Chapter 1

Numerical series

1.1. Definitions and generalities

Definition 1. Let $(u_n)_{n\geqslant 0}$ be a real sequence.

The expression $u_0+u_1+\cdots+u_n+\cdots=\sum_{n\geqslant 0} u_n$ is called a numerical series with general term u_n, and we note $\sum u_n$ or $\sum_n u_n$.

The sequence $(S_n)_{n\geqslant 0}$ where $S_n=\sum_{k=0}^{n} u_k=u_0+u_1+\cdots+u_n$, is called the sequence of partial sums of the series $\sum u_n$.

Definition 2. We say that the series $\sum u_n$**converges** (resp. **diverges**), if the sequence of its partial sums (S_n) converges (resp. diverges). If the series $\sum u_n$ converges, the limit of (S_n) denoted $\sum_{k=0}^{+\infty} u_k=S$ is called the sum of the series $\sum u_n$.

Example 1.

1. The geometric series $\sum q^n$ is convergent if and only if $|q|<1$.

   Indeed: if $q=1$, then $S_n=n+1 \underset{n\to+\infty}{\longrightarrow} +\infty$.

   If $q\neq 1$, then $S_n=\dfrac{1-q^{n+1}}{1-q}$ which has a finite limit if and only if $|q|<1$.

2. If the series $\sum_{n\geqslant 0} u_n$ where $u_n=a_n-a_{n+1}$ and the numerical sequence $(a_n)_n$ converges

to l, then the series $\sum u_n$ converges to $a_0 - l$.

Indeed,

$$S_n = \sum_{k=0}^{n} u_k = \sum_{k=0}^{n} (a_k - a_{k+1}) = a_0 - a_{n+1},$$

hence $\lim_{n\to+\infty} S_n = a_0 - l$. So the series $\sum_{n\geqslant 0} u_n$ is convergent and its sum is $S = a_0 - l$. For example

the series $\sum_{n\geqslant 1} \frac{1}{n(n+1)}$ is convergent and its sum $S = 1 - \lim_{n\to+\infty} \frac{1}{n+1} + 1 = 1$.

1.1.1. Series with complex terms

Definition 3. We call complex series $\sum_n u_n$, any series whose general term is written in the form $u_n = a_n + i b_n$ where a_n and b_n are real.

The partial sum S_n of the first n terms is written

$$S_n = A_n + iB_n, \text{ where } \begin{cases} A_n = a_0 + a_1 + \ldots + a_n \\ B_n = b_0 + b_1 + \ldots + b_n. \end{cases}$$

The complex series $\sum_n u_n$ is said to be convergent if, and only if, the series with general terms a_n and b_n, the real and imaginary parts of u_n, respectively, are separately convergent.

We have

$$\left(\lim_{n\to+\infty} S_n = A + iB \right) \Longleftrightarrow \begin{cases} \lim_{n\to+\infty} A_n = A \\ \lim_{n\to+\infty} B_n = B. \end{cases}$$

The study of a series with complex terms $\sum_n u_n = \sum_n (a_n + i b_n)$ can therefore be reduced to that of the two series with real terms $\sum_n a_n$ and $\sum_n b_n$. Thus

$$\begin{cases} \sum_n a_n & \text{converges with sum A} \\ \sum_n b_n & \text{converges with sum B} \end{cases} \Longleftrightarrow \begin{cases} \sum_n (a_n + i b_n) & \text{converges} \\ \text{with sum A+iB.} & \end{cases}$$

Remark 1. According to the previous equivalence, we deduce that,

$$\sum_n a_n \text{ and } \sum_n b_n \text{ diverges} \Leftrightarrow \sum_n (a_n + ib_n) \text{ diverges.}$$

1.1.2. General convergence criteria

Definition 4. Definition[Cauchy criterion] A numerical series $\sum u_n$ is said to be Cauchy if its sequence $(S_n)_n$ of partial sums is Cauchy i.e.,

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} / \forall p, q \in \mathbb{N} : \ (p > q \geqslant N_\epsilon \Rightarrow |S_p - S_q| < \epsilon)$$

or again

$$\boxed{\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} / \forall p, m \in \mathbb{N} : (p \geqslant N_\epsilon \Rightarrow \left| \sum_{k=p+1}^{p+m} u_k \right| < \epsilon).}$$

Example 2 (Harmonic series)**.** The general term of a harmonic series is $u_n = \frac{1}{n}$, $(n \in \mathbb{N}^*)$. Since the sequence $S_n = \sum_{k=1}^{n} \frac{1}{k}$ is not Cauchy (see Example 1.30), then the harmonic series $\sum_{n\geqslant 1} \frac{1}{n}$ is not Cauchy. Thus it is divergent.

A necessary condition for a numerical series to be convergent is given by the following Proposition.

Proposition 1. Let the numerical series be $\sum u_n$, then,

$$\text{the series } \sum u_n \text{converges } \Rightarrow \lim_{n\to+\infty} u_n = 0.$$

Proof. In fact, if $\sum u_n$ converges, then the sequence (S_n) converges to a limit S.

Since

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} S_{n-1} = S,$$

then,

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} (S_n - S_{n-1}) = 0.$$

■

Remark 2.

i) By the contrapositive, we deduce that if the general term of a series does not tend to 0, then the series is divergent.

ii) The reciprocal of the previous Proposition is false.

Example 3. The harmonic series diverges, but $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{1}{n} = 0$.

1.1.3. Operations on series

Definition 5. Let $(u_n)_n$ and $(v_n)_n$ be two numerical sequences, and let λ be a real number.

The series $\sum(u_n + v_n)$ is called, *the series sum* of the two series $\sum u_n$ and $\sum v_n$.

The series $\sum \lambda u_n$ is called, *the series product* of the series $\sum u_n$ by the scalar λ.

Remark 3. The set of numerical series is a vector space on $\mathbb{R}$.

Proposition 2. Let $\sum u_n$ and $\sum v_n$ be two numerical series, then,

1. if $\sum u_n$ converges and $\sum v_n$ converges, then the series $\sum(u_n + v_n)$ converges,
2. if $\sum u_n$ converges and $\sum v_n$ diverges, then the series $\sum(u_n + v_n)$ diverges,
3. if $\sum u_n$ diverges and $\sum v_n$ diverges, then we cannot conclude anything about the nature of the series $\sum(u_n + v_n)$.

