M'sila University	**Series n° : 1** **Numerical series**	Faculty of Mathematics and Computer Science
L 2 Mathematics		Module: Analyse 3

Remark 1

The exercise marked with $(\star)$ or additional will not be corrected in the session of **TD**.

Correction of exercise 1

1 The sequence with general term $\left(\frac{2}{3}\right)^n$ is a geometric sequence with reason $\frac{2}{3}$, then,

$$S_n = \sum_{k=0}^{n}\left(\frac{2}{3}\right)^k = 3\left(1-\left(\frac{2}{3}\right)^{n+1}\right),$$

therefore,

$$\lim_{n\to+\infty} S_n = 3,$$

so the series $\sum_{n\geq 0}\left(\frac{2}{3}\right)^n$ is convergent and it's sum is

$$S = \sum_{n=0}^{+\infty}\left(\frac{2}{3}\right)^n = \lim_{n\to+\infty} S_n = 3.$$

2 For all $n\in\mathbb{N}^*$, we have

$$\frac{1}{4n^2-1} = \frac{1}{2}\times\frac{1}{2n-1} - \frac{1}{2}\times\frac{1}{2n+1},$$

by consequently,

$$\begin{aligned} S_n &= \sum_{k\geq 1}\frac{1}{4k^2-1} = \frac{1}{2}\sum_{k=1}^{k=n}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right) = \frac{1}{2}\left[\sum_{k=0}^{k=n-1}\frac{1}{2k+1} - \sum_{k=1}^{k=n}\frac{1}{2k+1}\right] \\ &= \frac{1}{2}\left(1-\frac{1}{2n+1}\right), \end{aligned}$$

so we find it,

$$\lim_{n\to+\infty} S_n = \frac{1}{2}.$$

Then the series $\sum_{n\geq 1}\frac{1}{4n^2-1}$ is convergent.

3 For all $n\in\mathbb{N}^*$, we have

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)},$$

so we get,

$$S_n = \sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)} = \sum_{k=1}^{k=n}\left(\frac{1}{2k}-\frac{1}{k+1}+\frac{1}{2(k+2)}\right).$$

And yet,

$$\begin{aligned}\sum_{k=1}^{k=n}\frac{1}{2k}-\sum_{k=1}^{n}\frac{1}{2(k+2)} &= \frac{1}{2}\Big[\sum_{k=1}^{k=n}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{k+2}\Big]\\ &= \frac{1}{2}+\frac{1}{4}+\sum_{k=3}^{k=n}\frac{1}{k}+\frac{1}{2(n+1)}+\frac{1}{2(n+2)},\end{aligned}$$

and

$$\sum_{k=1}^{k=n}\frac{1}{k+1}=\sum_{k=2}^{k=n+1}\frac{1}{k}=\frac{1}{2}+\sum_{k=3}^{k=n}\frac{1}{k}+\frac{1}{n+1}.$$

As a result, we obtain

$$S_n=\frac{1}{2}\Big[\sum_{k=1}^{k=n}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{k+2}\Big]-\sum_{k=1}^{k=n}\frac{1}{k+2}=\frac{1}{4}-\frac{1}{2(n+1)}+\frac{1}{2(n+2)}.$$

Hence

$$\lim_{n\to+\infty}S_n=\frac{1}{4}.$$

Then the series $\sum_{n\geq 1}\frac{1}{n(n+1)(n+2)}$ is convergent.

4 For all $n\in\mathbb{N}$, we have

$$\frac{1}{\sqrt{n+2}+\sqrt{n+1}}=\sqrt{n+2}-\sqrt{n+1},$$

Consequently, we obtain,

$$S_n=\sum_{k=0}^{n}\frac{1}{\sqrt{k+2}+\sqrt{k+1}}=\sum_{k=0}^{n}\left(\sqrt{k+2}-\sqrt{k+1}\right)=\sqrt{n+2}-1.$$

Then, it follows that,

$$\lim_{n\to+\infty}S_n=+\infty,$$

so the series $\sum_{n\geq 0}\frac{1}{\sqrt{n+2}+\sqrt{n+1}}$ is divergent.

5 For any $n\geq 3$, we have

$$\ln\left(1-\frac{2}{n}\right)=\ln(n-2)-\ln(n),$$

then,

$$\begin{aligned}S_n &= \sum_{k=3}^{n}\ln\left(1-\frac{2}{n}\right)=\sum_{k=3}^{n}\Big(\ln(k-2)-\ln(k)\Big)=\sum_{k=1}^{n-2}\ln(k)-\sum_{k=3}^{n}\ln(k)\\ &= \ln 2-\ln(n-1)-\ln n=\ln 2-\ln\left(\frac{n-1}{n}\right),\end{aligned}$$

hence,

$$\lim_{n\to+\infty}S_n=\ln 2,$$

so the series $\sum_{n\geq 3}\ln\left(1-\frac{2}{n}\right)$ is divergent.

6 Knowing that, for all $\alpha,\ \beta \in]-\frac{\pi}{2},\frac{\pi}{2}[$, we have

$$\tan(\alpha-\beta)=\frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}.$$

Let's put,

$$\alpha=\arctan a \quad \text{and} \quad \beta=\arctan b,$$

where $a,\ b\in\mathbb{R}$, so we get

$$\arctan a-\arctan b=\arctan\left(\frac{a-b}{1+ab}\right).$$

Then, for $a=k+1$ and $b=k$ we deduce that

$$\arctan(k+1)-\arctan(k)=\arctan\left(\frac{1}{1+k(k+1)}\right)=v_{n+1}-v_n,$$

where $v_k=\arctan(k)$ is the general term of the sequence $(v_k)_{k\geq 1}$. Therefore, for all $k\in\mathbb{N}^*$, we find,

$$S_n=\sum_{k=1}^{k=n}\arctan\left(\frac{1}{k^2+k+1}\right)=\arctan(n+1)-\arctan(1)=\arctan(n+1)-\frac{1}{4}.$$

Passing to the limit, we obtain,

$$\lim_{n\to+\infty}S_n=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}.$$

Hence, the series $\sum\limits_{n\geq 1}\arctan\left(\frac{1}{n^2+n+1}\right)$ is convergent and it's sum is $\frac{\pi}{4}$.

