M'sila University		Faculty of Mathematics and Computer Science
L 2 Mathematics	**Series n° : 2** **Correction of functions sequences and series**	Module: Analyse 3

Remark 1

The exercise marked with $(\star)$ or additional will not be corrected in the session of **TD**.

Correction of exercise 1

1 Study of $f_n(x) = \dfrac{2n|x|}{1+n^2x^2}$.

a The sequence $(f_n)_{n\geq 0}$ defined on $\mathbb{R}$, for all $n \in \mathbb{N}$. Therefore, we study the simple convergence on $\mathbb{R}$.
For $x = 0$, we have $f_n(0) = 0$, so, $\lim\limits_{n\to+\infty} f_n(0) = 0$.
For $x \neq 0$, we have $\lim\limits_{n\to+\infty} f_n(x) = \lim\limits_{n\to+\infty} \dfrac{2|x|}{nx^2} = 0$.
So the sequence $(f_n)_{n\geq 0}$ simply converges to the zero function f on $\mathbb{R}$, defined by $f(x) = 0$.

b Uniform convergence on $\mathbb{R}$.
For $x = \dfrac{1}{n}$, we have, $\lim\limits_{n\to+\infty} f_n(x) = \lim\limits_{n\to+\infty} \dfrac{2n|x|}{1+n^2x^2} = \lim\limits_{n\to+\infty} \dfrac{2}{1+1} = 1 \neq 0$. Then, the convergence is not uniform on $\mathbb{R}$.
We study uniform convergence on intervals of the form $I =]-\infty, a] \cup [a, +\infty[$ for $a > \dfrac{1}{n}$. The function f_n is even, so we need only study it on $[a, +\infty[$. For all $x > 0$, we have,

$$f_n'(x) = 2n\frac{1-n^2x^2}{(1+n^2x^2)^2},$$

and the table of variations is given by

x	0		$\frac{1}{n}$	a		$+\infty$
$f_n'(x)$		$+$	0		$-$	
$f_n(x)$	0	$\nearrow$	1		$\searrow$	0

$$\|f_n\|_\infty = \sup_{x\in I} |f_n(x)| = f_n(a) = \frac{2na}{1+n^2a^2},$$

Therefore, $\lim\limits_{n\to+\infty} \|f_n\|_\infty = 0$, which shows the uniform convergence of the sequence $(f_n)_{n\geq 0}$ on I.

2 Study of $f_n(x) = \dfrac{1}{1+nx}$.

a We study the simple convergence of $(f_n)_{n\geq 0}$ on $[0,1]$.
For all $n\in\mathbb{N}$, we have $f_n(0)=1$, therefore, $\lim_{n\to+\infty} f_n(0)=1$.

Pour $x\in]0,1]$, we have $\lim_{n\to+\infty} f_n(x)=\lim_{n\to+\infty}\dfrac{1}{1+nx}=0$.
So the sequence $(f_n)_{n\geq 0}$ simply converges to the function f defined on $[0,1]$ by

$$f(x)=\begin{cases} 1 & : \ x=0 \\ 0 & : \ x\in]0,1]. \end{cases}$$

b Uniform convergence on $[0,1]$.
Since for all $n\in\mathbb{N}$, the functions $f_n(x)$ are continuous on $[0,1]$ and the limit function f is discontinuous on $[0,1]$, then convergence is not uniform on $[0,1]$.

3 Study the converging of $f_n(x)=n(e^{\frac{x}{n}}-1)$ on $I=\mathbb{R}$.

a The simple convergence of $(f_n)_{n\geq 0}$ on $I=\mathbb{R}$.

