M'sila University	**Correction of exam**	Faculty : Maths-Computer Sc
Mathematics L2		Academic year : 2025/2026
Module : Analysis 3	Normal session	Duration : 1 h 30m

Scale	

Exercise correction : 1 (4 pts)

Let the numerical series $\sum_{n\geq 0} u_n$, and S its sum, where $u_n = \dfrac{2}{(n+1)(n+3)}$.

0.5×2 ① We have $\forall n \in \mathbb{N}:\ u_n \underset{+\infty}{\sim} \dfrac{1}{n^2}$. Since, the Riemann series $\sum_{n\geq 1} \dfrac{1}{n^2}$ is convergent, because

0.5 $\alpha = 2 > 1$. Thus, according to the comparison theorem, the series $\sum_{n\geq 0} u_n$ is convergent.

0.5 ② $\forall n \in \mathbb{N}:\ \left(\dfrac{1}{n+1} - \dfrac{1}{n+2}\right) + \left(\dfrac{1}{n+2} - \dfrac{1}{n+3}\right) = \dfrac{1}{n+1} - \dfrac{1}{n+3} = u_n$

③ (a) So its partial sums S_n is

1

$$\begin{aligned}
S_n &= \sum_{k=0}^{k=n} \left(\frac{1}{k+1} - \frac{1}{k+2}\right) + \left(\frac{1}{k+2} - \frac{1}{k+3}\right) \\
&= \sum_{k=0}^{k=n} \left(\frac{1}{k+1} - \frac{1}{k+2}\right) + \sum_{k=0}^{k=n} \left(\frac{1}{k+2} - \frac{1}{k+3}\right). \\
&= \left(1 - \frac{1}{n+2}\right) + \left(\frac{1}{2} - \frac{1}{n+3}\right)
\end{aligned}$$

1 (b) Therefore, $S = \lim_{n\to+\infty} S_n = 1 + \dfrac{1}{2} = \dfrac{3}{2}$.

Scale	

Exercise correction : 2 (4 pts)

Let the sequence of functions $(f_n)_{n\geq 1}$ be defined on $\mathbb{R}$ by $f_n(x) = \sqrt{x^2 + \dfrac{1}{n}}$.

① The simple convergence of $(f_n)_{n\geq 1}$.

1 For all $x \in \mathbb{R}$, we have $\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \sqrt{x^2 + \dfrac{1}{n}} = \sqrt{x^2} = |x|$.

0.5 This implies that, $f_n \xrightarrow{S.C} f$ (simply converges) where $f(x) = |x|$ on $\mathbb{R}$.

② The absolutely convergence of $(f_n)_{n\geq 1}$.

0.5 For all $x \in \mathbb{R}$, and for all $n \in \mathbb{N}^*$, we have $|f_n(x)| = \sqrt{x^2 + \dfrac{1}{n}} = f_n(x) \geq 0$.

0.5 Then, $f_n \xrightarrow{Abs.C} f$ (absolutely converges) on $\mathbb{R}$.

③ The uniform convergence of $(f_n)_{n\geq 1}$.

0.5 On $\mathbb{R}$ we have $|f_n(x) - f(x)| = \sqrt{x^2 + \dfrac{1}{n}} - |x| = \dfrac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}} + |x|}$.

0.5 This means that, $\sup_{x\in[0,+\infty[} \left|f_n(x) - f(x)\right| \leq \dfrac{1}{\sqrt{n}} \underset{n\to+\infty}{\longrightarrow} 0$.

0.5 Consequently, $(f_n)_{n\geq 1}$ is uniformly convergent to f on $\mathbb{R}$.

Exercise correction : 3 (6 pts)

Let the function series $\sum_{n\geq 1} (-1)^n f_n$ be defined on $\mathbb{R}_+$ where $f_n(x) = \ln\left(1 + \dfrac{x}{n(1+x)}\right)$.

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1. Show that the series $\sum_{n\geq 1}(-1)^n f_n$ is uniformply converges on $\mathbb{R}_+$.

