M'sila University		Faculty of Mathematics and Computer Science
L 2 Mathematics	**Series n° : 3** **Integers series**	Module: Analysis 3

Remark 1

The exercise marked with $(\star)$ or additional will not be corrected in the session of **TD**.

Exercise 1

Calculate the radius of convergence of the following integer series of general terms

1 $u_n(z) = \dfrac{n^2+1}{3^n} z^n$.

2 $u_n(z) = \dfrac{n^n}{n!} z^{3n}$.

3 $u_n(z) = \dfrac{\ln(n)}{n^2} z^{2n}$.

4 $u_n(z) = e^{-n^2} z^n$.

Exercise 2

Consider the integer series of general terms

a $u_n(z) = z^n,\ z \in \mathbb{C}$.

b $u_n(z) = nz^n,\ z \in \mathbb{R}$.

c $u_n(z) = \dfrac{1}{n} z^n,\ z \in \mathbb{R}$.

d $u_n(z) = \dfrac{5n}{n+2} z^n,\ (\star)$.

e $u_n(z) = \dfrac{(z-1)^{2n}}{4^n},\ z \in \mathbb{R}$.

1 Determine the radius and domain of convergence of each of the integer series and their convergence at points on the boundary of the domain of convergence.

2 Calculate the sum of the integer series in the convergence interval.

Exercise 3

Let S be the sum of the integer series $\sum\limits_{n\geq 0} \dfrac{(-1)^n x^{2n+2}}{(n+1)(2n+1)}$.

1 Find the radius of convergence of this series.

2 Show that S is continuous on $[1,1]$.

3 Determine the function $S'(x)$ for all $x \in]1,1[$. Deduce $S(x)$.

4 Calculate the sum $\sum\limits_{n=0}^{+\infty} \dfrac{(-1)^n}{(n+1)(2n+1)}$, $(\star)$.

Exercise 4

Develop (Expand) the following functions in integer series in the neighbourhood of 0, and then calculate the radius of convergence.

1. $f(x) = \dfrac{1}{(1-x)(2+x)}$.
2. $f(x) = \arctan\left(\dfrac{x-1}{1+x}\right)$.
3. $f(x) = \ln\left(\dfrac{x-1}{x+2}\right)$.
4. $f(x) = e^x \sin x$, and $g(x) = e^x \cos x$.

Exercise 5

Let be the functions sequence $(f_n)_{n\geq 0}$ such that $f_n(x) = \dfrac{1}{(4n)!}x^{4n}$

1. Find the radius of convergence of the series $\sum\limits_{n=0}^{+\infty} f_n(x)$.
2. Develop the function $f(x) = \cos x + chx$ as an integer series on $\mathbb{R}$.
3. Deduct the sum of the series $\sum\limits_{n=0}^{+\infty} f_n(x)$.

Supplementary exercise 1

We propose to calculate the sum $S(x)$ of the series $\sum\limits_{n=0}^{+\infty} n^2 x^n$.

1. Determine the open interval of convergence of this series.
2. We deduce the radius of convergence of integer series
   a. $\sum\limits_{n\geq 0} x^n$
   b. $\sum\limits_{n\geq 0} n x^{n-1}$
   c. $\sum\limits_{n\geq 0} n(n-1)x^{n-2}$
3. Deduce $S(x)$.
4. Deduct the sum $\sum\limits_{n\geq 0} \dfrac{n^2}{2^n}$.

Supplementary exercise 2

Let be the differential equation

$$(E): \quad x^2(1-x)y'' - x(1+x)y' + y = 0 \tag{1}$$

Determine the solution of the equation (1) that can be developed into an integer series in the neighbourhood of 0.

Supplementary exercise 3

Show that $\sum\limits_{n=1}^{+\infty} \dfrac{(-1)^n}{n(n+1)} = 1 - 2\ln 2$.

Indication: using the sum of the following integer series $\sum\limits_{n=1}^{+\infty} x^n$.