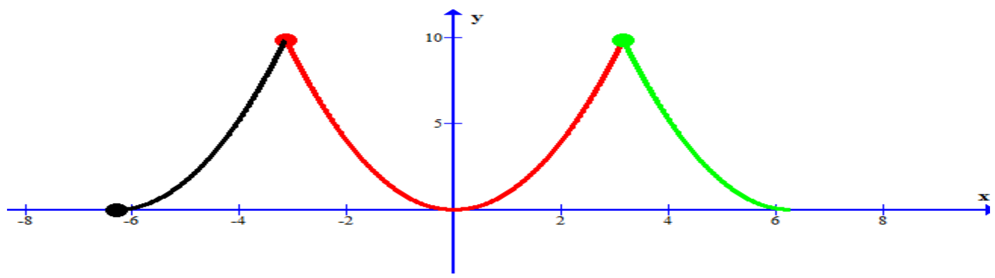
University of M'sila	Rattrapage exam	Faculty : Mathematics-Computer Sc
Mathematics-L2		Academic year : 2024/2025
Module : Analysis 3	Third semester	Duration : 1 h 30 m

Scale	**Exercise : 1** (3pts)
3×1	Study the nature of the following numerical series : (a) $\sum_{n\geq 0}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$. (b) $\sum_{n\geq 1}\frac{n!}{n^n}$. (c) $\sum_{n\geq 1}\frac{1}{n\sqrt[3]{n}}$.
Scale	**Exercise : 2** (7pts)
	Let the alternating series of functions be defined on $\mathbb{R}_+$ by $\sum_{n\geq 1}\frac{(-1)^n}{n+n^2x}$.
2.5	(1) Show that the series $\sum_{n\geq 1}\frac{(-1)^n}{n+n^2x}$ does not normally converges on $\mathbb{R}_+$.
2.5	(2) Study the uniform convergence of this series on $\mathbb{R}_+$.
1	(3) (a) Show that the function S is continuous on $\mathbb{R}_+$, where $S(x)=\sum_{n=1}^{+\infty}\frac{(-1)^n}{n+n^2x}$.
0.5×2	(b) Calculate $\lim_{x\to+\infty} S(x)$ and $\lim_{x\to 0} S(x)$.
Scale	**Exercise : 3** (7pts)
	Let f be a $2\pi-$ periodic function defined on $[-\pi,\pi]$, by $f(x)=x^2$.
1.5	(1) Draw the graph of f on the interval $[-2\pi,2\pi[$.
2.5	(2) Calculate the trigonometric Fourier coefficients of f.
1	(3) Give the Fourier series associated with f.
$1+1$	(4) Deduct the sums of the series $\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}$, and $\sum_{n=1}^{+\infty}\frac{1}{n^2}$.
Scale	**Exercise : 4** (3pts)
1.5×2	Study the convergence of the following generalized integrals (a) $I_1=\int_0^{+\infty}\frac{1}{(1+e^{-x})(1+e^x)}dx$. (b) $I_2=\int_1^{+\infty}\frac{\ln x}{x^2}dx$. Indication : $\forall x\in\mathbb{R}_+ :\ \frac{1}{(1+e^{-x})(1+e^x)}=\frac{e^x}{(1+e^x)^2}$.
End	**8 juin 2025** — **Good luck**

