\infty}\frac{1}{n}z^n-z-\frac{z^2}{2}\Big)=\frac{5z}{1-z}-\frac{10}{z^2}\Big(-\ln(1-z)-z-\frac{z^2}{2}\Big)\\ &= \frac{10\ln(1-z)}{z^2}-\frac{5(2-z)}{z(1-z)}.\end{aligned}$$

e For the series $\sum\limits_{n\geq1}\frac{n^2+n-1}{n!}z^n$ on the set $\mathbb{R}$.
We have for all $z\in\mathbb{R}$ and for all $n\in\mathbb{N}$,

$$\begin{aligned}\sum_{n=0}^{+\infty}\frac{n^2+n-1}{n!}z^n &= \sum_{n=0}^{+\infty}\frac{n^2}{n!}+\sum_{n=0}^{+\infty}\frac{n}{n!}z^n-\sum_{n=0}^{+\infty}\frac{1}{n!}z^n\\ &= \sum_{n=1}^{+\infty}\frac{n}{(n-1)!}z^n+\sum_{n=1}^{+\infty}\frac{}{(n-1)!}z^n-e^z\\ &= \sum_{n=2}^{+\infty}\frac{z^n}{(n-2)!}+z\sum_{n=1}^{+\infty}\frac{z^{n-1}}{(n-1)!}+z\sum_{n=0}^{+\infty}\frac{z^n}{n!}-e^z\\ &= z^2\sum_{n=2}^{+\infty}\frac{z^{n-2}}{(n-2)!}+z\sum_{n=1}^{+\infty}\frac{z^{n-1}}{(n-1)!}+z\sum_{n=0}^{+\infty}\frac{z^n}{n!}-e^z\\ &= z^2\sum_{n=0}^{+\infty}\frac{z^n}{n!}+z\sum_{n=0}^{+\infty}\frac{z^n}{n!}+z\sum_{n=0}^{+\infty}\frac{z^n}{n!}-e^z=(z^2+2z-1)e^z.\end{aligned}$$

f Calculate the sum of the series $\sum\limits_{n=0}^{+\infty}(-1)^n(4z+1)^n$ on the set $\Big\{z\in\mathbb{C}:\ \Big|z+\frac{1}{4}\Big|<\frac{1}{4}\Big\}$.
For $z\in\overset{\circ}{D}(A,\frac{1}{4})$, we have

$$\sum_{n=0}^{+\infty}(-1)^n(4z+1)^n=\sum_{n=0}^{+\infty}(-(4z+1))^n=\frac{1}{1+4z+1}=\frac{1}{2+4z}.$$

g For the series $\sum\limits_{n=0}^{+\infty}\frac{(z-1)^{2n}}{4^n}$ where z in the set $]-1,4[$, the sum is given by,

$$\sum_{n=0}^{+\infty}\frac{(z-1)^{2n}}{4^n}=\sum_{n=0}^{+\infty}\left(\frac{(z-1)^2}{4}\right)^n=\frac{1}{1-\frac{(z-1)^2}{4}}=\frac{4}{(3-z)(z+1)}.$$

Correction of exercise 4

1 Let $f(x)=\frac{1}{(1-x)(2+x)}$. For $|x|<1,\ |x|<2$, and near 0 we find,

$$\begin{aligned}f(x)=\frac{1}{(1-x)(2+x)} &= \frac{1}{3}\times\frac{1}{1-x}+\frac{1}{3}\frac{1}{2+x}=\frac{1}{3}\times\frac{1}{1-x}+\frac{1}{6}\times\frac{1}{1+\frac{x}{2}}\\ &= \sum_{n=0}^{+\infty}\Big(\frac{1}{3}+\frac{(-1)^n}{6\times2^n}\Big)x^n.\end{aligned}$$

Hence, for all $x \in]-1,1[$, we have, $f() = \sum_{n=0}^{+\infty}\left(\frac{1}{3} + \frac{(-1)^n}{6\times 2^n}\right)x^n$.