Proof. 1. and 2. are obvious.

For 3. here are two examples,

1. Let $\sum u_n$ and $\sum v_n$ be such that for all $n \in \mathbb{N}:\ u_n = 1$ and $v_n = -1$, then the two series $\sum u_n$ and $\sum v_n$ are divergent but the series $\sum (u_n + v_n)$ is a series with the general term constant 0, so it is convergent.

2. Let $\sum u_n$ and $\sum v_n$ be such that for all $n \in \mathbb{N}:\ u_n = 1$ and $v_n = \frac{1}{n}$, then the two series $\sum u_n$ and $\sum v_n$ are divergent but the series $\sum (u_n + v_n)$ is a series that does not tend to 0, so it is divergent.

■

Definition 6. The space of convergent series is a vector subspace of the space of numerical series, i.e., if the two series $\sum u_n$ and $\sum v_n$ are convergent, then the series $\sum (\alpha u_n + \beta v_n)$ $(\alpha, \beta \in \mathbb{R})$ converges and we have

$$\sum_{n=0}^{+\infty} (\alpha u_n + \beta v_n) = \alpha \sum_{n=0}^{+\infty} u_n + \beta \sum_{n=0}^{+\infty} v_n.$$

Definition 7. Let $\sum u_n$ be a numerical series and let $n \in \mathbb{N}$, the series $\sum_{k \geqslant n+1} u_k = R_n$ is called *the remainder of order n of the series* $\sum u_n$.

Proposition 3. The series $\sum u_n$ converges, if and only if the sequence $(R_n)_n$ converges to 0.

Remark 4.

i) If the two series $\sum u_n$ and $\sum v_n$ differ only by a finite number of terms, then the two series are of the same nature. In case of convergence, they do not necessarily have the same sum.

ii) The nature of a series $\sum u_n$ does not change if a finite number of terms are added or subtracted.

iii) The nature of a series does not depend on its first terms.

1.2. Series with positive terms

Definition 8. The series $\sum u_n$ is said to have positive terms if for all $n \geqslant 0 : u_n \geqslant 0$.

Note that the sequence of partial sums $(S_n)_n$ is increasing, so we have,

Proposition 4. A series with positive terms $\sum u_n$ converges if and only if its sequence of partial sums $(S_n)_n$ upper bounded(bonnded above).

Furthermore, in this case, we have

$$\sum_{n=0}^{+\infty} u_n = \lim_{n\to+\infty} S_n = \sup_{n\in\mathbb{N}} S_n.$$

We also note $\sum u_n < +\infty$ to say that the series $\sum u_n$ converges.

Proof.

($\Leftarrow$) It is clear that the sequence of general term $S_n = \sum_{k=0}^{n} u_k$ is increasing, then if S_n is bounded above, therefore S_n is convergent, thus the series $\sum u_n$ converges.

($\Rightarrow$)

$$\begin{aligned}\sum u_n \text{ converges } &\Leftrightarrow (S_n)_n \text{ converges}\\ &\Rightarrow (S_n)_n \text{ bounded}\\ &\Rightarrow (S_n)_n \text{ upper bounded(bonnded above)}.\end{aligned}$$

∎

Remark 5. Let $\sum u_n$ be a series with positive real terms. If $(S_n)_n$ is not upper bounded, then

$$\lim_{n\to+\infty} S_n = +\infty,$$

so the series $\sum u_n$ diverges $\Leftrightarrow \lim_{n\to+\infty} S_n = +\infty$.

In this case we write $\sum_{n=0}^{+\infty} u_n = +\infty$.

1.2.1. Comparison criteria

Direct comparison convergence

Theorem 1. Let $\sum u_n$ and $\sum v_n$ be two series with positive real terms such that,

$$\exists n_0 \in \mathbb{N}, \forall n \geqslant n_0 : \; u_n \leqslant v_n,$$

then,

i) the series $\sum v_n$ converges $\Rightarrow$ the series $\sum u_n$ converges,

ii) the series $\sum u_n$ diverges $\Rightarrow$ the series $\sum v_n$ diverges.

Proof. We pose $S_n^{(u)} = \sum_{k=0}^{n} u_k$ and $S_n^{(v)} = \sum_{k=0}^{n} v_k$.

- If the series $\sum v_n$ converges, then the sequence of partial sums $(S_n^{(v)})_n$ is is not upper bounded by some $M > 0$. So the sequence $\left(\sum_{k=n_0}^{n} v_k\right)_n$ is upper bounded by M (> 0). Hence
$$S_n^{(u)} = u_0 + u_1 + \ldots + u_{n_0-1} + \sum_{k=n_0}^{n} u_k \leqslant u_0 + u_1 + \ldots + u_{n_0-1} + M.$$
Thus the sequence of positive terms $(S_n^{(u)})_n$ is upper bounded by $M_1 = u_0 + u_1 + \ldots + u_{n_0-1} + M$. Consequently, the series $\sum u_n$ converges.

- Py contrapositive of i).

■

Example 4. Consider the series $\sum_{n\geqslant 0} \sin\left(\frac{1}{2^n}\right)$ and $\sum_{n\geqslant 0} \frac{1}{2^n}$. It is clear that the two series have positive terms, and moreover we have

$$\forall n \in \mathbb{N}, 0 < \sin\left(\frac{1}{2^n}\right) \leqslant \frac{1}{2^n}.$$

Since $\sum_{n\geqslant 0} \frac{1}{2^n}$ is a convergent geometric series, then the series $\sum_{n\geqslant 0} \sin\left(\frac{1}{2^n}\right)$ is convergent.

Corollary 1. Let $\sum u_n$ and $\sum v_n$ be two series with positive real terms such that $u_n = O(v_n)$, for $n \to +\infty$. Then

i) the series $\sum v_n$ converges $\Rightarrow$ the series $\sum u_n$ converges,

ii) the series $\sum u_n$ diverges $\Rightarrow$ the series $\sum v_n$ diverges.

