University of M'sila	**Interrogation**	Faculty : Mathematics-Computer Sc
L2 Mathematics	Correction	Academic year : 2024/2025
Module : Analysis 3		Duration : 1hour

Scale	
	Correction of exercise : 1 4pts
	(1) We study the nature of the following numerical series :
1	(a) We put $u_n = \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$. By using the partial sums, $S_n = \sum_{k=0}^{k=n} u_k = 1 - \frac{1}{n+2}$.
0.5	Since, $\lim_{n\to+\infty} S_n = 1$. Then, the numerical series $\sum_{n\geq 0} u_n$ is convergent.
1	(b) Here $u_n = \frac{n!}{n^n}$. We apply the D'Alembert criterion, we obtain, $\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to+\infty} \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!} = \lim_{n\to+\infty} \left(\frac{n}{n+1}\right)^n$
0.5	$= \lim_{n\to+\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1.$
	Therefore, the series $\sum_{n\geq 1} u_n$ is convergent.
0.5	(c) For the series $\sum_{n\geq 1} \frac{1}{n\sqrt[3]{n}}$, we have : $\forall n \in \mathbb{N}^* : \frac{1}{n\sqrt[3]{n}} = \frac{1}{n^{\frac{4}{3}}}$.
0.5	Since, the Riemann series $\sum_{n\geq 1} \frac{1}{n^{\frac{4}{3}}}$ converges, because $\alpha = \frac{4}{3} > 1$. Thus, according to the comparison theorem, the series $\sum_{n\geq 1} u_n$ is convergent.
	Remark :
0	(1) For question 1, we can apply the equivalence theorem directly, so, $\lim_{n\to+\infty} \frac{u_n}{\frac{1}{n^2}} = 1$. Then, the series $\sum_{n\geq 1} \frac{1}{n^2}$, and $\sum_{n\geq 1} u_n$ have the same nature. Since the Riemann's series is convegent. Consequently, the series $\sum_{n\geq 1} u_n$ is convergent.
	(2) For question 3, we can apply the equivalence theorem directly, so, $\lim_{n\to+\infty} \frac{u_n}{\frac{1}{n^{\frac{4}{3}}}} = 1$.
	(3) Then, the series $\sum_{n\geq 1} \frac{1}{n^{\frac{4}{3}}}$, and $\sum_{n\geq 1} u_n$ have the same nature. Consequently, the series $\sum_{n\geq 1} u_n$ is convergent.
Scale	**Correction d'exercice : 2** 5pts
	Let the sequence of functions $(f_n)_{n\geq 0}$ be defined on $\mathbb{R}_+$ by $f_n(x) = e^{-nx}$.
	(1) Let us show that the series $\sum_{n\geq 0} f_n$ converges simply on $\mathbb{R}_+$.
0.5	(a) For $x = 0$, we have $f_n(0) = 1$. So, $\lim_{n\to+\infty} f_n(0) = 1$.

1	(b) For $x \neq 0$ we have : $\lim\limits_{n\to+\infty} f_n(x) = \lim\limits_{n\to+\infty} e^{-nx} = 0.$
0.5	Then, the sequence of functions $(f_n)_{n\geq 0}$ simply converges on $\mathbb{R}_+$ to the function f such that,
	$f(x) = \begin{cases} 1 & : \quad x = 0 \\ 0 & : \quad x > 0 \end{cases}$
	(2) Uniform convergence of the series of functions $\sum\limits_{n\geq 0} f_n$.
1	Since $\lvert f_n(x) - f(x) \rvert = \begin{cases} 0 & : \quad x = 0 \\ e^{-nx} & : \quad x > 0 \end{cases}$
0.5	So, $\lim\limits_{n\to+\infty} \sup\limits_{x\in[0,+\infty[} \lvert f_n(x) - f(x)\rvert = \lim\limits_{n\to+\infty} \sup\limits_{x\in[0,+\infty[} e^{-nx} = \lim\limits_{n\to+\infty} 1 = 1 \neq 0.$
0.5	Therefore, $(f_n)_{n\geq 0}$ does not uniformly convergent to f on $\mathbb{R}_+$.
	Remark :
	We can see that the sequence of functions $(f_n)_{n\geq 0}$ simply converges to a discontinuous function f on $\mathbb{R}_+$, thus, the sequence of functions does not converge uniformly on $\mathbb{R}_+$.
End	**Dr. D. Bouafia** **5 janvier 2025**