For all $n\in\mathbb{N}$, we have, $\lim_{n\to+\infty} f_n(x)=\lim_{n\to+\infty} n(e^{\frac{x}{n}}-1)=\begin{cases} 0 & : \ x=0 \\ x & : \ x\neq 0. \end{cases}$
So the sequence $(f_n)_{n\geq 0}$ simply converges to the function f defined on $\mathbb{R}$ by

$$f(x)=\begin{cases} 0 & : \ x=0 \\ x & : \ x\neq 0. \end{cases}$$

b Uniform convergence of $(f_n)_{n\geq 0}$ on $\mathbb{R}$.
For $x=n$, we have $\lim_{n\to+\infty}|f_n(x_n)-f(x_n)|=\lim_{n\to+\infty} n(e-1)=+\infty\neq 0$. Therefore, convergence is not uniform on $\mathbb{R}$.

4 If $f_n(x)=\dfrac{x^n}{1+ax^n}$. Then we have,

a Simple convergence of $(f_n)_{n\geq 0}$ on $I=\mathbb{R}$.

For all $x\in\mathbb{R}$, we get, $\lim_{n\to+\infty} f_n(x)=\begin{cases} \dfrac{1}{a} & : \ |x|>1 \\ 0 & : \ |x|<1 \\ \dfrac{1}{1+a} & : \ x=1 \\ \text{no limit} & : \ x=-1. \end{cases}$
So the sequence $(f_n)_{n\geq 0}$ simply converges to the function f defined on $]-1,+\infty[$ by

$$f(x)=\begin{cases} \dfrac{1}{a} & : \ x>1 \\ 0 & : \ -1<x<1 \\ \dfrac{1}{1+a} & : \ x=1 \end{cases}$$

b Uniform convergence on $]1,+\infty[$. Since the functions f_n, for $n\geq 0$, are continuous on $]1,+\infty[$ and the simple limit function is discontinuous on $]-1,+\infty[$ then convergence is not uniform on $]1,+\infty[$.

5 Study the simple and uniform convergence of $f_n(x)=x^n(1-x),\ n\in\mathbb{N}$, on $I=[0,1]$.

a Simple convergence on $[0,1]$.
For $x=1$ and all $n\in\mathbb{N}$, we have $f_n(1)=0$.
for $x\in[0,1[$ we have, $\lim_{n\to+\infty} f_n(x)=\lim_{n\to+\infty}(x^n-x^{n+1})=0$. So the functions sequence $(f_n)_{n\geq 1}$ simply converges on $[0,1]$ to the null function.

b Uniform convergence on $[0,1]$.
For all $n \in \mathbb{N}$, the function f_n is differentiable on $]0,1[$ and we have,

$$f_n'(x) = x^{n-1}(n-(n+1)x).$$

x	0		$\frac{1}{n+1}$		1
$f_n'(x)$		$+$	0	$-$	
$f_n(x)$	0	$\nearrow$	$f\left(\frac{1}{n+1}\right)$	$\searrow$	0

From the previous table of variations we can deduce that,

$$\lim_{n\to+\infty} \sup_{x\in[0,1]} |f_n(x)-f(x)| \leq \lim_{n\to+\infty} f\left(\frac{1}{n+1}\right) = \lim_{n\to+\infty} \left(\frac{1}{(n+1)^n}\left(1-\frac{1}{(n+1)^n}\right)\right) = 0.$$

Hence the uniform convergence of the sequence $(f_n)_{n\geq 1}$ on $[0,1]$ to the zero function.

6 Let $f_n(x) = nx^2e^{-nx}$, $I = [0,+\infty[$.

a Simple convergence on $[0,+\infty[$.
Simple convergence on $n \in \mathbb{N}$, $f_n(0) = 0$.
For $x > 0$, and we have $\lim\limits_{n\to+\infty} nx^2e^{-nx} = 0$. So the functions sequence $(f_n)_{n\geq 0}$ simply converges on $[0,+\infty[$ to the null function $f \equiv 0$.

b uniform Converges on $[0,+\infty[$.
For all $n \in \mathbb{N}$, the function f_n is differentiable on $]0,+\infty[$ and we have

$$f_n'(x) = (2-nx)ne^{-nx}.$$

x	0		$\frac{2}{n}$		$+\infty$
$f_n'(x)$		$+$	0	$-$	
$f_n(x)$	0	$\nearrow$	$\frac{4e^{-2}}{n}$	$\searrow$	0

From the previous table of variations we can deduce that,

$$\lim_{n\to+\infty} \sup_{x\geq 0} |f_n(x)-f(x)| \leq \lim_{n\to+\infty} f\left(\frac{2}{n}\right) = \lim_{n\to+\infty} \frac{4e^{-2}}{n} = 0.$$

Hence the uniform convergence of the sequence $(f_n)_{n\geq 1}$ on $[0,+\infty[$ towards the zero function.