Scale	**Correction of exercise : 1** (5pts)
	1 We study the nature of the following numerical series :
0.5	a We put $u_n = \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$. By using the partial sums, $S_n = \sum_{k=0}^{k=n} u_k = 1 - \frac{1}{n+2}$.
0.5	Since, $\lim_{n\to+\infty} S_n = 1$. Then, the numerical series $\sum_{n\geq 0} u_n$ is convergent.
1	b Here $u_n = \frac{n!}{n^n}$. We apply the D'Alembert criterion, we obtain,
0.5	$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to+\infty} \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!} = \lim_{n\to+\infty} \left(\frac{n}{n+1}\right)^n$
0.5	$= \lim_{n\to+\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1.$
	Therefore, the series $\sum_{n\geq 1} u_n$ is convergent.
0.5	c For the series $\sum_{n\geq 1} \frac{1}{n\sqrt[3]{n}}$, we have : $\forall n \in \mathbb{N}^* : \frac{1}{n\sqrt[3]{n}} = \frac{1}{n^{\frac{4}{3}}}$.
0.5	Since, the Riemann series $\sum_{n\geq 1} \frac{1}{n^{\frac{4}{3}}}$ converges, because $\alpha = \frac{4}{3} > 1$. Thus, according to
	the comparison theorem, the series $\sum_{n\geq 1} u_n$ is convergent.
	Remark :
0	1 For question 1, we can apply the equivalence theorem directly, so, $\lim_{n\to+\infty} \frac{u_n}{\frac{1}{n^2}} = 1$.
	Then, the series $\sum_{n\geq 1} \frac{1}{n^2}$, and $\sum_{n\geq 1} u_n$ have the same nature. Since the Riemann's series is
	convegent. Consequently, the series $\sum_{n\geq 1} u_n$ is convergent.
	2 For question 3, we can apply the equivalence theorem directly, so, $\lim_{n\to+\infty} \frac{u_n}{\frac{1}{n^{\frac{4}{3}}}} = 1$.
	3 Then, the series $\sum_{n\geq 1} \frac{1}{n^{\frac{4}{3}}}$, and $\sum_{n\geq 1} u_n$ have the same nature.
	Consequently, the series $\sum_{n\geq 1} u_n$ is convergent.

Scale	**Correction of exercise : 2** (8pts)
	Let the alternating series of functions be defined on $\mathbb{R}_+$ by $\sum_{n\geq 1} \frac{(-1)^n}{n+n^2x}$.
	1 We study the normal convergencethat of the series $\sum_{n\geq 1} \frac{(-1)^n}{n+n^2x}$.

1.5	We have : $\sup\limits_{x\in\mathbb{R}_+}\|f_n(x)\| = \sup\limits_{x\in\mathbb{R}_+}\left\|\frac{(-1)^n}{n+n^2x}\right\| = \frac{1}{n}$
0.5 × 2	The Riemann series (harmonic series) $\sum\limits_{n\geq 1}\frac{1}{n}$ is divergent, so the series $\sum\limits_{n\geq 1}\frac{(-1)^n}{n+n^2x}$ does not converge normally on $\mathbb{R}_+$.
	2 We study the uniform convergence of this series on $\mathbb{R}_+$.
0.5	For all $x\in\mathbb{R}_+$, we put $u_n(x)=(-1)^n$ and $v_n(x)=\frac{1}{n+n^2x}$, $(n\in\mathbb{N}^*)$, so we have
1	a Since, the partial sums S_n of the series $\sum\limits_{n\geq 1}u_n(x)$ is bounded, because $\forall n\in\mathbb{N}:\ \|S_n\| = \left\|\sum\limits_{k=0}^{k=n}u_k\right\| \leq 1 > 0.$
0.5	b On the other hand, the sequence $(v_n(x))_n$ is decreasing and uniformly converges to zero, because, $\sup\limits_{x\in\mathbb{R}_+}\frac{1}{n+n^2x} \leq \frac{1}{n} \underset{n\to\infty}{\longrightarrow} 0,$
0.5	and for all $x\in\mathbb{N}^*:\ v_n(x)=\frac{1}{n+n^2x} \geq \frac{1}{(n+1)+(n+1)^2x} = v_{n+1}(x),$ This means that $\Big(v_n(x)\Big)$ is decreasing.
0.5	Thus, according to Abel's theorem, the series $\sum\limits_{n\geq 1}u_n(x)v_n(x)$ is uniformly convergent.
1	3 a From question 2, the series $\sum\limits_{n\geq 1}\frac{(-1)^n}{n+n^2x}$ is uniformly convergent, and the functions $f_n,\ n\in\mathbb{N}$ are continuous on $\mathbb{R}_+$. Then, According to continuity theorem, its sum $S(x)=\sum\limits_{n=1}^{+\infty}\frac{(-1)^n}{n+n^2x}$ is continuous on $\mathbb{R}_+$,
	b As the function S is continuous. So, we have,
0.5 × 2	$\lim\limits_{x\to+\infty}S(x) = \sum\limits_{n=1}^{+\infty}\lim\limits_{x\to+\infty}\frac{(-1)^n}{n+n^2x} = 0.$
1	and $\lim\limits_{x\to 0}S(x) = \sum\limits_{n=1}^{+\infty}\lim\limits_{x\to 0}\frac{(-1)^n}{n+n^2x} = \sum\limits_{n=1}^{+\infty}\frac{(-1)^n}{n} = -\sum\limits_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n} = -\ln 2.$
Scale	**Correction of exercise : 3** 7pts
1.5	Let f be a $2\pi-$ periodic function defined on $[-\pi,\pi]$, by $f(x)=x^2$. 1 We draw the graph of f on the interval $[-2\pi,2\pi[$. 