2 Now if $f(x) = \ln\left(\frac{x-1}{x+2}\right)$. For $|x| < 1$, and neighboring 0 we'll have,

$$\begin{aligned} f(x) = \ln\left(\frac{x-1}{x+2}\right) &= \ln(x-1) - \ln(2+x) = \ln(x-1) - \ln 2 - \ln(1+\frac{x}{2}) \\ &= -\ln 2 + \sum_{n=1}^{+\infty}\frac{x^n}{n} - \sum_{n=1}^{+\infty}(-1)^n\frac{x^n}{n2^n}. \end{aligned}$$

3 $f(x) = \arctan\left(\frac{x-1}{1+x}\right)$. For all $x \in \mathbb{R}$, $x \neq -1$, we have $f'(x) = \frac{1}{1+x^2}$, and since x is in the neighborhood of 0, we obtain,

$$f'(x) = \frac{1}{1+x^2} = \sum_{n=1}^{+\infty}(-1)^n x^{2n}.$$

Consequently,

$$\begin{aligned} f(x) &= \int_0^x \frac{1}{1+t^2}dt = \int_0^x \left(\sum_{n=0}^{+\infty}(-1)^n t^{2n}\right)dt \\ &= \sum_{n=0}^{+\infty}\left(\int_0^x (-1)^n t^{2n}dt\right) = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{2n+1} + c. \end{aligned}$$

Since, $f(0) = c = \arctan(-1) = -\frac{\pi}{4}$. Therefore, the integer series expansion of f is given by

$$f(x) = -\frac{\pi}{4} + \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{2n+1}.$$

4 If $f(x) = \frac{1}{6z^2 - 5z + 1}$. Then, for all $z \in \mathbb{C}$, we have

$$f(x) = \frac{1}{6z^2 - 5z + 1} = \frac{1}{(3z-1)(2z-1)} = \frac{3}{1-3z} - \frac{2}{1-2z}.$$

So, for $|z| < \frac{1}{3}$, we have $\frac{1}{1-3z} = \sum_{n=0}^{+\infty} 3^n z^n$.
For $|z| < \frac{1}{2}$, we have $\frac{1}{1-2z} = \sum_{n=0}^{+\infty} 2^n z^n$.
As a result, the radius of convergenceve of the integer series expansion of f is $R = \inf\left\{\frac{1}{3}, \frac{1}{2}\right\} = \frac{1}{3}$. (The sum of two series with a different the radii of convergence).
So, for $|z| < \frac{1}{3}$, we find the expansion of f

$$f(x) = 3 \times \frac{1}{1-3z} - 2 \times \frac{1}{1-2z} = \sum_{n=0}^{+\infty}\left(3^{n+1} - 2^{n+1}\right)z^n.$$

5 $f(x) = e^x \sin x$, and $g(x) = e^x \cos x$.
For all $x \in \mathbb{R}$, we have

$$f(x) + ig(x) = e^x(\cos x + i \sin x) = e^{(i+1)x} = \sum_{n=0}^{+\infty} \frac{(i+1)^n}{n!} x^n.$$

According to Moiver's formula, we have

$$(i+1)^n = (\sqrt{2})^n \Big(\cos(\frac{\pi}{4}) + i \sin(\frac{\pi}{4}) \Big)^n = (\sqrt{2})^n \left(\cos\left(n\frac{\pi}{4}\right) + i \sin\left(n\frac{\pi}{4}\right) \right).$$

Hence the development of $f + ig$,

$$f(x) + ig(x) = e^{(i+1)x} = \sum_{n=0}^{+\infty} \frac{(i+1)^n}{n!} x^n = \sum_{n=0}^{+\infty} (\sqrt{2})^n \cos\left(n\frac{\pi}{4}\right) \frac{x^n}{n!} + i \sum_{n=0}^{+\infty} (\sqrt{2})^n \sin\left(n\frac{\pi}{4}\right) \frac{x^n}{n!}.$$

That means,

$$\begin{cases} f(x) = \sum_{n=0}^{+\infty} (\sqrt{2})^n \cos\left(n\frac{\pi}{4}\right) \frac{x^n}{n!} \\ g(x) = \sum_{n=0}^{+\infty} (\sqrt{2})^n \sin\left(n\frac{\pi}{4}\right) \frac{x^n}{n!} \end{cases}$$

Correction of exercise 5

We calculate the radius of convergence of integer series,

1 (a) For the series $\sum_{n\geq 0} \frac{n+2}{n+1} x^n$, $x \in \mathbb{R}$. We have $a_n = \frac{n+2}{n+1}$, so, according to d'Almbert's criterion, the radius is,

$$R = \lim_{n\to+\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to+\infty} \frac{a_n}{a_{n+1}} \frac{n+2}{n+1} \times \frac{n+2}{n+3} = \lim_{n\to+\infty} \frac{n^2}{n^2} = 1.$$