Correction of exercise 2

1 Study the uniform convergence of the sequence $\left(\dfrac{x}{1+n^2x^2}\right)_{n\geq 1}$ on $\mathbb{R}$.
If $x=0$, and for all $n\in\mathbb{N}$, we have $f_n(0)=0$. Therefore, $\lim\limits_{n\to+\infty} f_n(0)=0$.
If $x\in]0,1]$, we have

$$\lim_{n\to+\infty} f_n(x)=\lim_{n\to+\infty}\frac{x}{n^2x^2}=\lim_{n\to+\infty}\frac{1}{n^2x}=0.$$

Then $(f_n)_{n\geq 0}$ simply converges on $[0,1]$ to the null function.
For all $n\in\mathbb{N}$, the function f_n is differentiable on $[0,1]$ and we have,

$$f_n'(x)=\frac{1-n^2x^2}{(1+n^2x^2)^2}.$$

Therefore, it's table of variations on $[0,1]$ is given by,

x	0		$\frac{1}{n}$		1
$f_n'(x)$		$+$	0	$-$	
$f_n(x)$	0	$\nearrow$	$\frac{1}{2n}$	$\searrow$	$\frac{1}{n^2+1}$

As a result,

$$\sup_{x\in[0,1]}|f_n(x)-f(x)|=f_n\left(\frac{1}{n}\right)=\frac{1}{2n}\underset{n\longrightarrow+\infty}{\longrightarrow}0$$

$(f_n)_{n\geq 1}$ converges uniformly on $[0,1]$ to a function f which is differentiable with $f'=0$.

2 Let's study the uniform convergence of the sequence of derivative functions $f_n'(x)=\dfrac{1-n^2x^2}{(1+n^2x^2)^2}$.
If $x=0$, and for all $n\in\mathbb{N}^*$, we have $f_n'(0)=1$. Therefore, $\lim\limits_{n\to+\infty} f_n'(0)=1$.
If $x\in]0,1]$, we have,

$$\lim_{n\to+\infty} f_n'(x)=\lim_{n\to+\infty}\frac{-n^2x^2}{n^4x^4}=\lim_{n\to+\infty}\frac{-1}{n^4x^4}=0.$$

So the sequence $(f_n')_{n\geq 1}$ simply converges on $[0,1]$ to the function h defined by.

$$h(x)=\begin{cases}1 & : x=0\\ 0 & : x\in]0,1].\end{cases}$$

It is therefore obvious that $f'\neq h$

Correction of exercise 3

We deduce from this exercise is therefore to highlight the importance of the hypotheses in the theorem of differentiability of a functions sequence, that is to say, we do not generally have,

$$\lim_{n\to+\infty}\frac{d}{dx}\Big(f_n(x)\Big) \neq \frac{d}{dx}\Big(\lim_{n\to+\infty} f_n(x)\Big).$$

Correction of exercise 4

Study of the simple and uniform convergence of the sequence $(f_n)_{n\geq 1}$ on $[0,1]$, we have,

1 **a** Simple convergence on $[0,1]$. For all $x\in[0,1]$, we find,

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty}\frac{ne^x + xe^{-x}}{x+n} = \lim_{n\to+\infty}\frac{ne^x}{n} = e^x.$$

So the sequence $(f_n)_{n\geq 1}$, converges simply on $[0,1]$ to a function f defined by,

$$f(x) = e^x,\ x\in[0,1].$$

b The uniform convergence of $(f_n)_{n\geq 1}$ on $[0,1]$. Then, for all $x\in[0,1]$, we have,

$$\lim_{n\to+\infty}\sup_{x\in[0,1]}|f_n(x)-f(x)| = \lim_{n\to+\infty}\sup_{x\in[0,1]}\left|\frac{xe^{-x}-xe^x}{n+x}\right| \leq \lim_{n\to+\infty}\sup_{x\in[0,1]}\frac{1+e}{n} = 0.$$

This shows that convergence is uniform on $[0,1]$.