0.5	2 (a) Since f is even, then $\forall n \in \mathbb{N}:\ b_n = 0.$
0.5	(b) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x^2dx = \frac{2}{3\pi}\left[x^3\right]_0^{\pi} = \frac{2\pi^2}{3}.$
1.5	(c) For all $n \in \mathbb{N}^*$, we have : $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = 2\int_0^{\pi} x^2\cos(nx)dx.$ Using the integral by parts twice, we find $a_n = \frac{4(-1)^n}{n^2}.$
1	3 The Fourier series associated with f is $F(f)(x) = \frac{\pi}{3} + 4\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}\cos(nx).$
2	4 f is C^1 piecewise and continuous on $\mathbb{R}$, so according to Dirichlet's Theorem, the Fourier series of f simply converges to f on $\mathbb{R}$. So at any point x where the function f is continuous, we have : $f(x) = \frac{\pi}{3} + 4\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}\cos(nx).$
1	(a) In particular, for $x = 0$, we have : $f(0) = 0 = \frac{\pi}{3} + 4\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}$, hence, $\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$
1	(b) Similarly for $x = \pi$ we have : $f(\pi) = \pi^2 = \frac{\pi}{3} + 4\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}(-1)^2,$ Thus, $\sum_{n=1}^{+\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$
Scale	**Correction of exercise : 4** 3pts
0.5	(a) For all $x \in \mathbb{R}_+$, we have $\frac{1}{(1+e^{-x})(1+e^x)} = \frac{e^x}{(1+e^x)^2}.$ We know that the primitive of the function $x \mapsto \frac{u'(x)}{(1+u(x))^2}$ is the function $x \mapsto -\frac{1}{1+u(x)}$ and, as a result (here $u(x) = e^x$), so, we obtain
1	$\int_0^{+\infty}\frac{1}{(1+e^{-x})(1+e^x)}dx = \lim_{t\to+\infty}\int_0^t\frac{e^x}{(1+e^x)^2}dx = \lim_{t\to+\infty}\left[\frac{-1}{1+e^x}\right]_0^t$ $= \frac{1}{2} - \lim_{t\to+\infty}\frac{1}{1+e^t} = \frac{1}{2}$. Consquently I_1 is convergent.
0.5	(b) Integrating by parts, we find $\int\frac{\ln x}{x^2}dx = -\frac{\ln x}{x} + \int\frac{dx}{x^2} = -\frac{\ln x}{x} - \frac{1}{x}$. Then
1	$I_2 = \int_1^{+\infty}\frac{\ln x}{x^2}dx = \lim_{t\to+\infty}\int_1^t\frac{\ln x}{x^2}dx = \lim_{t\to+\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t = 1 - \lim_{t\to+\infty}\left(\frac{\ln t}{t} + \frac{1}{t}\right) = 1$ Thus, I_2 is convergent.
End	**8 juin 2025** **Good luck**