M'sila University		Faculty : Maths-Computer Sc
Mathematics L2	**Correction of exam**	Academic year : 2024/2025
Module : Analysis 3	Normal session	Duration : 1 h 30m

Scale

Exercise correction : 1

5pts

We study the nature of the following numerical series :

0.5 (1) We put $u_n = \left(1+\frac{1}{n}\right)^n$. Since $\lim\limits_{n\to+\infty} u_n = e \neq 0$. Then according to the necessary

0.5 conditions, the numerical series $\sum\limits_{n\geq 1} u_n$ is not convergent.

0.5×3 (2) According to the indication, we have $\forall n \in \mathbb{N}: \; u_n = \ln\left(1+\frac{1}{n}\right) = \ln(1+n) - \ln n.$

So its partial sums is $S_n = \sum\limits_{k=1}^{k=n}\Big(\ln(k+1)-\ln(k)\Big) = \ln(n+1) \underset{n\longrightarrow+\infty}{\longrightarrow} +\infty.$

0.5 Therefore, the series $\sum\limits_{n\geq 1} u_n$ is divergent.

0.5 (3) For the series $\sum\limits_{n\geq 0}\frac{e^{-n}}{n+1}$, we put : $\forall n \in \mathbb{N}: \; u_n = e^{-n}$, and $v_n = \frac{1}{n+1}$.

0.5 (a) Since, the partial sums S_n of the series $\sum\limits_{n\geq 0} u_n$ is bounded, because

$$\forall n \in \mathbb{N}: \; |S_n| = \left|\sum_{k=0}^{k=n} u_k\right| = \left|\frac{1-e^{-n-1}}{1-e^{-1}}\right| \leq \frac{1}{1-e^{-1}} = M > 0.$$

0.5 (b) On the other hand, the sequence $(v_n)_n$ is decreasing and tends to zero.

0.5 Thus, according to Abel's theorem, the series $\sum\limits_{n\geq 0} u_n v_n$ is convergent.

Remark :

(1) For question 2, we can apply the equivalence theorem, so, $\lim\limits_{n\to+\infty}\frac{u_n}{\frac{1}{n}} = 1.$

Then, the series $\sum\limits_{n\geq 1}\frac{1}{n}$, and $\sum\limits_{n\geq 1} u_n$ have the same nature. Since the harmonic series is divegent.

Consequently, the series $\sum\limits_{n\geq 1} u_n$ is divergent.

(2) For question 3, we put $u_n = \frac{e^{-n}}{n+1}$. Then by applying the D'Alembert criterion, we have,

$\lim\limits_{n\to+\infty}\frac{u_{n+1}}{u_n} = e^{-1} < 1.$ Thus, the series $\sum\limits_{n\geq 0} u_n$ is convergent.

Scale

Exercise correction : 2

8pts

(I) Let the sequence of functions $(f_n)_{n\geq 0}$ be defined on $\mathbb{R}_+$ by $f_n(x) = e^{-n(x+1)}$.

(1) The simple convergence of $(f_n)_{n\geq 0}$.

1 Since for all $x \in \mathbb{R}_+$, we have $x+1>0$, So, $\lim\limits_{x\to+\infty} f_n(x) = 0.$

2×0.5 Then, the sequence $(f_n)_{n\geq 0}$ is simply converges to the null function $f = 0$ on $[0,+\infty[$.

(2) The uniform convergence of $(f_n)_{n\geq 0}$.

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0.5 On $[0, +\infty[$ the functions f_n, $(n \in \mathbb{N})$ are positive and decreasing, therefore,

1
$$\sup_{x\in[0,+\infty[} \left|f_n(x) - f(x)\right| = e^{-n} \underset{n\to+\infty}{\longrightarrow} 0.$$

0.5 This means that $(f_n)_{n\geq 0}$ is uniformly convergent to the nul function on $[0, +\infty[$.

II Let the function series $\sum\limits_{n\geq 0} e^{-n(x+1)}$ and S its sum on $\mathbb{R}_+$.