Proof. We have,

$$u_n = O(v_n)\text{in the neighbourhood of infinity} \Leftrightarrow \exists M > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N} : (n \geqslant N \Rightarrow u_n \leqslant M v_n),$$

so it is clear from theorem 1 that i) and ii) are true. ■

Corollary 2. Let $\sum u_n$ and $\sum v_n$ be two series with positive real terms such that $\lim_{n\to+\infty} \frac{u_n}{v_n} = l$ $(l \in \mathbb{R}_+)$.

i) If $l = 0$ (i.e., $u_n = o(v_n)$), then,

a) the series $\sum v_n$ converges $\Rightarrow$ the series $\sum u_n$ converges,

b) the series $\sum u_n$ diverges $\Rightarrow$ the series $\sum v_n$ diverges.

ii) If $l \neq 0$, then the two series $\sum u_n$ and $\sum v_n$ are of the same nature. In particular for $l = 1$ the two series are said to be equivalent (i.e., $u_n \sim v_n$).

Proof.

i)

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = o \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N} : n \geqslant N_\epsilon \Rightarrow \frac{u_n}{v_n} < \epsilon,$$

for $\epsilon = 1$, $\exists N \in \mathbb{N}, \forall n \geqslant N : u_n \leqslant v_n$. So, a) and b) follow from theorem 1.

ii)

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = l(\neq 0) \Leftrightarrow \forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall n \in \mathbb{N} : \left(n \geqslant N_\epsilon \Rightarrow \left| \frac{u_n}{v_n} - l \right| < \epsilon \right).$$

For $\epsilon = \alpha < l$, $\exists N \in \mathbb{N} / \forall n \geqslant N : l - \alpha < \frac{u_n}{v_n} < l + \alpha$, so

$$\exists N_\epsilon \in \mathbb{N} / \forall n \geqslant N : \begin{cases} u_n \leqslant (l+\alpha) v_n & (2.1) \\ (l-\alpha) v_n \leqslant u_n & (2.2), \end{cases}$$

Thus, by applying the theorem 1 we find,

- if $\sum u_n$ converges, then according to (2.2) the series $\sum v_n$ converges,
- if $\sum u_n$ diverges, then according to (2.1) the series $\sum v_n$ diverges.

Hence the two series are of the same nature.

■

Example 5.

1. Let $\sum_{n\geqslant 0} u_n$ and $\sum_{n\geqslant 0} v_n$ be series such that,
$$u_n = \ln\left(1 + \frac{1}{2^n}\right) \quad \text{and} \quad v_n = \frac{3}{2^n}.$$
We have $\lim_{n\to+\infty} \frac{u_n}{v_n} = \frac{1}{3}$, and as $\sum v_n$ is convergent, so is $\sum u_n$.

2. Let $\sum u_n$ and $\sum v_n$ be series such that $u_n = \frac{1}{n}$ and $v_n = \arctan\left(\frac{1}{\sqrt{n(1+\ln(n))}}\right)$. We have
$$\lim_{n\to+\infty} \frac{u_n}{v_n} = \lim_{n\to+\infty} \frac{1}{n}\arctan\left(\frac{1}{\sqrt{n(1+\ln(n))}}\right) = \lim_{n\to+\infty} \frac{1+\ln(n)}{\sqrt{n}} = 0,$$
because,
$$\arctan\left(\frac{1}{\sqrt{n(1+\ln(n))}}\right) \underset{+\infty}{\sim} \frac{1}{\sqrt{n(1+\ln(n))}}.$$
Since the series $\sum u_n$ is divergent, so it is the same for the series $\sum v_n$.

1.3. Integral criterion

Proposition 5. Let $f : [1,+\infty[\longrightarrow \mathbb{R}^+$ be a continuous and decreasing function. Then

the series $\sum f(n)$ converges $\iff \displaystyle\int_1^{+\infty} f(t)\,dt$ converges (i.e., $\displaystyle\int_1^{+\infty} f(t)\,dt < +\infty$).

Proof. Let us set $f(n) = u_n$.

Since f is decreasing, then for $n \geqslant 2$, we have

$$f(t) \leqslant f(n) = u_n, \forall t \in [n, n+1]$$

and

$$f(n) \leqslant f(t), \forall t \in [n-1, n],$$

hence

$$\int_n^{n+1} f(t)\,dt \leqslant \int_n^{n+1} u_n\,dt = u_n$$

and

$$u_n = \int_{n-1}^{n} u_n\,dt \leqslant \int_{n-1}^{n} f(t)\,dt.$$

namely

$$\int_n^{n+1} f(t)\,dt \leqslant u_n \leqslant \int_{n-1}^{n} f(t)\,dt.$$

We set $v_n = \displaystyle\int_{n-1}^{n} f(t)\,dt$. Then for all $n \geqslant 2$:

$$v_{n+1} \leqslant u_n \leqslant v_n,$$

thus the series of general terms u_n and v_n are of the same nature. Since

$$\sum_{k=1}^{n} v_k = \int_1^2 f(t)\,dt + \int_2^3 f(t)\,dt + \ldots + \int_{n-1}^{n} f(t)\,dt = \int_1^n f(t)\,dt,$$

so,

$$\sum_{n=1}^{+\infty} v_n = \lim_{n\to+\infty} \sum_{k=1}^{n} v_k = \lim_{n\to+\infty} \int_1^n f(t)\,dt = \int_1^{+\infty} f(t)\,dt,$$

$$\sum_{n=1}^{+\infty} f(n) \text{ converges} \iff \int_1^{+\infty} f(t)\,dt \text{ converges}.$$

that is, the series $\sum v_n$ and the integral $\displaystyle\int_1^{+\infty} f(t)\,dt$ are of the same nature, and consequently the series $\sum u_n$ and the integral $\displaystyle\int_1^{+\infty} f(t)\,dt$ are of the same nature. ■

Example 6 (Riemann series).

Let the series $\sum_{n\geqslant 1} \frac{1}{n^\alpha} (\alpha > 0)$.

Let us set $f(t) = \frac{1}{t^\alpha}$. It is easy to see that the function f is continuous and decreasing on $[1, +\infty[$, and we have

$$\int_1^{+\infty} f(t)\,dt = \int_1^{+\infty} \frac{1}{t^\alpha}\,dt = \begin{cases} +\infty & \text{if } \alpha = 1 \\ +\infty & \text{if } \alpha < 1 \\ \dfrac{1}{\alpha - 1} & \text{if } \alpha > 1, \end{cases}$$

hence the Riemann series $\sum_{n\geqslant 1} \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$.