Correction of exercise 2

It suffices to show that the sequence of partial sums associated with $(S_n)_{n\geq 1}$ is not Cauchy, i.e.,

$$\exists\varepsilon>0,\ \forall n_0,\ \exists p,q\in\mathbb{N}:\ p>q\geq n_0 \text{ on a } |S_p-S_q|>\varepsilon.$$

In fact, $S_n=\sum\limits_{k=1}^{n}\frac{1}{k}$, then, for $p=2n$ and $q=n$, we have

$$\begin{aligned}|S_p-S_q| &= |S_{2n}-S_n|\\ &= \left|\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\right|\\ &\geq \frac{1}{2n}+\frac{1}{2n}+\ldots\frac{1}{2n}=\frac{2}{2n}=\frac{1}{2}.\end{aligned}$$

It suffices to choose $\varepsilon=\frac{1}{2}$, so that $(S_n)_{n\geq 1}$ is not Cauchy sequence.

Correction of exercise 3

1. This is a divergent Riemann series with $\alpha = 1 \leq 1$.

2. We have $\forall n \in \mathbb{N} : \dfrac{1}{3n+1} \sim \dfrac{1}{3n}$. Then, it is a divergent Riemann series with $\alpha = \dfrac{1}{2} \leq 1$.

3. Since, $\lim\limits_{n\to+\infty} \dfrac{n^2+2}{n^2} = 1 \neq 0$. Then, the necessary condition is not satisfied, so the series does not converge.

4. We have $\forall n \in \mathbb{N}^* : \dfrac{3}{\sqrt{n}} = \dfrac{3}{n^{\frac{1}{2}}}$. This is the general term of a divergent Riemann series with $\alpha = 1 \leq 1$.

5. For all $n \in \mathbb{N}$, we have
$$\frac{(4n+1)^3}{(5n^2+1)^3} \sim \frac{4^3}{5^3} \times \frac{1}{n^3}.$$
This is the general term of a convergent Riemann series with $\alpha = 3 > 1$.

6. For all $n \in \mathbb{N}^*$, we have
$$\left(1-\frac{1}{n}\right)^n \sim e^{n\ln(1-\frac{1}{n})} = e^{n(-\frac{1}{n}+o(\frac{1}{n}))} = e^{-1+o(1)} \underset{n\longrightarrow+\infty}{\longrightarrow} e^{-1} \neq 0.$$
So the series diverges.

7. For all $n \in \mathbb{N}^*$, we have
$$ne^{\frac{1}{n}} - n = n\left(e^{\frac{1}{n}}-1\right) \sim n\left(1+\frac{1}{n}+o\left(\frac{1}{n}\right)-1\right) \sim 1+o(1) \underset{n\longrightarrow+\infty}{\longrightarrow} 1 \neq 0.$$
Therefore, the necessary condition is not satisfied and the series is divergent.

8. $$\forall n \in \mathbb{N}^* : \ln\left(1+e^{-n}\right) \sim e^{-n} = \left(\frac{1}{e}\right)^n.$$
Therefore, the series is geometrically convergent, because of reason in $]-1,1[$.

Correction of exercise 4

1. For all $n \in \mathbb{N}$, we have
$$\begin{aligned} u_n &= \left(\frac{n}{n+1}\right)^{n^2} = e^{n^2\ln(\frac{n}{n+1})} = e^{-n^2\ln(\frac{n+1}{n})} = e^{-n^2\ln(1+\frac{1}{n})} \\ &\sim e^{-n^2\left(1+\frac{1}{n}+\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)\right)} \sim e^{-n+1+o(1)} \sim e^{-n}e^{1+o(1)} \sim e\left(\frac{1}{e}\right)^n. \end{aligned}$$
This is a geometric series with a reason in $]-1,1[$, and the series is convergent.

2. For all $n \in \mathbb{N}$, we have
$$u_n = \frac{1}{n\cos^2 n} > \frac{1}{n}$$
This is a series with positive terms greater than $\dfrac{1}{n}$, which is the general term of a divergent

Riemann series with $\alpha = 1 \leq 1$. So the series diverges.

3 We have, $\forall n \in \mathbb{N}^*$:

$$\sqrt[n]{u_n} = \frac{1}{\ln(n)} > \frac{1}{n} \underset{n \longrightarrow +\infty}{\longrightarrow} 0 < 1.$$

According to Cauchy's test, $0 < 1$, means that the series is convergent.

Correction of exercise 5

1 The sequence of partial sums associated with the series $\sum\limits_{n\geq 0} \frac{(n+1)}{3^n}$ is positive, increasing and bounded by 1,, so is convergent. Let's put $A_n = \sum\limits_{k=0}^{n} \frac{1}{3^n}$, then we have:

$$\begin{aligned} S_n &= \sum_{k=0}^{n} \frac{(k+1)}{3^k} = \sum_{k=0}^{n} \frac{1}{3^k} + \sum_{k=0}^{n} \frac{k}{3^k} \\ &= \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{n+1}\right) + \sum_{k=0}^{n} \frac{k}{3^k}. \end{aligned}$$

And yet,

$$\sum_{k=0}^{n} \frac{k}{3^k} = \sum_{k=1}^{n} \frac{k}{3^k} = \sum_{k=0}^{n-1} \frac{k+1}{3^{k+1}} = \frac{1}{3} S_{n-1}.$$

Thus, we get,

$$S_n = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{n+1}\right) + \frac{1}{3} S_{n-1} \Rightarrow \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} A_n + \frac{1}{3} \lim_{n\to+\infty} S_{n-1} \Rightarrow \lim_{n\to+\infty} S_n = \frac{9}{4}.$$

Consequently, the series $\sum\limits_{n\geq 0} \frac{(n+1)}{3^n}$ converges and it's sum is $\frac{9}{4}$.