For all $x \in]-1, 1[$, and as for all $n \geq 0$: $\frac{n+2}{n+1} = 1 + \frac{1}{n+1}$, we have

$$\sum_{n=0}^{+\infty} \frac{n+2}{n+1} x^n = \sum_{n=0}^{+\infty} x^n + \frac{1}{x} \sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1} = \frac{1}{1-x} + \frac{\ln(1-x)}{x}.$$

(b) Consider the series $\sum_{n\geq 0} \frac{n^3}{n!} x^n$, $x \in \mathbb{R}$. We have $a_n = \frac{n^3}{n!}$, so, according to d'Almbert's criterion, the radius is,

$$R = \lim_{n\to+\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to+\infty} \frac{(n+1)!}{(n+1)^3} \times \frac{n^3}{n!} = \lim_{n\to+\infty} (n+1) = +\infty.$$

We know that, for any $n \geq 0$

$$\frac{n^3}{n!} = \frac{1}{(n-3)!} + \frac{3}{(n-2)!} + \frac{1}{(n-1)!}.$$

Consequently, for all $x \in \mathbb{R}$, we have,

$$\begin{aligned}\sum_{n=0}^{+\infty} \frac{n^3}{n!}x^n &= \sum_{n=3}^{+\infty} \frac{1}{(n-3)!}x^n + \sum_{n=2}^{+\infty} \frac{3}{(n-2)!}x^n + \sum_{n=1}^{+\infty} \frac{1}{(n-1)!}x^n \\ &= \sum_{n=0}^{+\infty} \frac{1}{n!}x^{n+3} + 3\sum_{n=0}^{+\infty} \frac{1}{n!}x^{n+2} + \sum_{n=0}^{+\infty} \frac{1}{n!}x^{n+1} = (x^3+3x^2+x)e^x.\end{aligned}$$

2 Let be the integer series $\sum\limits_{n\geq 1} \left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right)z^n$.

a We determine the radius of convergence of this series.

We have for all $n \in \mathbb{N}^*$: $1 \leq +\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n} \leq n$, and since the radii of convergence of the series $\sum\limits_{n\geq 1} z^n$ and $\sum\limits_{n\geq 1} nz^n$ are equal to 1 then the radius of convergence is of the series $\sum\limits_{n\geq 1} \left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right)z^n$ is $R = 1$.

b We calculate the sum $\sum\limits_{n\geq 1} \left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right)z^n$.

Note that for all $n \geq 1$ that

$$\sum_{n\geq 1} \left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right)z^n = \sum_{n\geq 1} a_n z^n \sum_{n\geq 1} b_n z^n = \sum_{n\geq 1} c_n z^n,$$

where $a_n = \frac{1}{n}$, $b_n = 1$ and $c_n = \sum\limits_{n=1}^{n} a_k b_{n-k} z^n$. This gives us the sum,

$$\sum_{n=1}^{+\infty} \left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right)z^n = \sum_{n=1}^{+\infty} \frac{1}{n}z^n \sum_{n=1}^{+\infty} z^n = \sum_{n\geq 1} c_n z^n = \ln(1-z) \times \frac{z}{1-z}.$$

Correction of exercise 6

Firstly, assume that $a_{2k+1} = 0$ for all $k \in \mathbb{N}$, S is even as the sum of a series of even functions. Conversely, suppose that S is even, and let $T(x) = S(-x)$.
Then, for all $x \in]-R, R[$, we have

$$T(x) = \sum_{n=0} (-1)^n a_n x^n.$$

Moreover, since S is even, T and S are identical on $]-R, R[$. So for any integer $n \in \mathbb{N}$, we have, $(-1)^n a_n = a_n$, thus $a_n = 0$ if n is odd.

Correction of exercise 7

Let be the functions serie $\sum\limits_{n\geq 0} f_n(x)$ such that $f_n(x) = \frac{1}{(4n)!}x^{4n}$

1 First, we determine the convergence radius of the series $\sum\limits_{n\geq 0} \frac{1}{(4n)!}x^n$.

So if we put $a_n = \dfrac{1}{(4n)!}$, then we have

$$R = \lim_{n\to+\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to+\infty} \frac{(4(n+1))!}{(4n)!} = \lim_{n\to+\infty} \frac{(4n+4)(4n+3)(4n+2)(4n+1)(4n)!}{(4n)!} = +\infty.$$

Hence, the radius of convergence of the series $sum_{n\geq 0}\dfrac{1}{(4n)!}x^n$ is $R = +\infty$.Thus, according to a previous result, the radius of convergence of the series $\sum\limits_{n\geq 0} \dfrac{1}{(4n)!}x^{4n}$ is $\sqrt[4]{R} = +\infty$.