For $\alpha = 1$: the series $\sum_{n\geqslant 1} \frac{1}{n}$ is called a harmonic series. Using the Cauchy criterion, we can show that the series $\sum_{n\geqslant 1} \frac{1}{n}$ is divergent.

$$\text{The Riemann series } \sum \frac{1}{n^\alpha} \text{ is } \begin{cases} \text{convergent if } \alpha > 1 \\ \text{divergent if } \alpha \leqslant 1 \end{cases}$$

For $\alpha \leqslant 0$, the series diverges because $\frac{1}{n^\alpha} \not\to 0$ when $n \to +\infty$.

1.3.1. Convergence by comparing with Riemann

Proposition 6. Let $\sum u_n$ be a series with strictly positive terms.

i) If for $\alpha > 1$: $\lim_{n\to+\infty} n^\alpha u_n = 0 \Rightarrow$ the series $\sum u_n$ is convergent.

ii) If for $\alpha \leqslant 1$: $\lim_{n\to+\infty} n^\alpha u_n = 0 \Rightarrow$ the series $\sum u_n$ is divergent.

Proof. This follows immediately from the Corollary 2. ∎

Proposition 7. Let $\sum u_n$ be a series with positive terms. If there exists $l > 0$ such that $u_n \sim \frac{l}{n^\alpha} (i.e. \lim_{n\to+\infty} \frac{u_n}{\frac{1}{n^\alpha}} = l)$, we have,

i) if $\alpha > 1$, the series $\sum u_n$ is convergent,

ii) if $\alpha \leqslant 1$, the series $\sum u_n$ is divergent.

Proof. Following immediately from the Corollary 2. ■

1.4. Convergence by logarithmic comparison

Proposition 8. Suppose there exists $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0 : u_n > 0, v_n > 0$ and

$$\frac{u_{n+1}}{u_n} \leqslant \frac{v_{n+1}}{v_n}.$$

Then,

1. the series $\sum v_n$ converges $\Rightarrow$ The series $\sum u_n$ converges,
2. the series $\sum u_n$ diverges $\Rightarrow$ The series $\sum v_n$ diverges.

Proof. We have for all $n \geqslant n_0$:

$$\frac{u_{n+1}}{u_n} \leqslant \frac{v_{n+1}}{v_n} \quad \text{so} \quad \frac{u_{n+1}}{v_{n+1}} \leqslant \frac{u_n}{v_n},$$

in other words, the sequence $\left(\frac{u_n}{v_n}\right)_n$ is decreasing from rank n_0. Thus

$$\text{for all } n \geqslant n_0 : \frac{u_n}{v_n} \leqslant \frac{u_{n_0}}{v_{n_0}},$$

hence,

$$\text{for all} n \geqslant n_0 : u_n \leqslant \frac{u_{n_0}}{v_{n_0}} v_n$$

and consequently according to theorem 1, we find the result. ■

Example 7. Study the convergence of the series $\sum_{n\geqslant 0} \frac{n+2}{3^n}$ using the logarithmic comparison rule with the series $\sum_{n\geqslant 0} \frac{1}{2^n}$. Let $u_n = \frac{n+2}{3^n}$ and $v_n = \frac{1}{2^n}$, then, for all $n \in \mathbb{N}$, we have

$$\frac{u_{n+1}}{u_n} = \frac{n+3}{3(n+2)} \leqslant \frac{1}{2} = \frac{v_{n+1}}{v_n},$$

because $2n+6 \leqslant 3n+6$, $\forall n \in \mathbb{N}$. Hence the series $\sum_{n\geqslant 0} \frac{n+2}{3^n}$ is convergent because $\sum_{n\geqslant 0} \frac{1}{2^n}$ is ($\sum_{n\geqslant 0} \frac{1}{2^n}$ is a geometric series of reason $\frac{1}{2} < 1$).

D'Alembert's criterion

Proposition 9 (D'Alembert's rule). Let $\sum u_n$ be a series with strictly positive terms.

1. If there exists $n_0 \in \mathbb{N}$ such that, $\forall n \geqslant n_0$, $\frac{u_{n+1}}{u_n} \geqslant 1$, then $sumu_n$ diverges.
2. If there exists $n_0 \in \mathbb{N}$ and $0 < \lambda < 1$ such that, $\forall n \geqslant n_0$, $\frac{u_{n+1}}{u_n} \leqslant \lambda$, then $\sum u_n$ converges.

Proof.

1. It is immediate because u_n does not tend to 0. Indeed: For $n \geqslant n_0$,
$$\frac{u_{n+1}}{u_n} \geqslant 1 \Rightarrow u_{n+1} \geqslant u_n,$$
that is to say from rank n_0, the sequence $(u_n)_n$ is increasing. Thus
$$\forall n \geqslant n_0 : u_n \geqslant u_{n_0} > 0,$$
by passing to the limit, we find that
$$\lim_{n\to+\infty} u_n \geqslant u_{n_0} \text{ which is strictly positive,}$$
thus the series $\sum u_n$ is divergent.
2. It suffices to take $v_n = \lambda^n$, and apply proposition 8.

∎

Corollary 3 (Usual D'Alembert criterion). Let $\sum u_n$ be a series with strictly positive terms, such that $\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = \lambda$.

1. If $\lambda < 1$, then the series $\sum u_n$ converges.
2. If $\lambda > 1$, then the series $\sum u_n$ diverges.
3. If $\lambda = 1$, we can say nothing about the nature of the series $\sum u_n$.

Proof.

1. For $\lambda < 1$, it is clear that there exists $\epsilon_0 > 0$ such that $\lambda + \epsilon_0 < 1$ (for example $\epsilon_0 = \dfrac{1-\lambda}{2}$) and therefore
$$\frac{u_{n+1}}{u_n} < \lambda + \epsilon_0 < 1 \quad \text{from a certain rank } n_0.$$
Thus according to the previous Proposition, the series $\sum u_n$ is convergent.