2 Let's calculate the limit $\lim_{n\to+\infty}\int_0^1 (x^3+1)f_n(x)dx$. Since convergence is uniform on $[0,1]$, then,

$$\lim_{n\to+\infty}\int_0^1 (x^3+1)f_n(x)dx = \int_0^1 (x^3+1)\lim_{n\to+\infty} f_n(x)dx = \int_0^1 (x^3+1)e^x dx.$$

By integration by parts we obtain,

$$\int_0^1 (x^3+1)\lim_{n\to+\infty} f_n(x)dx = \int_0^1 (x^3+1)e^x dx = 5-e.$$

Correction of exercise 5

1 Let us show that $(f_n)_{n\geq 1}$ converges uniformly to f on $[0,+\infty[$.

2 **a** Simple convergence of $(f_n)_{n\geq 0}$. Let $x\in[0,+\infty[$. Then, there exists $n_0\in\mathbb{N}$ such that for all $n>n_0>x$, we have $f_n(x)=\frac{1}{n}$, consequently, we find,

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty}\frac{1}{n} = 0.$$

(b) Study of uniform convergence of $(f_n)_{n\geq 1}$. For any $x \in [0, +\infty[$ we have,

$$\lim_{n\to+\infty} \sup_{x\in[0,+\infty[} |f_n(x) - f(x)| = \lim_{n\to+\infty} \sup_{x\in[0,+\infty[} \frac{1}{n} = \lim_{n\to+\infty} \frac{1}{n} = 0.$$

Therefore, the sequence $(f_n)_{n\geq 1}$ converges uniformly on $[0, +\infty[$.

(3) We have,

$$\lim_{n\to+\infty} \int_0^{+\infty} f_n(x)dx = \lim_{n\to+\infty} \int_0^{n} \frac{1}{n}dx = \lim_{n\to+\infty} \frac{1}{n}\Big[x\Big]_0^n = \lim_{n\to+\infty} 1 = 1.$$

But we have

$$\int_0^{+\infty} \lim_{n\to+\infty} f_n(x)dx = \lim_{a\to+\infty} \int_0^{a} \lim_{n\to+\infty} f_n(x)dx = \lim_{a\to+\infty} \int_0^{a} 0.dx = 0.$$

We note that

$$\lim_{n\to+\infty} \int_0^{+\infty} f_n(x)dx \neq \int_0^{+\infty} \lim_{n\to+\infty} f_n(x)dx.$$

This does not allow us to use the integration theorem to invert the limit and the integral on an unbounded interval.

Correction of exercise 6

(1) If $x = 0$, $f_n(0) = \dfrac{(-1)^n}{n+1}$. This is an alternating series, so the sequence $\left(\dfrac{1}{n+1}\right)_{n\geq 0}$ tends to 0 and decreasing, so the numerical series of general term $\dfrac{(-1)^n}{n+1}$ converges.
If $x > 0$,

$$\lim_{n\to+\infty} n^2|f_n(x)| = \lim_{n\to+\infty} \frac{n^2e^{-nx}}{n+1} = 0.$$

The numerical series of the general term $f_n(x)$ converges absolutely, so it converges, in other words the functions series of the general term simply converges on $[0, +\infty[$.