2 We have the development for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$

$$\cos x = \sum_{n=0}^{+\infty} (-1)^n \frac{1}{(2n)!} x^{2n} \quad \text{and} \quad chx = \sum_{n=0}^{+\infty} \frac{1}{(2n)!} x^{2n}.$$

Then, the development of the function f is,

$$f(x) = \cos x + chx = \sum_{n=0}^{+\infty} \frac{(-1)^n + 1}{(2n)!} x^{2n}$$

3 Depending on the value of number n (even or odd), we have

$$\begin{aligned} f(x) &= \sum_{n=0}^{+\infty} \frac{(-1)^n + 1}{(2n)!} x^{2n} \\ &= \sum_{k=0}^{+\infty} \frac{(-1)^{2k} + 1}{(4k)!} x^{4k} + \sum_{k=0}^{+\infty} \frac{(-1)^{2k+1} + 1}{(2(2k+1))!} x^{4k+1} \\ &= 2\sum_{k=0}^{+\infty} \frac{1}{(4k)!} x^{4k}. \end{aligned}$$

Hence, the result,

$$\frac{1}{2}\frac{\cos x + chx}{2} = \frac{1}{2}\sum_{k=0}^{+\infty} \frac{(-1)^{2k}+1}{(4k)!} x^{4k} = \sum_{n\geq 0} f_n(x).$$

Correction of exercise 8

Let S be the sum of series $\sum\limits_{n\geq 0} \dfrac{x^{2n+2}}{(n+1)(2n+1)}$.

1 We pose, $f_n(x) = \dfrac{x^{2n+2}}{(n+1)(2n+1)}$, then, we have

$$\lim_{n\to+\infty} \left| \frac{f_{n+1}(x)}{f_n(x)} \right| = \lim_{n\to+\infty} \frac{|x|^{2n+4}}{(n+2)(2n+3)} \times \frac{(n+1)(2n+1)}{|x|^{2n+2}} = |x|^2.$$

2 **a** If $|x|^2 < 1$ that is, $|x| < 1$, the series $\sum\limits_{n=0}^{+\infty} f_n(x)$ absolutely converges,

b If $|x|^2 > 1$ i.e., $|x| > 1$, it diverges.

c For $|x| = 1$, the series $\sum_{n=0}^{+\infty} \frac{1}{(n+1)(2n+1)} \underset{n\to+\infty}{\sim} \frac{1}{n^2}$, converges, since $\sum_{n=0} \frac{1}{n^2}$ is a Reimann series that converges for $\alpha = 2 > 1$.We deduce that the radius of convergence of the series $\sum_{n=0}^{+\infty} \frac{x^{2n+2}}{(n+1)(2n+1)}$ is $R = 1$. So, $D = [-1, 1]$. Then, for all $x \in D$, we have

$$S(x) = \sum_{n=0}^{+\infty} \frac{x^{2n+2}}{(n+1)(2n+1)}.$$

3 Studying the continuity of S.} Since for all $n \in \mathbb{N}$, the functions $f_n(x) = \frac{x^{2n+2}}{(n+1)(2n+1)}$ are continuous on the domain $[-1, 1]$. Then, the series $\sum_{n=0}^{+\infty} \frac{x^{2n+2}}{(n+1)(2n+1)}$ is normally (i.e., uniformly) convergent on $[-1, 1]$, because

$$\sup_{x\in[-1,1]} \frac{x^{2n+2}}{(n+1)(2n+1)} = \frac{1}{(n+1)(2n+1)} \underset{n\to+\infty}{\sim} \frac{1}{2n^2}.$$

Therefore, according to the continuity theorem, S is continuous on $[1, 1]$.

4 For all $x \in]1, 1[$, we have

$$S'(x) = \sum_{n=0}^{+\infty} \frac{(2n+2)x^{2n+1}}{(n+1)(2n+1)} = 2\sum_{n=0}^{+\infty} \frac{x^{2n+1}}{2n+1} = 2\arctan(x) = \ln\left(\frac{1+x}{1-x}\right).$$

So by integrating by parts, we obtain,

$$S(x) = \int \ln\left(\frac{1+x}{1-x}\right)dx = x\ln\left(\frac{1+x}{1-x}\right) + \ln(1-x^2) + c.$$

Since, $S(0) = 0$, so $c = 0$. Hence,

$$S(x) = x\ln\left(\frac{1+x}{1-x}\right) + \ln(1-x^2).$$

Correction of exercise 9

Let be the differential equation (E) : $\quad 3xy' + (2-5x)y = x$.