2. For $\lambda > 1$, it is clear that there exists $\epsilon_1 > 0$ such that $\lambda - \epsilon_1 > 1$ (for example $\epsilon_1 = \dfrac{\lambda-1}{2}$) and
$$\frac{u_{n+1}}{u_n} > \lambda - \epsilon_1 > 1 \quad \text{from a certain rank } n_1.$$
Thus, according to the previous Proposition, the series $\sum u_n$ is divergent.

3. For $\lambda = 1$, we cannot conclude anything. For example, if we take the divergent series $\sum \dfrac{1}{n}$ and the convergent series $\sum \dfrac{1}{n^2}$, we have
$$\lim_{n\to+\infty} \frac{n+1}{n} = \lim_{n\to+\infty} \frac{n}{n+1} = 1 \text{ and } \lim_{n\to+\infty} \frac{1}{(n+1)^2} = \lim_{n\to+\infty} \frac{n^2}{(n+1)^2} = 1.$$

■

Example 8. Study the nature of the series $\sum_{n\geqslant 1} \dfrac{n!}{n^n}$. We have
$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}.$$
So,
$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = e^{-1} < 1.$$
Thus, according to D'Alembert's Criterion, the series $\sum \dfrac{n!}{n^n}$ converges.

Cauchy's rule

Proposition 10. Let $\sum u_n$ be a series with positive terms.

1. If there exists $M \in \mathbb{R}$, $0 < M < 1$ such that $\sqrt[n]{u_n} \leqslant M$ for n large enough, then the series

$\sum u_n$ converges.

2. If $\sqrt[n]{u_n} \geqslant 1$ for n large enough, then the series $\sum u_n$ diverges.

Proof.

1. From the growth of the function x^n on $\mathbb{R}_+$, we have

$$\sqrt[n]{u_n} \leqslant M \Rightarrow u_n \leqslant M^n,$$

since $\sum M^n$ is a convergent geometric series $(0 < M < 1)$, according to theorem 1, the series $\sum u_n$ converges.

2. We have $\sqrt[n]{u_n} \geqslant 1 \Rightarrow u_n \geqslant 1$, because the function x^n is increasing on $\mathbb{R}_+$, so $\lim_{n\to\infty} u_n \geqslant 1$, thus, the series $\sum u_n$ diverges.

■

Corollary 4. [Usual Cauchy rule]. Let $\sum u_n$ be a series with positive terms, let's put,

$$\lambda = \lim_{n\to\infty} \sqrt[n]{u_n}.$$

i) If $\lambda < 1$, then the series $\sum u_n$ converges.

ii) If $\lambda > 1$, then the series $\sum u_n$ diverges.

iii) If $\lambda = 1$, we can say nothing about the nature of the series $\sum u_n$.

Proof.

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lambda \Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} / \forall n \in \mathbb{N} : (n \geqslant N_\varepsilon \Rightarrow |\sqrt[n]{u_n} - \lambda| < \varepsilon)$$
$$\Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} / \forall n \in \mathbb{N} : (n \geqslant N_\varepsilon \Rightarrow \lambda - \varepsilon < \sqrt[n]{u_n} < \lambda + \varepsilon).$$

1. For $\lambda < 1$, it is clear that there exists $\varepsilon_0 > 0$ such that $0 < \lambda + \varepsilon_0 < 1$ (for example $\varepsilon_0 = \frac{1-\lambda}{2}$) and $\sqrt[n]{u_n} < \lambda + \varepsilon_0 < 1$ from a certain rank n_0. Thus by the previous Proposition, the series $\sum u_n$ is convergent.

2. For $\lambda > 1$, it is clear that there exists $\varepsilon_1 > 0$ such that $\lambda - \varepsilon_1 > 1$ (for example $\varepsilon_1 = \frac{\lambda - 1}{2}$) and $\sqrt[n]{u_n} > \lambda - \varepsilon_1 > 1$ from a certain rank n_1. Thus according to the previous Proposition, the series $\sum u_n$ is divergent.

3. For $\lambda = 1$, we can't say anything, for example if we take the two series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$, we have
$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{e^{\frac{\ln(n)}{n}}} = \lim_{n\to\infty} e^{-\frac{\ln(n)}{n}} = e^0 = 1$$
and
$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{e^{\frac{2\ln(n)}{n}}} = \lim_{n\to\infty} e^{-\frac{2\ln(n)}{n}} = e^0 = 1.$$
Now the first series is divergent and the second series is convergent

■

Example 9. Let the series $\sum u_n$ be of general term $u_n = \left(a + \frac{1}{n^p}\right)^n$, with $a > 0$ and $p > 0$. It is clear that the series $\sum u_n$ has positive terms and
$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \sqrt[n]{\left(a + \frac{1}{n^p}\right)^n} = a.$$
Then the series $\sum u_n$ is convergent for $a < 1$ and divergent for $a > 1$. If $a = 1$, we cannot conclude anything using Cauchy's rule. But we have
$$\begin{aligned} \lim_{n\to\infty} u_n &= \lim_{n\to\infty} \left(1 + \frac{1}{n^p}\right)^n \\ &= \lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{n^p}\right)}. \\ &= \lim_{n\to\infty} e^{n\frac{1}{n^p}} = \begin{cases} 1 & \text{si } p > 1 \\ e & \text{si } p = 1 \\ +\infty & \text{si } 0 < p < 1 \end{cases} \end{aligned}$$
So the general term does not tend to zero, thus the series is divergent.

A question now arises, can we have different limits by applying the two Criteria, of D'Alembert and that of Cauchy? The answer is given by the following Proposition.

Proposition 11. Let $\sum u_n$ be a series with strictly positive terms.

1. if $\lim\limits_{n\to+\infty} \frac{u_{n+1}}{u_n} = l_1 \neq 0$ and $\lim\limits_{n\to+\infty} \sqrt[n]{u_n} = l_2 \neq 0$, then $l_1 = l_2$.
2. If $\lim\limits_{n\to+\infty} \frac{u_n}{u_{n+1}} = l\,(l \in \mathbb{R}_+)$, then $\lim\limits_{n\to+\infty} \sqrt[n]{u_n} = l$.

3. If $\lim\limits_{n\to+\infty} \dfrac{u_{n+1}}{u_n} = +\infty$, then $\lim\limits_{n\to+\infty} \sqrt[n]{u_n} = +\infty$.