(2) If $x > a > 0$, we have for all $n \in \mathbb{N}$,

$$f_n(x) = \frac{e^{-nx}}{n+1} \leq \frac{e^{-na}}{n+1} = \alpha_n.$$

As a result, we find

$$\lim_{n\to+\infty} n^2\alpha_n = \lim_{n\to+\infty} \frac{n^2e^{-na}}{n+1} = 0.$$

Using Riemann's rules with $\alpha > 1$, this shows that the numerical series of general term α_n converges, so the function series of general term f_n converges normally on $[a, +\infty[$.

(3) (a) Pour $x \in [a, +\infty[$, we find,

$$\sup_{x\geq 0} |f_n(x)| = \sup_{x\geq 0} \frac{e^{-nx}}{n+1} = \frac{1}{n+1}.$$

Or $\dfrac{1}{n+1} \sim \dfrac{1}{n}$ is the general term of a divergent Riemann series with $\alpha = 1 \leq 1$, so the functions series of general term f_n does not converge normally on $[0, +\infty[$.

b Si $x > 0$, we'll have

$$\sup_{x>0} |f_n(x)| = \sup_{x>0} \frac{e^{-nx}}{n+1} = \frac{1}{n+1}.$$

It makes no difference if you open the interval.

4 We will use the rest increase of the theorem of alternating sequences

$$|R_n(x)| \leq |f_{n+1}(x)| = \frac{e^{-(n+1)x}}{n+2} \leq \frac{1}{n+1}.$$

So,

$$\sup_{x\geq 0} \left| \sum_{n=0}^{+\infty} f_n(x) - \sum_{n=0}^{N} f_n(x) \right| = \sup_{x\geq 0} \left| R_N(x) \right| \leq \frac{1}{N+1} \underset{n\longrightarrow+\infty}{\longrightarrow} 0.$$

The functions series of general term converges uniformly on $[0, +\infty[$.

Correction of exercise 7

1 This is an alternating series, for any $x > 0$, $|f_n(x)| = \dfrac{1}{n+x}$ tends towards 0 and decreasing, so according to the alternating series theorem, the numerical series with general term $f_n(x)$ converges, in other words, the function series with general term $f_n(x)$ simply converges.

2 For all $x > 0$, we have $f_n'(x) = \dfrac{(-1)^{n+1}}{(n+x)^2}$, so,

$$|f_n'(x)| = \frac{1}{(n+x)^2} \leq \frac{1}{n^2}.$$

$\dfrac{1}{n^2}$ is the general term of a convergent Riemann series with $\alpha = 2 > 1$ so the functions serie of general term f_n' converges normally on $]0, +\infty[$, so uniformly on $]0, +\infty[$, since the functions f_n are derivable, the series derivation theorem implies that the sum is derivable and that

$$f_n'(x) = \sum_{n=0}^{+\infty} f_n'(x) = \sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{(n+x)^2}.$$

Correction of supplementary exercise 1

1 Let's put $f_n(x) = \dfrac{(-1)^n}{n + \ln(1+nx)} = a_n(x)b_n(x)$, such that $a_n = (-1)^n$ and $b_n = \dfrac{1}{n + \ln(1+nx)}$. For x fixed, $(b_n(x))_{n\geq 1}$ is positive, decreasing and $\lim\limits_{n\to+\infty} \sup\limits_{x\geq 0} |b_n(x)| = \lim\limits_{n\to+\infty} \dfrac{1}{n} = 0$.

Moreover the partial sum of $(a_n)_{n\geq 1}$ is bounded. In fact

$$|S_n| = \left|\sum_{k=1}^{k=n}(-1)^n\right| \leq 1.$$

Consequently, according to Leibniz's theorem (alternating series theorem), the functions series $\sum_{n=1}^{+\infty}\frac{(-1)^n}{n+\ln(1+nx)}$ converge uniformément sur $]0,+\infty[$ and since the function f_n is continuous on $]0,+\infty[$ for all $n\geq 1$. Then, the continuity theorem guarantees the continuity of the function f on $]0,+\infty[$.