Let $f(x) = \sum_{n=0}^{+\infty} a_n x^n$. Then, we have $f'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$. So, equation (**??**) is equivalent to

$$\begin{aligned}
\sum_{n=0}^{+\infty} 3na_n x^n + \sum_{n=0}^{+\infty} 2a_n x^n - \sum_{n=0}^{+\infty} 5a_n x^{n+1} &= x \\
\sum_{n=0}^{+\infty} 3na_n x^n + \sum_{n=0}^{+\infty} 2a_n x^n - \sum_{n=1}^{+\infty} 5a_{n-1} x^n &= x \\
2a_0 + \sum_{n=1}^{+\infty} (3na_n + 2na_n - 5a_n)x^n &= \\
5a_1 - 3a_0 + \sum_{n=2}^{+\infty} (3na_n + 2na_n - 5a_n)x^n &= x
\end{aligned}$$

Therefore, $a_0 = 0,\ a_1 = \frac{1}{5}$ et $a_n = \frac{5}{3n+2}a_{n-1}$ for all $n \geq 1$.
So, for all $n\mathbb{N}^*$, we have

$$a_n = \frac{5^{n-2}}{\prod\limits_{nk=2}^{k=n}(3k+2)} \text{ and } f(x) = \sum_{n=1}^{+\infty}\Big(5^{n-2}/\prod_{nk=2}^{k=n}(3k+2)\Big)x^n.$$

Correction of exercise 10

We propose to calculate the sum $S(x)$ of the series $\sum\limits_{n=0}^{+\infty} n^2x^n$.

1 For $x \neq 0$, we put $f_n(x) = n^2x^n$, then, for all $n > 1$, we find,

$$\left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \left|\frac{(n+1)^2x^{n+1}}{n^2x^n}\right| = \left(1+\frac{1}{n}\right)^2|x| \underset{n\longrightarrow+\infty}{\sim} |x|.$$

If $|x| < 1$, the series $\sum\limits_{n=0}^{+\infty} n^2x^n$ converges absolutely, and if $|x| > 1$ it roughly diverges.

We deduce that the radius of convergence of the series $\sum\limits_{n=0}^{+\infty} n^2x^n$ is $R = 1$.

Therefore, the domain of convergence $D =]-1,1[$, because for $|x| = 1$, the general term of the series does not tend to 0.

2 By the series derivation theorem, the three integer series

$$\sum_{n=0}^{+\infty} x^n, \quad \sum_{n=0}^{+\infty} nx^{n-1} \quad \text{and} \quad \sum_{n=0}^{+\infty} n(n-1)x^{n-2}$$

Have the same radius of convergence $R = 1$.
In addition, we know the sum of the geometric series

$$\forall x \in]-1,1[:\ \sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}.$$

Using the derivation theorem, we find the sums of the other two series (which are the first and second derivatives).

$$\forall x \in]-1,1[:\ \sum_{n=1}^{+\infty} nx^{n-1} = \frac{1}{(1-x)^2} \text{ and } \sum_{n=2}^{+\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}.$$

We can immediately deduce the relations

$$\forall x \in]-1,1[:\ \sum_{n=0}^{+\infty} nx^n = \frac{x}{(1-x)^2} \text{ and } \sum_{n=0}^{+\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}.$$

3 For all $xin]-1,1[$ and for all natural $n \geq 0$, we have,

$$\sum_{n=0}^{+\infty} n^2x^n = \sum_{n=0}^{+\infty}\Big(n(n-1)+n\Big)x^n = \sum_{n=0}^{+\infty} n(n-1)x^n + \sum_{n=0}^{+\infty} nx^n.$$

So, according to the question below, we have,

$$\sum_{n=0}^{+\infty} n^2 x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x+x^2}{(1-x)^3}.$$

Hence the results,

$$S(x) = \frac{x+x^2}{(1-x)^3}.$$

4 We note that $S\left(\frac{1}{2}\right) = \sum_{n=0}^{+\infty} \frac{n^2}{2^n}$. We deduce,

$$\sum_{n=0}^{+\infty} \frac{n^2}{2^n} = 6.$$

Correction of exercise 11

Integer series expansion of the real x, for the following functions

1 Consider the function $f_1(x) = (2+x)e^x$. Then, since f_1 is developable as an integer series in the neighbourhood of 0, because it is of class C^∞ so it's integer series expansion in the neighborhood 0 corresponds to it's Taylor expansion $\sum_{n=0}^{+\infty} \frac{f_1^{(n)}(0)}{n!} x^n$.