Remark 6. The converse of ii) is false. Indeed, it suffices to consider the series $\sum\limits_{n\geqslant 0} u_n$ where

$$u_n = \begin{cases} \left(\dfrac{2}{3}\right)^n & \text{if } n \text{ is even} \\ 2\left(\dfrac{2}{3}\right)^n & \text{if } n \text{ is odd} \end{cases}$$

We have,

$$\frac{u_{n+1}}{u_n} = \begin{cases} \dfrac{4}{3} & \text{if } n \text{ is even} \\ \dfrac{1}{3} & \text{if } n \text{ is odd} \end{cases},$$

then, the numerical sequence $\left(\dfrac{u_{n+1}}{u_n}\right)_n$ does not admit a limit, so the D'Alembert Criterion does not apply. However, $\lim\limits_{n\to+\infty} \sqrt[n]{u_n} = \dfrac{2}{3} < 1$, so the Cauchy Criterion applies and the series $\sum u_n$ converges. In particular, if $\lim\limits_{n\to+\infty} \dfrac{u_{n+1}}{u_n} = 1$, there is no point in trying Cauchy's rule.

Remark 7. The Cauchy and D'Alembert Criteria are valid only if $\lim\limits_{n\to+\infty} \sqrt[n]{u_n}$ and $\lim\limits_{n\to+\infty} \dfrac{u_{n+1}}{u_n}$ exist. On the other hand, the quantity $l = \lim\limits_{n\to+\infty} \sqrt[n]{u_n} = \lim\limits_{n\to+\infty} \sup \sqrt[n]{u_n}$ is always defined. Then we have,

1. if $l < 1$, the series $\sum u_n$ is convergent.
2. if $l > 1$, the series $\sum u_n$ is divergent.
3. if $l = 1$, we cannot conclude anything about the nature of the series $\sum u_n$.

In the following, we give rules for exploring the case where $\lim\limits_{n\to+\infty} \dfrac{u_{n+1}}{u_n} = 1$, in D'Alembert's Criterion.

1.4.1. Raabe and Duhamel criteria

Proposition 12. Let $\sum u_n$ be a series with strictly positive terms.

1. If $\left(n\left(\frac{u_n}{u_{n+1}}-1\right)\right) \geqslant \lambda > 1$ for n large enough, then the series $\sum u_n$ is convergent.
2. If $\left(n\left(\frac{u_n}{u_{n+1}}-1\right)\right) \leqslant \lambda < 1$ for n large enough, then the series $\sum u_n$ is divergent.

Corollary 5. [Raabe criterion]Let $\sum u_n$ be a series with strictly positive terms, such that

$$\lim_{n\to+\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \mu.$$

i) If $\mu > 1$, then the series $\sum u_n$ is convergent.

ii) If $\mu < 1$, then the series $\sum u_n$ is divergent.

iii) If $\mu = 1$, we cannot conclude anything about the nature of the series $\sum u_n$.

Proof.

$$\lim_{n\to+\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \mu \Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} / \forall n \in \mathbb{N} : \left(n \geqslant N_\varepsilon \Rightarrow \left|n\left(\frac{u_n}{u_{n+1}}-1\right)-\mu\right| < \varepsilon\right)$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} / \forall n \in \mathbb{N} : \left(n \geqslant N_\varepsilon \Rightarrow \mu - \varepsilon < n\left(\frac{u_n}{u_{n+1}}-1\right) < \mu + \varepsilon\right).$$

i) For $\mu > 1$, it is clear that there exists $\varepsilon_0 > 0$ such that $\mu - \varepsilon_0 > 1$ (for example $\varepsilon_0 = \frac{\mu-1}{2}$) and $n\left(\frac{u_n}{u_{n+1}}-1\right) > \mu - \varepsilon_0 > 1$ from a certain rank n_0. Thus by the previous Proposition, the series $\sum u_n$ is convergent.

ii) For $\mu < 1$, it is clear that there exists $\varepsilon_1 > 0$ such that $\mu + \varepsilon_1 < 1$ (for example $\varepsilon_1 = \frac{1-\mu}{2}$) and $n\left(\frac{u_n}{u_{n+1}}-1\right) < \mu + \varepsilon_1 < 1$ from a certain rank n_1. Thus, according to the previous Proposition, the series $\sum u_n$ is divergent.

iii) For $\mu = 1$, we cannot conclude anything because for example if we take the two series $\sum \frac{1}{n}$ and $\sum \frac{1}{n\ln^2 n}$, the first is divergent and the second (Bertrand series) is convergent, bien que

$$\lim_{n\to+\infty} n\left(\frac{\frac{1}{n}}{\frac{1}{n+1}}-1\right) = \lim_{n\to+\infty} n\left(\frac{1}{n}\right) = 1,$$

and

$$\lim_{n\to+\infty} n\left(\frac{\frac{1}{n\ln^2 n}}{\frac{1}{(n+1)\ln^2(n+1)}}-1\right) = \lim_{n\to+\infty} n\left(\frac{(n+1)\ln^2(n+1)}{n\ln^2 n}-1\right)$$
$$= \lim_{n\to+\infty} n\left(\frac{(n+1)\ln^2(n+1)}{n\ln^2 n}-1\right)$$
$$= \lim_{n\to+\infty} n\left(\left(1+\frac{1}{n}\right)-1\right) = 1.$$

■

Corollary 6. [Usual Duhamel criterion] Let $\sum u_n$ be a series with strictly positive terms, such that,

$$\frac{u_{n+1}}{u_n} = 1-\frac{\mu}{n}+o\left(\frac{1}{n}\right), \quad \text{for } n\to+\infty.$$

i) If $\mu > 1$, then the series $\sum u_n$ converges.

ii) If $\mu < 1$, then the series $\sum u_n$ diverges.

iii) If $\mu = 1$, we can conclude nothing for the nature of the series $\sum u_n$.

