M'sila University	Substitution exam	Faculty : Maths-Computer Sc
Mathematics L2	(With correction)	Academic year : 2024/2025
Module : Analysis 3		Duration : 1 h 30m

Scale	**Exercise : 1** 5pts
	Let the numerical series $\sum_{n\geq 1} u_n$, and S its sum, where $u_n = \dfrac{1}{n^3+3n^2+2n}$.
2	(1) Use the equivelent theorem to show that the series $\sum_{n\geq 1} u_n$ is convergent.
1	(2) Check that $\forall n \in \mathbb{N}^*:\ u_n = \dfrac{1}{2}\Big[\Big(\dfrac{1}{n} - \dfrac{1}{n+1}\Big) + \Big(\dfrac{1}{n+2} - \dfrac{1}{n+1}\Big)\Big]$.
2	(3) Simplify the partial sum $S_n = \sum_{k=1}^{k=n} u_k$, and then deduce the value of the sum S.

Scale	**Exercise : 2** 7pts
	Let the sequence of functions $(f_n)_{n\geq 0}$ be defined by $f_n(x) = e^{-nx}\sin(nx)$.
$1.5+2$	(1) Show that the sequence $(f_n)_{n\geq 0}$ simply converges in $\mathbb{R}_+$, but is not uniformly.
$1.5+2$	(2) Proof that the sequence $(f_n)_{n\geq 0}$ simply and uniformly converges in $[a,+\infty[,\ (a>0)$.

Scale	**Exercice : 3** 8pts
	Let be the functions sequence $(f_n)_{n\geq 0}$ such that $f_n(x) = \dfrac{1}{(4n)!}x^{4n}$.
2.5	(1) Find R the radius and D the domain of convergence for the series $\sum_{n=0} f_n(x)$.
3	(2) Develop the function $f(x) = \cos x + chx$ as an integer series in the neighbourhood of 0.
2.5	(3) Deduct the sum of the series $\sum_{n=0}^{+\infty} f_n(x)$.

End	8 juin 2025	Good luck

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Exercise correction : 1 (5pts)

Let the numerical series $\sum_{n\geq 1} u_n$, and S its sum, where $u_n = \dfrac{1}{n^3+3n^2+2n}$.

$1+0.5$ 1 We have, $\lim_{n\to+\infty} \dfrac{u_n}{\frac{1}{n^3}} = 1$. Since the Riemann's series $\sum_{n\geq 1} \dfrac{1}{n^3}$ is convergent.

0.5 Then according to the equivalence theorem, the series $\sum_{n\geq 1} u_n$ is convergent.

0.5×2 2 $\forall n \in \mathbb{N}^*$:

$$\frac{1}{2}\Big[\Big(\frac{1}{n}-\frac{1}{n+1}\Big)+\Big(\frac{1}{n+2}-\frac{1}{n+1}\Big)\Big] = \frac{1}{2}\Big[\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\Big] = \frac{1}{n(n+1)((n+2))} = u_n.$$

3 a So its partial sums S_n is

1

$$\begin{aligned} S_n &= \frac{1}{2}\sum_{k=1}^{k=n}\Big[\Big(\frac{1}{k}-\frac{1}{k+1}\Big)+\Big(\frac{1}{k+2}-\frac{1}{k+1}\Big)\Big] \\ &= \frac{1}{2}\sum_{k=1}^{k=n}\Big[\Big(\frac{1}{k}-\frac{1}{k+1}\Big)+\frac{1}{2}\sum_{k=1}^{k=n}\Big(\frac{1}{k+2}-\frac{1}{k+1}\Big)\Big]. \\ &= \frac{1}{2}\Big(1-\frac{1}{n+1}\Big)+\frac{1}{2}\Big(\frac{1}{n+2}-\frac{1}{2}\Big) \end{aligned}$$

1 b Therefore, $S = \lim_{n\to+\infty} S_n = \dfrac{1}{4}$.

Scale

Exercise correction : 2 (7pts)

1 Let the sequence of functions $(f_n)_{n\geq 0}$ be defined by $f_n(x) = e^{-nx}\sin(nx)$.

a The simple convergence of $(f_n)_{n\geq 0}$.