2 From the partial sum of the previous series we have:

$$\begin{aligned} S'_n &= \sum_{k=0}^{n} \frac{(k-(-1)^k)}{3^k} = \sum_{k=0}^{n} \frac{k}{3^k} - \sum_{k=0}^{n} \frac{(-1)^k}{3^k} \\ &= \sum_{k=0}^{n} \frac{k}{3^k} - \frac{3}{4}\left(1 - \left(\frac{-1}{3}\right)^{n+1}\right). \end{aligned}$$

And yet,

$$\sum_{k=0}^{n} \frac{k}{3^k} = \sum_{k=1}^{n} \frac{k}{3^k} = \sum_{k=0}^{n-1} \frac{k+1}{3^{k+1}} = \frac{1}{3} S_{n-1}.$$

Then, using the previous result, we obtain the following result,

$$\lim_{n\to+\infty} S'_n = -\lim_{n\to+\infty} \frac{3}{4}\left(1 - \left(\frac{-1}{3}\right)^{n+1}\right) + \frac{1}{3} \lim_{n\to+\infty} S_{n-1} = -\frac{3}{4} + \frac{1}{3} \times \frac{9}{4} = 0.$$

Consequently, the series $\sum\limits_{n\geq 0} b_n$ is convergent and it's sum is 0.

3 We know that for all $n \in \mathbb{N}^*$, we have

$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2.$$

Alors, pour tout $n \in \mathbb{N}^*$, on obtient

$$\begin{aligned} S_n = \sum_{k=1}^{n} c_k &= \sum_{k=1}^{n} \frac{1+2+\ldots+k}{1^3+2^3+\ldots+k^3} = \sum_{k=1}^{n} \frac{1+2+\ldots+k}{(1+2+\ldots+k)^2} = \sum_{k=1}^{n} \frac{2}{k(k+1)} \\ &= 2\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 2\left(1 - \frac{1}{n+1}\right) \underset{n\longrightarrow+\infty}{\longrightarrow} 2\,. \end{aligned}$$

Consequently, the series $\sum\limits_{n\geq 1} c_n$ converges and has as sum the value 2.

4. For all $n \in \mathbb{N}$, we have,

$$S_n = \sum_{k=0}^{n} u_k = \sum_{k=0}^{n} \frac{k}{}4k! - \sum_{k=0}^{n} \frac{k+1}{}3k! = \frac{(n+1)^3}{n!} \underset{n\longrightarrow+\infty}{\longrightarrow} 0\,.$$

Consequently, the series $\sum\limits_{k=0}^{n} u_k$ converges and has a sum value of zero.

5. For all $n \in \mathbb{N}$, we have

$$\begin{aligned} S_n = \sum_{k=0}^{n} v_k &= \sum_{k=0}^{n} \frac{k}{(k^2+k+1)(k^2-k+1)} = \frac{1}{2} \times \sum_{k=0}^{n} \Big(\underbrace{\frac{1}{k^2-k+1}}_{x_k} - \underbrace{\frac{1}{k^2+k+1}}_{x_{k+1}} \Big) \\ &= \frac{1}{2} \times \sum_{k=0}^{n} \left(x_k - x_{k+1}\right) = \frac{1}{2} \times (x_0 - x_{n+1}) = \frac{1}{2} \times \left(1 - \frac{1}{n^2+n+1}\right) \underset{n\longrightarrow+\infty}{\longrightarrow} \frac{1}{2}\,. \end{aligned}$$

Consequently, the series $\sum\limits_{n\geq 1} v_n$ converges and has for sum the value $\frac{1}{2}$.

6. For all $n \geq 2$, we have

$$\begin{aligned} S_n = \sum_{k=2}^{n} w_k &= \sum_{k=2}^{n} \ln\left(1 - \frac{1}{k^2}\right) = \sum_{k=2}^{n} \Big(\ln(k+1) + \ln(k-1) - 2\ln(k)\Big) \\ &= -\ln 2 + \ln(n+1) - \ln(n) = -\ln 2 + \ln\left(\frac{n+1}{n}\right) \underset{n\longrightarrow+\infty}{\longrightarrow} -\ln 2\,. \end{aligned}$$

Consequently, the series $\sum\limits_{n\geq 1} w_n$ converges and has for sum the value $-\ln 2$.

Correction of exercise 6

1. The function f is defined on $\mathbb{R}_+$ by:

$$f(x) = \frac{1}{\ln(2+\sqrt{x})}.$$

Then, we have

$$f'(x) = -\frac{(\ln(2+\sqrt{x}))'}{(\ln(2+\sqrt{x}))^2} = -\frac{\frac{1}{2\sqrt{x}}}{(\ln(2+\sqrt{x}))^3} < 0.$$

So the sequence with general term $u_n = \dfrac{(-1)^n}{\ln(2+\sqrt{n})}$ is decreasing, tending towards 0, according to the alternating series test (Leibniz's Theorem), the series converges.