We must therefore start by calculating $f_1^{(n)}$ for all n. Next, we'll study on which interval f_1 is equal to it's Taylor expansion. Thus, we have

$$f_1'(x) = (2+x)e^x + e^x = f_1(x) + e^x, \text{and } f_1''(x) = f_1(x) + 2e^x.$$

By recurrence we obtain for all $n \geq 0$,

$$f_1^{(n)}(x) = f_1(x) + ne^x.$$

Thus $f_1^{(n)}(0) = f_1(0) + ne^0 = 2 + n$. So the Taylor expansion of f_1 is

$$f_1(x) = \sum_{n=0}^{+\infty} \frac{n+2}{n!} x^n.$$

D'Alembert's rule tells us that the radius of convergence of this series is $R = +\infty$. It remains to find on which interval of $\mathbb{R}$, f_1 it's development is equal to an integer series in the neighborhood of 0. To do this we can use Taylor-Cauchy's formula. We have

$$f_1(x) = \sum_{n=0}^{n} \frac{f_1^{(n)}(0)}{n!} x^n + \frac{f_1^{(n+1)}(c)}{(n+1)!} \quad \text{where c is between, 0 and x.}$$

Let's put

$$S(x) = \sum_{n=0}^{n} \frac{f_1^{(n)}(0)}{n!} x^n = \sum_{n=0}^{n} \frac{n+2}{n!} x^n.$$

We realize that

$$\left| s(x) - f_1(x) \right| = \left| \frac{2+c}{(n+1)!} x^{n+1} \right| \text{ où c comprise entre, 0 et x.}$$

Depending on the sign of x, one has,

$$\left|\frac{2+c}{(n+1)!}x^{n+1}\right| \leq \left|\frac{2+|x|}{(n+1)!}x^{n+1}\right| \underset{n\longrightarrow+\infty}{\longrightarrow} 0.$$

So, for all $x \in \mathbb{R}$, we have,

$$f_1(x) = \sum_{n=0}^{+\infty} \frac{n+2}{n!}x^n.$$

2 Let $f_2(x) = \ln(2+x)$ be the function.
Then we know that

$$\forall x \in]-1,1[: \quad \ln(1+x) = \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1}{n!}x^n.$$

Therefore, for all $|x| < 2$, we have,

$$\ln(2+x) = \ln 2 + \ln\left(1+\frac{x}{2}\right) = \ln 2 + \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1}{2^n n!}x^n.$$

3 For the function $f_3(x) = \dfrac{1}{(x-1)(x-2)}$, we have for all $x \neq 1$ and $x \neq 2$,

$$f_3(x) = \frac{1}{(x-1)(x-2)} = x\left(\frac{1}{1-x} - \frac{1}{2-x}\right).$$

Since the function $x \mapsto \dfrac{1}{1-x}$ has an integer series expansion for all $|x| < 1$ and

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n.$$

So, according to the above, for all $|x| < 2$, we have

$$\frac{1}{2-x} = \frac{1}{2} \times \frac{1}{1-\frac{x}{2}} = \frac{1}{2}\sum_{n=0}^{+\infty}\frac{x^n}{2^n}.$$

At the end, for all $x \in]-1,1[\cap]-2,2[=]-1,1[$, we have the development,

$$\begin{aligned} f_3(x) &= x\left(\frac{1}{1-x} - \frac{1}{2-x}\right) = x\left(\sum_{n=0}^{+\infty}x^n - \frac{1}{2}\sum_{n=0}^{+\infty}\frac{x^n}{2^n}\right) \\ &= \sum_{n=0}^{+\infty}\left(1-\frac{1}{2^{n+1}}\right)x^{n+1} = \sum_{n=1}^{+\infty}\left(1-\frac{1}{2^n}\right)x^n. \end{aligned}$$

