M'sila University		Faculty of Mathematics and Computer Science
L 2 Mathematics	**Series n° : 1** **Function sequences and series**	Module: Analyse 3

Exercise 1

Study the simple and uniform convergence of a functions sequence on the given domain.

1. $f_n(x) = \dfrac{1}{1+nx}$, $I = [0,1]$.
2. $f_n(x) = n(e^{\frac{x}{n}} - 1)$, $I = \mathbb{R}$ $(\star)$.
3. $f_n(x) = x^n(1-x)$, $I = [0,1]$.
4. $f_n(x) = nx^2e^{-nx}$, $I = [0,+\infty[$ $(\star)$.

Exercise 2

Let be the functions sequence defined on $\mathbb{R}$ for all $n \in \mathbb{N}^*$, by $f_n(x) = \dfrac{x}{1+n^2x^2}$.

1. Show that the sequence (f_n) converges uniformly to a function f on $[0,1]$.
2. Show that we have $f'(x) = \lim\limits_{x\to+\infty} f_n'(x)$ everywhere except at $x = 0$. What can we deduce

Exercise 3

Let $(f_n)_{n\geq1}$ be a functions sequence defined on $[0,1]$ by $f_n(x) = \dfrac{ne^x + xe^{-x}}{x+n}$.

1. Study the simple and uniform convergence of the series $(f_n)_{n\geq1}$ on $[0,1]$.
2. Calculate $\lim\limits_{n\to+\infty} \int\limits_0^1 (x^3+1)f_n(x)dx$.

Exercise 4

Let be the functions sequence defined on $[0,+\infty[$ for all $n \in \mathbb{N}^*$, by $f_n(x) = \begin{cases} \dfrac{1}{n} & : x \in [0,n] \\ 0 & : x > n. \end{cases}$

1. Show that the sequence (f_n) converges uniformly to a function f on $[0,+\infty[$.
2. Compute $\lim\limits_{n\to+\infty} \int\limits_0^{+\infty} f_n(x)dx$ et $\int\limits_0^{+\infty} \lim\limits_{n\to+\infty} f_n(x)dx$. What can we deduce ?

Exercise 5

Let the series defined on the interval $I = [1,+\infty[$ be $\sum\limits_{n\geq1} \dfrac{(-1)^n}{n+x}$.

1. Show that the series converges to a continuous limit S on I.

2. Show that S is derivable on I and calculate it's derivative S' and deduce that S is increasing on I.

3. Then calculate $\lim_{x\to -1} S(x)$ and $\lim_{x\to +\infty} S(x)$.

Exercise 6

Let (f_n) be the functions sequence defined on $\mathbb{R}_+$ by, $f_n(x) = (-1)^n \dfrac{e^{-nx}}{n+1}$.

1. Show that the functions serie $\sum_{n\geq 0} f_n$ simply converges to a function on $\mathbb{R}_+$.

2. Show that the functions serie $\sum_{n\geq 0} f_n$ converges normally on $[a, +\infty[,\ a > 0$. Does this result hold true on $[0, +\infty[$ or on $]0, +\infty[$.

3. Show that the series $\sum_{n\geq 0} f_n$ converges uniformly on $\mathbb{R}_+$.

Exercise 7

Let the functions series $\sum_{n=0} f_n$ be defined on the interval $]0, +\infty[$ such that, $f_n(x) = \dfrac{(-1)^n}{n+x}$.

1. Show that the series $\sum_{n\geq 0} f_n$ simply converges to a function f.

2. Show that $\forall x \in]0, +\infty[:\ f'(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^{n+1}}{(n+x)^2}$.

Supplementary exercise 1

Let the function f be defined on $\mathbb{R}_+^*$ for all $n \in \mathbb{N}^*$, by $f(x) = \sum_{n=1}^{+\infty} \dfrac{(-1)^n}{n + \ln(1+nx)}$.

1. Justify that f is defined and continuous on $\mathbb{R}_+^*$.

2. Determine the limit $\lim_{x\to +\infty} f(x)$.

3. Establish that f is of class C^1 on $\mathbb{R}_+^*$ and determine f'.

Supplementary exercise 2

Let be the functions sequence $(f_n)_{n\geq 0}$, defined by $f_n(x) = \sqrt{1-x^2} + e^{-nx^2}\sin(nx)$.

1. Show that the sequence $(f_n)_{n\geq 0}$, converges simply on $[-1, 1]$ to a function to be determined.

2. Show that the sequence $(f_n)_{n\geq 0}$, converges uniformly to a function f on any interval $[a, 1]$ where $a > 0$.

3. Show that the sequence $(f_n)_{n\geq 0}$, does not converge uniformly to f on $[0, 1]$.