Example 10. Study the nature of the series $\sum \frac{n!\,e^n}{n^{n+1}}$. We have

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty} e\left(\frac{n}{n+1}\right)^{n+1} e$$
$$= \lim_{n\to\infty} e\left(1-\frac{1}{n+1}\right)^{n+1} = ee^{-1} = 1.$$

We apply the Duhamel Criterion, by the development, we have

$$\frac{u_{n+1}}{u_n} = e\left(1-\frac{1}{n+1}\right)^{n+1} = e^{\left[(n+1)\ln\left(1-\frac{1}{n+1}\right)\right]}$$
$$= 1-\frac{\frac{1}{2}}{n+1}+o\left(\frac{1}{n+1}\right)$$
$$= 1-\frac{\frac{1}{2}}{n}+o\left(\frac{1}{n}\right).$$

In this case $\mu = \frac{1}{2} < 1$, so the series $\sum \frac{n!\,e^n}{n^{n+1}}$ is divergent.

Gaussian criteria

Proposition 13. [Gaussian criterion] Let $\sum u_n$ be a series with strictly positive terms such that,

$$\exists(\alpha,\beta)\in\mathbb{R}\times]1,+\infty[, \frac{u_{n+1}}{u_n}=1-\frac{\alpha}{n}+O\left(\frac{1}{n^\beta}\right).$$

Then,

$$\exists k\in\mathbb{R}_+^*, u_n\sim\frac{k}{n^\alpha},$$

and consequently the series $\sum u_n$ converges if $\alpha>1$ and diverges if $\alpha\leqslant 1$.

Example 11. Study the nature of the series

$$\sum\left(\frac{\prod_{k=1}^n(2k-1)}{\prod_{k=1}^n 2k}\frac{1}{\sqrt{n}}\right)$$

For all $n\in\mathbb{N}^*$:

$$\frac{u_{n+1}}{u_n}=\frac{\prod_{k=1}^{n+1}(2k-1)}{\prod_{k=1}^{n+1}2k}\frac{\prod_{k=1}^n 2k}{\prod_{k=1}^n(2k-1)}\frac{\sqrt{n}}{\sqrt{n+1}}=\frac{2n+1}{2n+2}\sqrt{\frac{n}{n+1}}\underset{n\to+\infty}{\longrightarrow}1,$$

but we can write

$$\begin{aligned}\frac{u_{n+1}}{u_n}&=\frac{1+\frac{1}{2n}}{1+\frac{1}{n}}\left(1+\frac{1}{n}\right)^{-\frac{1}{2}}=\left(1+\frac{1}{2n}\right)\left(1+\frac{1}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^{-\frac{1}{2}}\\&=\left(1+\frac{1}{2n}\right)\left(1-\frac{1}{n}+o\left(\frac{1}{n^2}\right)\right)\left(1-\frac{1}{2n}+o\left(\frac{1}{n^2}\right)\right),\end{aligned}$$

hence,

$$\frac{u_{n+1}}{u_n}=\left(1-\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right),$$

so according to the previous Proposition, the series $\sum\left(\frac{\prod_{k=1}^n(2k-1)}{\prod_{k=1}^n 2k}\cdot\frac{1}{\sqrt{n}}\right)$ is divergent.

1.5. Series with arbitrary terms

1.5.1. Alternating series

Definition 9. Let $\sum u_n$ be a series with arbitrary terms.
The series $\sum u_n$ is said to be alternating if for all $n \in \mathbb{N}: \ u_n u_{n+1} < 0$.

Remark 8. Any alternating series can be written in the form $\sum(-1)^n u_n$, where u_n is of constant sign.

Theorem 2. [Leibniz's theorem] Soit $\sum(-1)^n u_n$ an alternating series, if $(u_n)_n$ is decreasing and tends to 0, then the series $\sum u_n$ is convergent. Moreover, its sum S is always between two consecutive terms S_n and S_{n+1} of the sequence of its partial sums, and the residue,

$$R_n = S - S_n = \sum_{k=n+1}^{+\infty} u_k$$

has the sign of u_{n+1} and verifies $|R_n| \leqslant |u_{n+1}|$.

Example 12. The series $\sum \frac{(-1)^n}{n^\alpha} (\alpha > 0)$ is alternating and verifies the hypotheses of the previous Theorem, so it converges. It is called **an alternating Riemann series**.

1.5.2. Abel's criterion for series of the form $\sum u_n v_n$

Theorem 3. [Abel's criterion] Let the series $\sum u_n v_n$ be such that,

1. the sequence $(v_n)_n$ decreases and converges to 0,
2. there exists $M > 0$ such that, for all $n \in \mathbb{N}: \left|\sum_{k=0}^{n} u_k\right| \leqslant M$.

Then, the series $\sum u_n v_n$ is convergent.

Proof. We pose $S_n = \sum_{k=0}^{n} u_k$ and $T_n = \sum_{k=0}^{n} u_k v_k$. Let $\varepsilon > 0$ et soient $p, q \in \mathbb{N}^*$, then,

$$\begin{aligned}
\left|T_{p+q} - T_p\right| &= \left|\sum_{k=p+1}^{p+q} u_k v_k\right| = \left|\sum_{k=p+1}^{p+q} (S_k - S_{k-1}) v_k\right| = \left|\sum_{k=p+1}^{p+q} S_k v_k - \sum_{k=p+1}^{p+q} S_{k-1} v_k\right| \\
&= \left|\sum_{k=p+1}^{p+q} S_k v_k - \sum_{k=p}^{p+q-1} S_k v_{k+1}\right| = \left|\sum_{k=p+1}^{p+q-1} S_k v_k + S_{p+q} v_{p+q} - S_p v_{p+1} - \sum_{k=p+1}^{p+q-1} S_k v_{k+1}\right| \\
&= \left|S_{p+q} v_{p+q} - S_p v_{p+1} + \sum_{k=p+1}^{p+q-1} S_k (v_k - v_{k+1})\right| \\
&\leqslant \left|S_{p+q}\right| \left|v_{p+q}\right| + \left|S_p\right| \left|v_{p+1}\right| + \sum_{k=p+1}^{p+q-1} |S_k| (|v_k - v_{k+1}|) \leqslant M \left|v_{p+q}\right| + M \left|v_{p+1}\right| + M \sum_{k=p+1}^{p+q-1} |v_k - v_{k+1}|.
\end{aligned}$$

Since the sequence $(v_n)_n$ is decreasing, then

$$\left|T_{p+q} - T_p\right| \leqslant M v_{p+q} + M v_{p+1} + M\left(v_{p+1} - v_{p+q}\right) \leqslant 2M v_{p+1},$$

and as the sequence $(v_n)_n$ converges to 0, then there exists $N_\varepsilon > 0$, such that for all $n \geqslant N_\varepsilon$ we have $v_n \leqslant \frac{\varepsilon}{2M}$. Thus

$$\forall \varepsilon > 0, \exists N_\varepsilon, \forall p, q \in \mathbb{N}^* : \left(p \geqslant N_\varepsilon \Rightarrow \left|T_{p+q} - T_p\right| < \varepsilon\right).$$

and consequently the sequence $(T_n)_n$ is Cauchy and therefore convergent. Hence the series $\sum u_n v_n$ is convergent. ■

Example 13.

