M'sila University	**Series n° : 4** **Fourier series**	Faculty of Mathematics and Computer Science
L 2 Mathematics		Module: Analysis 3

Remark 1

The exercise marked with $(\star)$ or additional will not be corrected in the session of **TD**.

Exercise 1

Let f be the $2\pi-$ periodic function defined on the interval $[-\pi, \pi[$, by

$$f(x) = \begin{cases} 0 & : \ x \in [-\pi, 0[, \\ x & : \ x \in [0, \pi[. \end{cases}$$

1 Find the Fourier coefficients a_n and b_n associated with f and give the Fourier series associated with f.

2 Deduct the sums $\sum\limits_{n=0}^{+\infty} \frac{1}{(2n+1)^2}$, and $\sum\limits_{n=1}^{+\infty} \frac{1}{n^2}$.

Exercise 2

Let f be the $2\pi-$ periodic function defined on the interval $[-\pi, \pi[$, by $f(x) = x$.

1 Determine the Fourier coefficients a_n and b_n associated with f and give the Fourier series associated with f.

2 Let $n \in \mathbb{N}$, determine, $\sin\left(n\frac{\pi}{2}\right)$.

3 Deduct the sums $\sum\limits_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^2}$, and $\sum\limits_{n=1}^{+\infty} \frac{1}{n^2}$.

Exercise 3

Let f be the $2\pi-$ periodic function defined on the interval $[-\pi, \pi]$, by $f(x) = |x|$.

1 Determine the Fourier coefficients a_n and b_n associated with f and give the Fourier series associated with f.

2 Let $n \in \mathbb{N}$, find, $\sin\left(n\frac{\pi}{2}\right)$.

3 Deduct the sums $\sum\limits_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^2}$, $\sum\limits_{n=0}^{+\infty} \frac{1}{n^2}$, $\sum\limits_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^4}$, and $\sum\limits_{n=0}^{+\infty} \frac{1}{n^4}$.

Exercise 4

Let f be the $2\pi-$ periodic, odd function defined on $\mathbb{R}$, by $f(x) = \begin{cases} 1 & : \ x \in \left]0, \pi\right[, \\ 0 & : \ x = \pi. \end{cases}$

1. Calculate the trigonometric Fourier coefficients of f.
2. Study the convergence (simple, uniform) of the Fourier series of f.
3. Deduce the values of the sums

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}, \ \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}, \ \sum_{n=1}^{+\infty} \frac{1}{n^2}, \text{ and } \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2}.$$

Exercise 5

Let f be a $2\pi-$ periodic function defined on $\mathbb{R}$, by

$$f(x) = (x-\pi)^2, \quad x \in [0, 2\pi[$$

1. Calculate the trigonometric Fourier coefficients of f.
2. Examine the convergence of the Fourier series of f.
3. Deduct the sums of the series $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}$, and $\sum_{n=1}^{+\infty} \frac{1}{n^2}$.

Exercise 6

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable $2\pi-$periodic function, and let λ be a non-zero real number. Consider the differential equation: $y'(x) + \lambda y(x) = f(x)$.
2. Find a 2 periodic solution to this equation by writing $y(x)$ and $f(x)$ in the form of trigonometric Fourier series.
3. Apply this result to the case where $\lambda = 1$ and

$$f(x) = \begin{cases} \left(x - \frac{\pi}{2}\right)^2 & : x \in [0, \pi[\\ \frac{\pi^2}{2} - \left(x - \frac{3\pi}{2}\right)^2 & : x \in [\pi, 2\pi[\end{cases}$$

Supplementary exercise 1

Let f be a periodic function of period 2, defined on the interval $]-1, 1]$, by $f(x) = x - x^3$.

1. Determine the Fourier coefficients a_n and b_n associated with f and give the Fourier series associated with f.
2. Deduct the sum of the series $\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3}$.