1 1 We apply the Weierstrass theorem. Indeed : $\forall n \in \mathbb{N}:\ \sup\limits_{x\geq 0}\left|e^{-n(x+1)}\right| \leq e^{-n}$. Since the

1 geometric series $\sum\limits_{x\geq 0} e^{-n}$ is convergent. Then, the series $\sum\limits_{n\geq 0} e^{-n(x+1)}$ is normally convergent on $\mathbb{R}_+$

1 2 As the series $\sum\limits_{n\geq 0} e^{-n(x+1)}$ is geometric with reason $q = e^{-(x+1)}$, then, we have :

1 $S(x) = \sum\limits_{n=0}^{+\infty} e^{-n(x+1)} = \dfrac{1}{1-e^{-(x+1)}}$.

Scale	**Exercise correction 3** 7pts

Let be the integer series $\sum\limits_{n\geq 0} \dfrac{1}{n+1} x^n$ and S its sum.

1 a According to Hadamard's rule, we have :

0.5 $a_n = \dfrac{1}{n+1}$ and the radius $R = \lim\limits_{n\to+\infty} \dfrac{a_n}{a_{n+1}} = \lim\limits_{n\to+\infty} \dfrac{n+2}{n+1} = 1$.

In other words, the series converges for all $|x| < 1$.

b To determine the domain D, we need to examine the case $|x| = 1$.

0.5 Therefore, for $x = 1$, we have the series $\sum\limits_{n\geq 0} \dfrac{1}{n+1}$ diverges, because $\dfrac{1}{n+1} \underset{+\infty}{\sim} \dfrac{1}{n}$, and the

0.5 harmonic series $\sum\limits_{n\geq 1} \dfrac{1}{n}$ is divergent.

0.5 If $x = -1$, the series $\sum\limits_{n\geq 0} \dfrac{(-1)^n}{n+1}$ is converges according to Leibniz theorem, because

0.5 the sequence $\left(\dfrac{1}{n+1}\right)_{n\geq 0}$ is decreasing to zero. Hence, $D = [-1, 1[$.

0.5 2 Let's calculate the sum S. Firstly for all $x \in D$, we have, $S(x) = \sum\limits_{n=0}^{+\infty} \dfrac{1}{n+1} x^n$.

0.5 a If $x = 0$, we have $S(0) = 0$.

b If $x \neq 0$, according to integral theorem, we obtain

0.5
$$S = \sum_{n=0}^{+\infty} \frac{1}{n+1} x^n = \frac{1}{x} \sum_{n=0}^{+\infty} \frac{1}{n+1} x^{n+1} = \frac{1}{x} \sum_{n=0}^{+\infty} \int_0^x t^n dt = \frac{1}{x} \int_0^x \Big(\sum_{n=0}^{+\infty} t^n\Big) dt$$

0.5
$$= \frac{1}{x} \int_0^x \frac{1}{1-t} dt = -\frac{1}{x} \int_0^x \frac{-1}{1-t} dt = -\frac{1}{x} \ln(1-x).$$

0.5	Hence, $S(x) = \begin{cases} 0 & : \quad x = 0 \\ -\dfrac{1}{x}\ln(1-x) & : \quad x \neq 0 \end{cases}$
0.5	3 Since $\sum\limits_{n\geq 0} \dfrac{(-1)^n}{n+1}$ is convergent, then according to Abel's continuity theorem we have,
0.5	$\lim\limits_{\substack{x \to -1 \\ >}} S(x) = \lim\limits_{\substack{x \to -1 \\ >}} \sum\limits_{n=0}^{+\infty} \dfrac{1}{(n+1)} x^n = \sum\limits_{n=0}^{+\infty} \dfrac{1}{(n+1)} \lim\limits_{\substack{x \to -1 \\ >}} x^n = \sum\limits_{n=0}^{+\infty} \dfrac{(-1)^n}{(n+1)}.$
0.5	On the other hand, we have $\lim\limits_{\substack{x \to -1 \\ >}} S(x) = \lim\limits_{\substack{x \to -1 \\ >}} \left[-\dfrac{1}{x}\ln(1-x)\right] = \ln 2.$
0.5	Consequently, $\sum\limits_{n=0}^{+\infty} \dfrac{(-1)^n}{(n+1)} = \ln 2.$