2 On the other hand, we have

$$|u_n| = \frac{1}{\ln(2+\sqrt{n})},$$

and as,

$$\lim_{n\to+\infty} n^{\frac{1}{2}}|u_n| = \lim_{n\to+\infty} \frac{n^{\frac{1}{2}}}{\ln(2+\sqrt{n})} = +\infty.$$

According to Riemann's rules, the series with general term $|u_n| = \dfrac{1}{\ln(2+\sqrt{n})}$ diverges, because $(\alpha = \dfrac{1}{2} < 1)$, which shows that the series $\sum\limits_{n\geq 0} u_n$ does not converge absolutely. This series is therefore semi-convergent.

Correction of exercise 7

1 Let $v_n = |u_n| = \dfrac{n^3}{n!}$. Then we have

$$\frac{v_{n+1}}{v_n} = \left(\frac{n+1}{n}\right)^3 \frac{1}{n+1} \underset{n\longrightarrow+\infty}{\longrightarrow} 0 .$$

According to D'Alembert's criterion, the series of general term v_n converges, so the series of general term u_n converges absolutely, so it converges.

2 We put, $v_n = |u_n| = \dfrac{|a|}{n!}$. Then we have

$$\frac{v_{n+1}}{v_n} = \frac{1}{n+1} \underset{n\longrightarrow+\infty}{\longrightarrow} 0 .$$

According to D'Alembert's criterion, the series of general term v_n converges, so the series of general term u_n converges absolutely, so it converges.

3 $u_n = na^{n-1},\ a \in \mathbb{C}$. We put, $v_n = |u_n| = n|a|^{n-1}$. Then we have

$$\frac{v_{n+1}}{v_n} = \frac{n}{n+1}|a| \underset{n\longrightarrow+\infty}{\longrightarrow} |a| .$$

If $|a| < 1$, according to D'Alembert's criterion, the series with general term v_n converges, so the series with general term u_n converges absolutely, so it converges.
If $|a| \geq 1$, $|u_n| \to +\infty$, then the series grossly divergent.

4 For all $n \in \mathbb{N}^*$, we have

$$u_n = \sin\left(\frac{n^2+1}{n}\pi\right) = \sin\left(n\pi + \frac{\pi}{n}\right) = (-1)^n \sin\left(\frac{\pi}{n}\right).$$

It is an alternating series because $\forall n \in \mathbb{N}^* :\ u_n \geq 0$, it is more or less obvious that is decreasing and tends towards 0, according to the alternating series theorem, the series $\sum\limits_{\geq 1} \sin\left(\dfrac{n^2+1}{n}\pi\right)$ converges.
Note that this series is semi-convergent because, for all $n \geq 1$, we have

$$|u_n| = \left|\sin\left(\frac{n^2+1}{n}\pi\right)\right| = |(-1)^n \sin\left(\frac{\pi}{n}\right)| \underset{+\infty}{\sim} \frac{\pi}{n}.$$

5. For all $n \in \mathbb{N}$, we have,

$$u_n = (-1)^n(\sqrt{n+1} - \sqrt{n}) = (-1)^n \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Since the series $\sum\limits_{n \geq 0} a_n$, with $a_n = \dfrac{1}{\sqrt{n+1} + \sqrt{n}}$ of positive terms, decreasing and tends to 0, according to the alternating series theorem, the series $\sum\limits_{\geq 1} u_n$ converges.

6. For all $n \geq 1$, we have:

$$\frac{u_{n+1}}{u_n} = \frac{\dfrac{(n+1)!}{(n+1)^{n+1}}}{\dfrac{n!}{n^n}} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n},$$

therefore,

$$\lim_{n \to +\infty} \frac{u_{n+1}}{u_n} = \lim_{n \to +\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1.$$

So, according to D'Alembert's criterion, the series $\sum\limits_{n \geq 1} u_n$ converges.

Correction of exercise 8

1. According to Riemann's rule, we have

$$\lim_{n \to +\infty} \frac{\dfrac{n^2}{2^n + 1}}{\frac{1}{n^2}} = \lim_{n \to +\infty} \frac{n^4}{2^n} = \lim_{n \to +\infty} \frac{n^4}{e^{n \ln 2}} = 0,$$

The Riemann series $\sum\limits_{n \geq 1} \dfrac{1}{n^2}$ converges, so the series $\sum\limits_{n \geq 0} \dfrac{n^2}{2^n + 1}$ is convergent.

2. For all $n \in \mathbb{N}$, we have

$$\frac{2 + \cos(n)}{n^3} \leq \frac{13}{n^3},$$

the Riemann series $\sum\limits_{n \geq 1} \dfrac{1}{n^3}$ is convergent, so according to the Comparison Theorem, the series $\sum\limits_{n \geq 1} \dfrac{2 + \cos(n)}{n^3}$ is convergent.

3. Applying Riemann's rule again, we get

$$\lim_{n \to +\infty} \frac{ne^{-n}}{\frac{1}{n^2}} = \lim_{n \to +\infty} \frac{n^3}{e^n} = 0,$$

The Riemann series $\sum\limits_{n \geq 1} \dfrac{1}{n^2}$ is convergent, so according to Riemenn's rule, the series $\sum\limits_{n \geq 0} ne^{-n}$ is convergent.

Correction of exercise 9

1 The series $\sum\limits_{n\geq 2}\dfrac{\cos(n)}{n(n-1)}$ whose general term does not keep a constant sign is convergent because it is absolutely convergent. Indeed, for all $n\geq 2$ we have

$$\left|\frac{\cos(n)}{n(n-1)}\right|\leq\frac{1}{n(n-1)}.$$

Now, we have already established the convergence of the series $\sum\limits_{n\geq 2}\dfrac{1}{n(n-1)}$. Based on the comparison criterion combined with the absolute convergence criterion, we can deduce the absolute convergence of $\sum\limits_{n\geq 2}\dfrac{\cos(n)}{n(n-1)}$, which immediately implies it's convergence.