2 Let's calculate the limit $\lim_{x\to+\infty} f(x)$.

Since the series $\sum_{n=1}^{+\infty} f_n(x)$ converges uniformly on $]0,+\infty[$ and f is continuous, then

$$\begin{aligned}\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty}\sum_{1}^{+\infty} f_n(x) = \sum_{1}^{+\infty}\lim_{x\to+\infty} f_n(x)\\ &= \sum_{1}^{+\infty}\lim_{x\to+\infty}\frac{(-1)^n}{n+\ln(1+nx)} = 0.\end{aligned}$$

3 Let's show that f is of class C^1 on $]0,+\infty[$.
We have $f_n\in C^1(]0,+\infty[)$ for all $n\geq 1$. Let's put $f_n'(x)=\alpha_n(x)\beta_n(x)$ with $\alpha_n(x)=(-1)^{n+1}$ and $\beta_n(x)=\frac{n}{(1+nx)(n+\ln(1+nx))^2}$. For x fixed, $(\beta_n(x))_{n\geq 1}$ is positive, decreasing and

$$\lim_{n\to+\infty}\sup_{x\geq 0}|\beta_n(x)| = \lim_{n\to+\infty}\frac{1}{(1+nx)(n+\ln(1+nx))^2} = 0.$$

Moreover, the partial sum of $(\alpha_n)_{n\geq 1}$ is bounded. Because,

$$|S_n| = \left|\sum_{k=1}^{k=n}(-1)^{n+1}\right| \leq 1.$$

Therefore, according to Leibniz's theorem (alternating series theorem), the functions series $\sum_{n=1}^{+\infty} f_n'(x)$ converges uniformly on $]0,+\infty[$ and since the series $\sum_{n=1}^{+\infty} f_n(x)$ is continuous on $]0,+\infty[$ for all $n\geq 1$. Then, the derivative theorem is given the result $f\in C^1(]0,+\infty[)$, and we obtain,

$$\forall x\in]0,+\infty[:\quad f_n'(x) = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}n}{(1+nx)(n+\ln(1+nx))^2}.$$

Correction of supplementary exercise 2

Let $(f_n)_{n\geq 0}$, be the functions sequence defined by $f_n(x)=\sqrt{1-x^2}+e^{-nx^2}\sin(nx)$.

1 For all $n\in\mathbb{N}$ the function f_n is defined for $x\in[-1,1]$, so the domain definition of f_n is $D_{f_n}=[-1,1]$. For all $x\in[-1,1]$, as $\lim_{n\to+\infty} e^{-nx^2}=0$ and the function $n\mapsto\sin(nx)$

bounded, then, we have

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \left(\sqrt{1-x^2} + e^{-nx^2}\sin(nx)\right) = \sqrt{1-x^2}.$$

Consequently, the sequence $(f_n)_{n\geq 0}$ simply converges on $[-1,1]$ to a function f defined by,

$$\forall x \in [-1,1] : \ f(x) = \sqrt{1-x^2}.$$

2 Let's show that, for any $a > 0$, the sequence $(f_n)_{n\geq 0}$ converges uniformly to f on $[a,1]$. Then, for all $x \in [a,1]$, and as $|\sin(nx)| \leq 1$, we have

$$\lim_{n\to+\infty} \sup_{x\in[a,1]} |f_n(x) - f(x)| = \lim_{n\to+\infty} \sup_{x\in[a,1]} |e^{-nx^2}\sin(nx)| \leq \lim_{n\to+\infty} \sup_{x\in[0a,1]} e^{-ax^2} = 0.$$

This shows that convergence is uniform on $[a,1]$.

3 Let's show that the sequence $(f_n)_{n\geq 0}$ does not converge uniformly to f on $[0,1]$.
Pour $x_n = \dfrac{1}{n}$, then,

$$\lim_{n\to+\infty} |f_n(x_n) - f(x_n)| = \lim_{n\to+\infty} |e^{-\frac{1}{n}}\sin(1)| = \sin(1) \neq 0.$$

Therefore, convergence is not uniform on $[0,1]$.