1 (a) Clearly, $f_n(x) = \ln\left(1+\frac{1}{n(1+x)}\right) \underset{n\to+\infty}{\to} 0.$

0.5 (b) For $x \geq 0,$ and for all $n \in \mathbb{N}^*,$ we have $1+\frac{1}{(n+1)(1+x)} \leq 1+\frac{1}{n(1+x)}.$

1 So as a result, for all $x \geq 0,$ and for all $n \in \mathbb{N}^*,$ we have

$$f_{n+1}(x) = \ln\left(1+\frac{1}{(n+1)(1+x)}\right) \leq \ln\left(1+\frac{1}{n(1+x)}\right) = f_n(x)$$

0.5 Therefore, $(f_n)_{n\geq 1}$ is decreasing.

2. Prove that the series $\sum_{n\geq 1}(-1)^n f_n$ is not normally convergent on $\mathbb{R}_+$.

1 We know that the function $f_n : x \mapsto \ln\left(1+\frac{1}{n(1+x)}\right)$ is decreasing on $\mathbb{R}_+$. Then

1 $\|f_n\| = \sup_{x\in[0,+\infty[} \ln\left(1+\frac{1}{n(1+x)}\right) = \ln\left(1+\frac{1}{n}\right) \underset{+\infty}{\sim} \frac{1}{n}$

1 Since the harmonic series $\sum_{n\geq 1}\frac{1}{n}$ is divergent then the serie $\sum_{n\geq 1}(-1)^n f_n$ is not normally convergent on $\mathbb{R}_+$.

Scale **Exercise correction 4** 6 pts

Let be the integer series $\sum_{n\geq 0}\frac{n+2}{n+1}x^n$ and S its sum.

1. (a) According to Hadamard's rule, we have :

0.5 $a_n = \frac{n+2}{n+1}$ and the radius $R = \lim_{n\to+\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to+\infty}\frac{(n+2)^2}{(n+1)(n+3)} = 1.$

In other words, the series converges for all $|x| < 1.$

(b) To determine the domain $D,$ we need to examine the case $|x| = 1.$

0.5 (α) : Therefore, for $x = 1,$ we have $\lim_{n\to+\infty}\frac{n+2}{n+1} = 1,$ so according to necessary condition, the serie $\sum_{n\geq 0}\frac{n+2}{n+1}$ is divergent.

0.5 (β) : If $x = -1,$ the series $\sum_{n\geq 0}(-1)^n\frac{n+2}{n+1}$ is divergent according to necessary condition,

0.5 because $\lim_{n\to+\infty}\frac{n+2}{n+1}$ does not exists. Hence, $D =]-1, 1[.$

2. Let's calculate the sum $S.$ Firstly for all $x \in D,$ we have, $S(x) = \sum_{n=0}^{+\infty}\frac{n+2}{n+1}x^n.$

According to integral theorem, we obtain

0.5 $$S = \sum_{n=0}^{+\infty}x^n + \sum_{n=0}^{+\infty}\frac{1}{n+1}x^n = \frac{1}{1-x} + \frac{1}{x}\sum_{n=0}^{+\infty}\frac{1}{n+1}x^{n+1} = \frac{1}{1-x} + \frac{1}{x}\sum_{n=0}^{+\infty}\int_0^x t^n dt$$

0.5	$= \frac{1}{1-x} + \frac{1}{x}\int_0^x \Big(\sum_{n=0}^{+\infty} t^n\Big)dt = \frac{1}{1-x} + \frac{1}{x}\int_0^x \frac{1}{1-t}dt = \frac{1}{1-x} - \frac{1}{x}\int_0^x \frac{-1}{1-t}dt$
0.5	$= \frac{1}{1-x} - \frac{1}{x}\ln(1-x).$
0.5	Hence, $S(x) = \frac{1}{1-x} - \frac{1}{x}\ln(1-x).$
1	3 a $\bar{S} = \sum_{n=0}^{+\infty} \frac{n-1}{n+1}x^n = \sum_{n=0}^{+\infty} x^n - 2\sum_{n=0}^{+\infty} \frac{1}{n+1}x^n = \frac{1}{1-x} + \frac{2}{x}\ln(1-x).$
1	b $\hat{S} = \sum_{n=0}^{+\infty} \frac{n+3}{n+1}x^n = \sum_{n=0}^{+\infty} x^n + 2\sum_{n=0}^{+\infty} \frac{1}{n+1}x^n = \frac{1}{1-x} - \frac{2}{x}\ln(1-x).$