2 For all $n>0$, we have

$$\arctan\left(\frac{1}{2n^2}\right)\underset{+\infty}{\sim}\frac{1}{2n^2},$$

the Riemann series $\sum\limits_{n\geq 1}\dfrac{1}{2n^2}$ is convergent, so according to the comparison theorem, the series $\sum\limits_{n\geq 1}\arctan\left(\dfrac{1}{2n^2}\right)$ is convergent.

3 For all $n>0$, we have

$$\sin\left(\frac{1}{n}\right)\underset{+\infty}{\sim}\frac{1}{n},$$

and the Riemann series $\sum\limits_{n\geq 1}\dfrac{1}{n}$ is divergent, so according to the comparison theorem, the series $\sum\limits_{n\geq 1}\sin\left(\dfrac{1}{n}\right)$ is divergent.

Correction of exercise 10

We know that for all $a,\ b\geq 0$, we have

$$ab\leq\frac{1}{2}(a^2+b^2).$$

We can therefore use the following comparisons

$$\max(u_n,v_n)\leq u_n+v_n,\ \sum\sqrt{u_nv_n}\leq\frac{1}{2}(u_n+v_n)\ \text{ and }\ \frac{u_nv_n}{u_n+v_n}=\frac{u_n}{u_n+v_n}v_n\leq v_n.$$

By comparing series with positive terms we can then conclude.

Correction of exercise 11

1 If $\alpha>1$, then we use the Riemann rule with $\beta\in]1,\alpha[$,

$$n^\beta\frac{\ln(n)}{n^\alpha}=\frac{\ln(n)}{n^{\alpha-\beta}}\underset{n\longrightarrow+\infty}{\longrightarrow}0<1.$$

This shows that the series with general term u_n converges because $\beta>1$.

2. If $\alpha < 1$, then we use Riemann's rule with $\beta \in]\alpha, 1[$,

$$n^{\beta}\frac{\ln(n)}{n^{\alpha}} = n^{\beta-\alpha}\ln(n) \underset{n\longrightarrow+\infty}{\longrightarrow} +\infty\ .$$

This shows that the series with general term u_n converges because $\beta < 1$.

3. When $\alpha = 1$, it's more complicated, Riemann's rules don't work. It is a series with positive terms. we can apply the comparison to an integral of the function

$$f:\ x \mapsto \frac{\ln x}{x}.$$

f is continuous on $]0, +\infty[$, so is integrable and we have

$$\int_2^A \frac{\ln x}{x}dx = \frac{1}{2}\Big[(\ln^2(x))\Big]_2^A \underset{A\longrightarrow+\infty}{\longrightarrow} +\infty\ .$$

Ce qui montre que l'intégrale est divergente, la fonction $x \mapsto \dfrac{\ln x}{x}$ est clairement décroissante et tend vers 0 en l'infini, donc la série de terme général $u_n = \dfrac{\ln(n)}{n^{\alpha}}$ diverge.

Correction of exercise 12

1. For all $n \in \mathbb{N}$, we have

$$\lim_{n\to+\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to+\infty}\frac{3}{n+1} = 0 < 1.$$

So, according to D'Alembert's criterion, the series $\sum\limits_{n\geq 1} u_n$ is convergent.

2. For all $n \in \mathbb{N}$, we have

$$\lim_{n\to+\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to+\infty}\frac{2n+1}{2n+2} = 1.$$

So D'Alembert's criterion gives no information about the nature of the series $\sum\limits_{n\geq 1} u_n$. But we can give a limited development of $\dfrac{u_{n+1}}{u_n}$ in the neighbourhood of of $+\infty$,

$$\frac{u_{n+1}}{u_n} = 1 + \frac{\frac{1}{2}}{n} + o\Big(\frac{1}{n}\Big),$$

hence, according to Duhamel's criterion, the series $\sum\limits_{n\geq 1} u_n$ is divergent.

3. $\lim\limits_{n\to+\infty}\sqrt[n]{u_n} = \lim\limits_{n\to+\infty}\left(1+\dfrac{a}{n}\right)^{-n} = e^{-a}$.

   a. If $a > 1$ the series $\sum\limits_{n\geq 1} u_n$ converges.

   b. If $a < 1$ the series $\sum\limits_{n\geq 1} u_n$ diverges.

   c. If $a = 1$, $u_n = 1$ and the series $\sum\limits_{n\geq 1} u_n$ converges.

4. $\lim\limits_{n\to+\infty} \sqrt[n]{u_n} = \lim\limits_{n\to+\infty} \left(\frac{n}{2^n}\right) = 0 < 1$. So the series $\sum\limits_{n\geq 1} u_n$ converges.

5. We have the limit
$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = 1,$$
d'Alembert's criterion does not allow us to conclude it's nature, so let's apply the Raabe-Duhamel rule.
$$\lim_{n\to+\infty} n\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to+\infty} n\left(\frac{1+2n}{n^2}\right) = 2 > 1,$$
which proves the convergence of the series.

6. For all $n \in \mathbb{N}$, we have
$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to+\infty} \left(\frac{(3n+1)}{(3n+3)}\right)^2 = 1.$$
Which is a doubtful case, so we calculate the limit,
$$\begin{aligned}\lim_{n\to+\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) &= \lim_{n\to+\infty} n\left(\left(\frac{(3n+3)}{(3n+1)}\right)^2 - 1\right)\\ &= \lim_{n\to+\infty} n\left(\frac{4}{3n+1} + \frac{4}{(3n+1)^2}\right) = \frac{4}{3} > 1.\end{aligned}$$
Hence, according to the Raabe-Duhamel rule, the series $\sum\limits_{n\geq 1} u_n$ converge.