0.5 If $x = 0$, we have $f_n(0) = 0 \underset{n\to+\infty}{\to} 0$.

0.5 If $x > 0$, so, $\forall n \in \mathbb{N} : |f_n(x)| \leq e^{-nx} \underset{n\to+\infty}{\to} 0$.

0.5 This implies that, $f_n \xrightarrow{S.C} f$ (simply converges) where $f(x) = 0$ on $[0, +\infty[$.

b The uniform convergence of $(f_n)_{n\geq 0}$.

On $[0, +\infty[$ we have $|f_n(x) - f(x)| = e^{-nx}|\sin(nx)|$.

1 We notice that, there exists $(x_n)_{n\geq 1}$ where $x_n = \dfrac{\pi}{2n}$ such that, $|f_n(x) - f(x)| = e^{\frac{-\pi}{2}}$.

0.5 This means that, $\sup_{x\in[0,+\infty[} \Big|f_n(x) - f(x)\Big| \geq |f_n(x) - f(x)| = e^{\frac{-\pi}{2}} \underset{n\to+\infty}{\not\to} 0$.

0.5 Consequently, $(f_n)_{n\geq 0}$ is not uniformly convergent to f (the nul function) on $[0, +\infty[$.

2 Now we define f on the interval $[a, +\infty[$, $(a > 0)$.

a The simple convergence of $(f_n)_{n\geq 0}$.

0.5 From the above we have $f_n \xrightarrow{S.C} f$ (simply converges) where $f(x) = 0$ on $[a, +\infty[$.

0.5×2 Becaus for all $x > a$, we have, $\forall n \in \mathbb{N} : |f_n(x)| \leq e^{-nx} \underset{n\to+\infty}{\to} 0$.

	(b) The uniform convergence of $(f_n)_{n\geq 0}$.
0.5×3	On $[a,+\infty[,\ (a>0)$ we have $\lvert f_n(x)-f(x)\rvert \leq e^{-nx}$. Since the function $x\mapsto e^{-nx}$ is decreasing on $[a,+\infty[$. Therefore, $\sup\limits_{x\in[a,+\infty[}\lvert f_n(x)-f(x)\rvert \leq e^{-na} \underset{n\to+\infty}{\to} 0$.
0.5	Consequently, $(f_n)_{n\geq 0}$ is uniformly convergent to f (the nul function) on $[a,+\infty[$.

Scale **Exercise correction 3** 8pts

Let be the integer series $\sum\limits_{n\geq 0}\frac{1}{(4n)!}x^{4n}$.

(1) According to D'Alembert's Criterion, we have :

$1+0.5$ $$\lim_{n\to+\infty}\left\lvert\frac{f_{n+1}(x)}{f_n(x)}\right\rvert = \lvert x\rvert^4 \lim_{n\to+\infty}\frac{(4n)!}{(4n+4)(4n+3)(4n+2)(4n+1)(4n)!} = 0 < 1.$$

0.5×2 In other words, the series converges for all $x\in\mathbb{R}$. So the radius $R=+\infty$ and $D=\mathbb{R}$.

(2) The development of the functions $\cos$ and ch for x near 0 are

1×2 $$\cos x = \sum_{n=0}^{+\infty}(-1)^n\frac{1}{(2n)!}x^{2n} \quad\text{and}\quad chx = \sum_{n=0}^{+\infty}\frac{1}{(2n)!}x^{2n}.$$

1 Then, the development of the function f is, $f(x)=\cos x + chx = \sum\limits_{n=0}^{+\infty}\frac{(-1)^n+1}{(2n)!}x^{2n}$.

0.5×2 (3) Depending on the value of number n (even or odd), we have

$$f(x) = \sum_{n=0}^{+\infty}\frac{(-1)^n+1}{(2n)!}x^{2n} = \sum_{k=0}^{+\infty}\frac{(-1)^{2k}+1}{(4k)!}x^{4k} + \sum_{k=0}^{+\infty}\frac{(-1)^{2k+1}+1}{(2(2k+1))!}x^{4k+1}$$

0.5 $$= 2\sum_{k=0}^{+\infty}\frac{1}{(4k)!}x^{4k}.$$

0.5 Hence, the result, $\frac{1}{2}f(x) = \frac{1}{2}\Big(\cos x + chx\Big) = \sum\limits_{k=0}^{+\infty}\frac{1}{(4k)!}x^{4k} = \sum\limits_{n=0}^{+\infty}f_n(x)$.