Correction of supplementary exercise 1

Let $y = f(x) = \sum\limits_{n\geq 0} a_n x^n$ be the integer series expansion of a function f. It is the solution of equation (**??**). So,

$$y' = \sum_{n\geq 1} a_n n x^{n-1}, \text{ and } y'' = \sum_{n\geq 2} a_n n(n-1)x^{n-2}.$$

If we substitute y' and y'' into the equation (**??**), we get

$$\begin{aligned} x^2(1-x)\sum_{n\geq 2} a_n n(n-1)x^{n-2} - x(1-x)\sum_{n\geq 1} a_n n x^{n-1} + \sum_{n\geq 0} a_n x^n &= 0 \\ 2a_2x^2 + \sum_{n\geq 3}(n-1)[na_n - (n-2)a_{n-1}]x^n - a_1x - \sum_{n\geq 2}[na_n + (n+1)a_{n-1}]x^n + \sum_{n\geq 0} a_n x^n &= 0 \\ a_0 + (a_2 - a_1)x^2 + \sum_{n\geq 3}(n-1)^2[a_n - a_{n-1}]x^n &= 0. \end{aligned}$$

Consequently, we find,

$$\begin{cases} a_0 = 0 \\ a_2 - a_1 = 0 \\ (n-1)^2[a_n - a_{n-1}] = 0 \quad \forall n \geq 3. \end{cases} \Rightarrow \begin{cases} a_0 = 0 \\ a_2 = a_1 \\ (a_n = a_{n-1} \quad \forall n \geq 3. \end{cases}$$

On the other hand, by making x zero in the expression of f, we have $f(0) = a_0 = 0$. So the series $\sum\limits_{n\geq 1} x^n$ has the radius of convergence $R = 1$. Therefore, any solution of (**??**), developable as an integer series, so, the solution f is

$$f(x) = \sum_{n\geq 1} x^n = -1 + \sum_{n\geq 0} x^n = -1 + \frac{1}{1-x} = \frac{x}{1-x}.$$

Correction of supplementary exercise 2

Suppose the integer series $\sum\limits_{n\geq 0} a_n x^n$ has a radius of convergence equal to R.
Then, for any $|x| < R$ the series converges, and diverges if $|x| > R$.

1 According to the hypothesis, the series $\sum\limits_{n\geq 0} a_n x^{2n}$ converges if and only if $|x^2| < R$, i.e., $|x| < \sqrt{R}$.
Therefore, the radius of convergence of the series $\sum\limits_{n\geq 0} a_n x^{2n}$ is $\sqrt{R}$.

2

- **a** We know that the radius of convergence of the series $\sum\limits_{n\geq 0} \frac{\ln n}{n^2} x^n$ is $R = 1$, so according to the previous result, the radius of convergence of $\sum\limits_{n\geq 0} \frac{\ln n}{n^2} x^{2n}$ is equal to $\sqrt{R} = 1$.
- **b** By D'Alembert's criterion, the radius of convergence of series $\sum\limits_{n\geq 0} \frac{n^2+n}{3^n} x^n$ is $R = 3$..
From this we deduce the radius of convergence of $\sum\limits_{n\geq 0} \frac{n^2+n}{3^n} x^{5n}$ is $\sqrt[5]{R} = \sqrt[5]{3}$.
- **c** We can calculate the radius of the series $\sum\limits_{n\geq 0} ch(n)x^n$, so we find, $R = \frac{1}{e}$. Therefore, the radius of convergence of the series $\sum\limits_{n\geq 0} ch(n)x^{3n}$ equals $\sqrt[3]{R} = \frac{1}{\sqrt[3]{e}}$.

Correction of supplementary exercise 3

We have for all $|x| < 1$, $\sum\limits_{n=1}^{+\infty} x^n = \frac{1}{1-x}$. So, we obtain

$$\sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1} = \int_0^x \frac{dt}{t} = -\ln(1-x) = \sum_{n=1}^{+\infty} \frac{x^n}{n}.$$

Consequently,

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{x^{n+1}}{n(n+1)} &= -\int_0^x \ln(1-t)dt = \Big[(1-t)\ln(1-t) - t\Big]_0^x \\
&= x - (1-x)\ln(1-x).
\end{aligned}$$

If $x = -1$, we get, $\sum\limits_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n(n+1)} = -1 + 2\ln 2$, therefore, we deduce that,

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n(n+1)} = -\sum_{n=1}^{+\infty} \frac{(-1)^n}{n(n+1)} = 1 - 2\ln 2.$$