7. $\lim\limits_{n\to+\infty} \sqrt[n]{u_n} = \lim\limits_{n\to+\infty} \left(\frac{3^{2-\frac{1}{n}}}{(n^2+1)^{\frac{1}{n}}}\right) = \lim\limits_{n\to+\infty} \left(\frac{3^{2-\frac{1}{n}}}{e^{\frac{1}{n}\ln(n^2+1)}}\right) = 9 > 1$. So the series $\sum\limits_{n\geq 0} u_n$ converges.

Correction of exercise 13

1. For $n \geq 2$, we have
$$|u_n| = \frac{1}{n^2 + (-1)^n} \underset{+\infty}{\sim} \frac{1}{n^2}.$$
So the series $\sum\limits_{n\geq 2} |u_n|$ is convergent, according to what we know about Riemann series, so that the alternating series $\sum\limits_{n\geq 2} u_n$ is absolutely convergent

2. For $n \geq 1$, we have
$$|u_n| = \frac{1}{\sqrt{n}} \underset{+\infty}{\sim} \frac{1}{n^{\frac{1}{2}}}.$$
So the series $\sum\limits_{n\geq 2} |u_n|$ is divergent (equivalent to the Riemann series with $\alpha < 1$). But according to Leibniz's rule, the sequence with general term $b_n = \frac{1}{\sqrt{n}}$ is decreasing and tends towards 0. Then the alternating series $\sum\limits_{n\geq 2} u_n$ is semi-convergent.

3. For $n \geq 2$, we have $u_n = \frac{(-1)^n}{n+(-1)^n} = (-1)^n v_n$, such that $v_n = \frac{1}{n+(-1)^n}$. Since the

sequence $(v_n)_{n\geq 2}$ is not decreasing, we cannot apply the Leibnitz criterion. But using limited development we find,

$$\begin{aligned} u_n &= \frac{(-1)^n}{n+(-1)^n} = \frac{(-1)^n}{n} \times \left(1+\frac{(-1)^n}{n}\right)^{-1} \\ &\simeq \frac{(-1)^n}{n} \times \left(1-\frac{(-1)^n}{n}\right) = \frac{(-1)^n}{n} - \frac{1}{n^2}. \end{aligned}$$

Because $(1+t)^\alpha \simeq 1+\alpha t$ in the neighbourhood of 0. Knowing that the two series of general terms $\dfrac{(-1)^n}{n}$ and $\dfrac{1}{n^2}$ converge according to Leibnitz and Riemann, then the series $\sum\limits_{n\geq 2} u_n$ converges.

For $n \geq 2$, we have $|u_n| = \dfrac{1}{n+(-1)^n} = (-1)^n v_n \underset{+\infty}{\sim} \dfrac{1}{n}$. So the series $\sum\limits_{n\geq 2} u_n$ semi-converges.

4 For all $n \in \mathbb{N}^*$, we have

$$|u_n| = \left|\frac{2+\sin(an)}{n^2}\right| \leq \frac{3}{n^2}.$$

Hence the series $\sum\limits_{n\geq 2} |u_n|$ converges (equivalent to the Riemann series with $\alpha > 2$). Consequently, the series $\sum\limits_{n\geq 2} u_n$ converges.

Correction of supplementary exercise 1

1 (a) We show the direct implication. If the series with general term u_n converges, then $\lim\limits_{n\to+\infty} u_n = 0$, so we have

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = \lim_{n\to+\infty} \frac{1}{1+u_n} = 1.$$

So, $u_n \sim v_n$, i.e. the two series are of the same nature.

(b) Conversely, if the series with general term v_n converges, then $\lim\limits_{n\to+\infty} v_n = 0$, so we have

$$v_n = \frac{u_n}{1+u_n} \Leftrightarrow u_n = \frac{v_n}{1-u_n}.$$

We also have $v_n \sim u_n$, so the series are of the same type.

2 (a) If $\sum\limits_{n\geq 1} u_n$ converges and is of sum S then $v_n \sim \dfrac{u_n}{S}$ and we can conclude.

(b) If $\sum\limits_{n\geq 1} u_n$ diverges, then,

$$\sum_{k=1}^{n} \ln(1-v_n) = \ln\left(\frac{u_1}{u_1+u_2+\ldots+u_n}\right) \underset{n\longrightarrow+\infty}{\longrightarrow} -\infty\,.$$

If $v_n \to 0$, $\ln(1-v_n) \sim -v_n$, therefore $\sum\limits_{n\geq 1} v_n$ diverges because the series have a constant sign.
If $v_n \to 0$, the series $\sum\limits_{n\geq 1} v_n$ grossly divergent

Correction of supplementary exercise 2

Study of the Bertrand series

$$\sum_{n\geq 2} \frac{1}{n^{\alpha}\ln^{\beta}(n)}, \quad (\alpha,\ \beta \in \mathbb{R}).$$

We pose $u_n = \dfrac{1}{n^{\alpha}\ln^{\beta}(n)}$. First we give a table which shows the limit of the general term u_n according to the real values of $\alpha,\ \beta$.

	$\alpha > 1$	$\alpha = 0$	$\alpha < 1$
$\beta > 1$	$\lim\limits_{n\to+\infty} u_n = 0$	$\lim\limits_{n\to+\infty} u_n = 0$	$\lim\limits_{n\to+\infty} u_n = +\infty$
$\beta = 0$	$\lim\limits_{n\to+\infty} u_n = 0$	$\lim\limits_{n\to+\infty} u_n = 1$	$\lim\limits_{n\to+\infty} u_n = +\infty$
$\beta < 1$	$\lim\limits_{n\to+\infty} u_n = 0$	$\lim\limits_{n\to+\infty} u_n = +\infty$	$\lim\limits_{n\to+\infty} u_n = +\infty$

From the previous table, it is clear that for $\alpha < 0$ or $(\alpha = 0$ and $\beta \leq 0)$, Bertrand's series $\sum\limits_{n\geq 2} \dfrac{1}{n^{\alpha}\ln^{\beta}(n)}$ is divergent.

1 For $\alpha = 0,\ \beta > 0$, the Bertrand series is divergent because

$$\lim_{n\to+\infty} \frac{\frac{1}{n}}{\frac{1}{\ln^{\beta} n}} = \lim_{n\to+\infty} \frac{\ln^{\beta} n}{n} \underset{n\longrightarrow+\infty}{\longrightarrow} 0,$$

so the series $\sum\limits_{n\geq 1} \dfrac{1}{n}$ is divergent. is divergent, so according to Riemann's rule the series $\sum\limits_{n\geq 1} \dfrac{1}{\ln^{\beta} n}$ is divergent. Hence the series $\sum\limits_{n\geq 1} \dfrac{1}{\ln^{\beta} n}$ is divergent for any $\beta \in \mathbb{R}$.

2 For $alpha = 0,\ \beta > 0$, the Bertrand series is divergent because,

$$\lim_{n\to+\infty} \frac{\frac{1}{n}}{\frac{1}{\ln^{\beta} n}} = \lim_{n\to+\infty} \frac{\ln^{\beta} n}{n} \underset{n\longrightarrow+\infty}{\longrightarrow} 0,$$

so the $\sum\limits_{n\geq 1} \dfrac{1}{n}$ series is divergent, so according to Riemann's rule, the series $\sum\limits_{n\geq 1} \dfrac{1}{\ln^{\beta} n}$ is divergent. Hence the series $\sum\limits_{n\geq 1} \dfrac{1}{\ln^{\beta} n}$ is divergent for all $\beta \in \mathbb{R}$.

3 For $\alpha > 1,\ \beta > 0$, we have,

$$\lim_{n\to+\infty} \frac{\dfrac{1}{n^{\alpha}\ln^{\beta} n}}{\dfrac{1}{n^{\alpha}}} = 0,$$

so the series $\sum\limits_{n\geq 1} \dfrac{1}{n^{\alpha}}$ is convergent, or according to Riemann's rule, the series $\sum\limits_{n\geq 1} \dfrac{1}{n^{\alpha}\ln^{\beta} n}$ is convergent.

4 For $0 < \alpha < 1$, we have,

$$\lim_{n\to+\infty} \frac{\frac{1}{n}}{\frac{1}{n^\alpha \ln^\beta n}} = \lim_{n\to+\infty} \frac{\ln^\beta n}{n^{1-\alpha}} = 0,$$

and since the series $\sum\limits_{n\geq 1} \frac{1}{n}$ is convergent, so according to Riemann's rule, the series $\sum\limits_{n\geq 1} \frac{1}{n^\alpha \ln^\beta n}$ is divergent.

5 **a** For $\alpha = 1$, the function $x \mapsto \frac{1}{x\ln^\beta x}$ is positive, continuous and decreasing on $[2, +\infty[$, therefore the integral $\int\limits_2^{+\infty} \frac{1}{x\ln^\beta x}dx$ and the series $\sum\limits_{n\geq 2} \frac{1}{n\ln^\beta n}$ are of the same nature. Now if $\beta \neq 1$ we have,

$$\begin{aligned}
\int\limits_2^{+\infty} \frac{1}{x^\alpha \ln^\beta x}dx &= \int\limits_{\ln 2}^{+\infty} u^\beta du = \lim_{t\to+\infty} \int\limits_{\ln 2}^{t} u^\beta du = \lim_{t\to+\infty} \left[\frac{1}{1-\beta}u^{1-\beta}\right]_{\ln 2}^{t} \\
&= \begin{cases} \frac{1}{1-\beta}(\ln 2)^{1-\beta} & : \ \beta > 1 \\ +\infty & : \ \beta < 1. \end{cases}
\end{aligned}$$

So in that case,

b If $\beta > 1$ the series $\sum\limits_{n\geq 2} \frac{1}{n\ln^\beta n}$ is convergent.

c If $\beta < 1$ the series $\sum\limits_{n\geq 2} \frac{1}{n\ln^\beta n}$ is divergent.

d If $\beta = 1$ the series $\sum\limits_{n\geq 2} \frac{1}{n\ln n}$ and the integral defined on $[2, +\infty[$, and the function $x \mapsto \frac{1}{x\ln^\beta x}$ are of the same nature, so it is divergent. Indeed,

$$\int\limits_2^{+\infty} \frac{1}{x\ln x}dx = \lim_{t\to+\infty} \int\limits_2^{t} \frac{1}{x\ln x} = \lim_{t\to+\infty} \int\limits_{\ln 2}^{t} \frac{1}{u}du = \lim_{t\to+\infty} \Big[\ln u\Big]_{\ln 2}^{t} = +\infty.$$

So if $\alpha > 1$ or $\alpha = 1$ and $\beta > 1$, the Bertrand series $\sum\limits_{n\geq 2} \frac{1}{n^\alpha \ln^\beta n}$ is convergent.

6 For $\alpha > 0$ and $\beta = 0$, the Bertrand series becomes a Riemann series $\sum\limits_{n\geq 2} \frac{1}{n^\alpha}$ which is convergent if and only if $\beta > 1$.

7 **a** For $\alpha > 0$ and $\beta < 0$, then, for all $\varepsilon > 0$, we find,

$$\lim_{n\to+\infty} \frac{\frac{1}{n^\alpha \ln^\beta n}}{\frac{1}{n^{\alpha-\varepsilon}}} = \lim_{n\to+\infty} \frac{1}{n^\varepsilon \ln^\beta n} = 0.$$

It is clear that for $\alpha - \varepsilon > 1$, (i.e. $\alpha > 1 + \varepsilon$), the Riemann series $\sum_{n\geq 2} \frac{1}{n^{\alpha-\varepsilon}}$ is convergent.

(b) If $\alpha > 1$ then there exists a $\varepsilon' > 0$ (for example $\varepsilon' = \frac{\alpha - 1}{2}$) such that $\alpha > 1 + \varepsilon'$, therefore the series $\sum_{n\geq 2} \frac{1}{n^\alpha \ln^\beta n}$ is convergent.

(c) If $\alpha \leq 1$ we get

$$\lim_{n\to+\infty} \frac{\frac{1}{n^\alpha}}{\frac{1}{n^\alpha \ln^\beta n}} = \lim_{n\to+\infty} \frac{\ln^\beta n}{n} \underset{n\longrightarrow+\infty}{\longrightarrow} 0,$$

Since the Riemann series $\sum_{n\geq 2} \frac{1}{n^\alpha}$ is convergent, because $\alpha \leq 1$, then according to Riemann's rule, the series $\sum_{n\geq 2} \frac{1}{n^\alpha \ln^\alpha} n$ is divergent.

So, from this study we conclude that

(i) If $\alpha > 1$ the series $\sum_{n\geq 2} \frac{1}{n^\alpha \ln^\beta n}$ converges for all $\beta \in \mathbb{R}$.

(ii) If $\alpha < 1$ the series $\sum_{n\geq 2} \frac{1}{n^\alpha \ln^\beta n}$ converges for all $\beta \in \mathbb{R}$.

(iii) If $\alpha = 1$ the series $\sum_{n\geq 2} \frac{1}{n^\alpha \ln^\beta n}$ converges if and only if $\beta > 1$.

Correction of supplementary exercise 3

1 For all $n \geq 1$ let $u_n = a_n b_n$, where the sequence $a_n = \frac{1}{n}$ is positive and decreasing to zero, and the sequence of partial sums associated with the sequence $b_n = \sin n$ is bounded. In fact, for all $n \geq 1$, we have,

$$\begin{aligned}
|S_n| &= \left|\sum_{k=1}^{n} b_k\right| = \left|\sum_{k=1}^{n} \sin k\right| = \left|\sum_{k=1}^{n} Im(e^{ik})\right| \\
&= \left|Im \sum_{k=1}^{n} e^{ik}\right| \leq \left|\sum_{k=1}^{n} e^{ik}\right| = \left|e^i \sum_{k=0}^{n-1} e^{ik}\right| = |e^i|\left|\frac{1-e^{in}}{1-e^i}\right| \\
&\leq \frac{2}{\sqrt{2-2\cos 1}} = \frac{1}{\sin(\frac{1}{2})} = M.
\end{aligned}$$

Then, according to Abel's criterion, the series $\sum_{n\geq 1} \frac{\sin n}{n}$ is convergent.

2 For all $n \geq 1$, let $u_n = a_n b_n$, where the sequence $a_n = \frac{1}{n}$ is positive and decreasing towards zero, and the sequence of partial sums associated with the sequence $b_n = \cos n$

is bounded. In fact, for all $n \geq 1$, we have,

$$\begin{aligned}
|S_n| &= \Big|\sum_{k=1}^{n} b_k\Big| = \Big|\sum_{k=1}^{n} \cos k\Big| = \Big|\sum_{k=1}^{n} Re(e^{ik})\Big| \\
&= \Big|Re\sum_{k=1}^{n} e^{ik}\Big| \leq \Big|\sum_{k=1}^{n} e^{ik}\Big| = \Big|e^{i}\sum_{k=0}^{n-1} e^{ik}\Big| = |e^{i}|\Big|\frac{1-e^{in}}{1-e^{i}}\Big| \\
&\leq \frac{2}{\sqrt{2-2\cos 1}} = \frac{1}{\sin(\frac{1}{2})} = M.
\end{aligned}$$

Then, according to Abel's criterion, the series $\sum\limits_{n\geq 1}\dfrac{\cos n}{n}$ is convergent.

Correction of supplementary exercise 4

1 For all $a \in]0,1[$, i.e. $|a| < 1$, then the geometric series $\sum\limits_{n\geq 0} a^{2n}$ converges (absolutely convergent). Then we have,

$$\begin{aligned}
\Big(\sum_{n\geq 0} a^{2n}\Big)^2 &= \Big(\sum_{n\geq 0} a^{n}\Big)\Big(\sum_{n\geq 0} a^{n}\Big) = \sum_{n\geq 0}\sum_{k=0}^{n} a^k a^{n-k} \\
&= \sum_{n\geq 0}\sum_{k=0}^{n} a^k a^{n-k} = \sum_{n\geq 0} x^n \sum_{k=0}^{n} 1 = \sum_{n\geq 0}(1+n)a^n.
\end{aligned}$$

2 For all $n \in \mathbb{N}^*$, we have,

$$w_n = \sum_{k=1}^{n-1} u_{nk}u_{n-k} = (-1)^n \sum_{k=1}^{n-1} \frac{1}{\sqrt{k(n-k)}}.$$

Since the terms of the sum $A_n = \sum\limits_{k=1}^{n-1} \dfrac{1}{\sqrt{k(n-k)}}$ are all greater than $\dfrac{1}{2n}$, we have,

$$A_n = \sum_{k=1}^{n-1} \frac{1}{\sqrt{k(n-k)}} \geq n\frac{1}{2n} = \frac{1}{2}.$$

From this we deduce that the series $(A_n)_{n\geq 1}$ does not converge to 0, and consequently $\lim\limits_{n\to+\infty} w_n \neq 0$. Therefore the series $\sum\limits_{n\geq 1} w_n$ is roughly divergent.

Note that the product series diverges despite the fact that the series $\sum\limits_{n\geq 1}\dfrac{(-1)^n}{\sqrt{n}}$ is convergent, but is not absolutely convergent.