1. **Study the nature of the series** $\sum \frac{\sin\left(n\frac{\pi}{2}\right)}{n}$.

Let us put: $v_n = \frac{1}{n}$ and $u_n = \sin\left(n\frac{\pi}{2}\right)$, then we have the sequence with positive terms $(v_n)_n$ is decreasing towards 0. On the other hand, let us consider the sequence $w_n = \cos\left(n\frac{\pi}{2}\right) + i\sin\left(n\frac{\pi}{2}\right)\cdot$. We have,

$$w_1 + w_2 + \ldots + w_n = e^{i\frac{\pi}{2}}\left(\frac{1 - e^{in\frac{\pi}{2}}}{1 - e^{i\frac{\pi}{2}}}\right) = e^{in\frac{\pi}{2}}$$

hence,

$$|w_1 + w_2 + \ldots + w_n| \leqslant \frac{2}{\left|1 - e^{i\frac{\pi}{2}}\right|} = \frac{2}{\sqrt{2}}$$

Thus the series $\sum \frac{\sin\left(n\frac{\pi}{2}\right)}{n}$ is convergent.

1.5.3. Absolutely convergent series

Definition 10. The series $\sum u_n$ is said to be **absolutely convergent**, if the series $\sum |u_n|$ is convergent.

Proposition 14. Any absolutely convergent series is convergent.

Proof. Let $\sum_{n\geqslant 0} u_n$ be an absolutely convergent series. We set

$$S_n = \sum_{k=0}^{n} u_k \text{ et } S_n^A = \sum_{k=0}^{n} |u_k|$$

then,

$$\begin{aligned}\sum_{n\geqslant 0} u_n \text{is an absolutely convergent series} &\Leftrightarrow \text{the series} \sum_{n\geqslant 0} |u_n| \text{ is convergent}\\ &\Leftrightarrow \text{the sequence} \left(S_n^A\right)_n \text{ is convergent}\\ &\Leftrightarrow \text{the sequence} \left(S_n^A\right)_n \text{ is Cauchy's.}\end{aligned}$$

The sequence $\left(S_n^A\right)_n$ is from Cauchy $\Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon, \forall n, p : \left(p \geqslant N_\varepsilon \Rightarrow \left|S_{n+p}^A - S_p^A\right| < \varepsilon\right)$.

However, we have

$$\left|S_{n+p} - S_p\right| = \left|\sum_{k=p+1}^{n+p} u_k\right| \leqslant \sum_{k=p+1}^{n+p} |u_k| = \left|S_{n+p}^A - S_p^A\right| < \varepsilon,$$

thus the sequence $(S_n)_n$ is Cauchy, and so it is convergent and consequently the series $\sum_{n\geqslant 0} u_n$ is convergent. ∎

Remark 9. The reciprocal of this Proposition is generally false, for example, the series $\sum_{n\geqslant 1} \frac{(-1)^n}{n}$, is convergent but not absolutely convergent.

1.5.4. Semi-convergent series

Definition 11. A series $\sum u_n$ is said to be semi-convergent if it converges and the series $\sum |u_n|$ diverges.

Example 14. The series $\sum_{n\geqslant 1} \frac{(-1)^n}{n^\alpha}(0 < \alpha \leqslant 1)$ is semi-convergent.

Definition 12. [Commutatively convergent series] We say that a series $\sum_{n\geqslant 0} u_n$ is commutatively convergent, if for any bijection $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$, the series $\sum_{n\geqslant 0} u_{\varphi(n)}$ is convergent.

Proposition 15. Any absolutely convergent series is commutatively convergent. In other words, an absolutely convergent series always converges even if we change the order of its terms, and the sum does not depend on the order of the terms.

Remark 10. The property referred to in the previous Proposition is not true if the series is semi-convergent, i.e., the order of the terms cannot be changed.

Example 15. We have $\sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} = \ln(2)$ because $\sum_{n\geqslant 1} \frac{x^n}{n} = -\ln(1-x)$. On the other hand, we have

$$\begin{aligned} S = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \ldots \\ &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \frac{1}{14} + \ldots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \ldots = \frac{S}{2}, \end{aligned}$$

thus, $S = 0 \neq \ln(2)$.

1.6. Product series

Definition 13. Let $\sum_{n\geqslant 0} u_n$ and $\sum_{n\geqslant 0} v_n$ be two numerical series.

the series $\sum_{n\geqslant 0} w_n$ avec $w_n = \sum_{k=0}^{n} u_k v_{n-k}$ is said to be the product of the series $\sum_{n\geqslant 0} u_n$ and $\sum_{n\geqslant 0} v_n$.

Example 16. Let $\sum_{n\geqslant 0} \frac{a^n}{n!}$ and $\sum_{n\geqslant 0} \frac{b^n}{n!}$ be two numerical series, then the general term of the product series $\sum_{n\geqslant 0} w_n$ is

$$w_n = \sum_{k=0}^{n} \frac{a^k}{k!} \frac{b^{n-k}}{(n-k)!} = \frac{(a+b)^n}{n!}.$$

Theorem 4. [Cauchy's theorem] If $\sum_{n\geqslant 0} u_n$ and $\sum_{n\geqslant 0} v_n$ are two absolutely convergent series, then their product series $\sum_{n\geqslant 0} w_n$ is absolutely convergent and has as its sum the product of the sums, namely,

$$\sum_{n=0}^{+\infty} w_n = \left(\sum_{n=0}^{+\infty} u_n\right)\left(\sum_{n=0}^{+\infty} v_n\right).$$

Theorem 5. [Mertens] If $\sum_{n\geqslant 0} u_n$ is an absolutely convergent numerical series and $\sum_{n\geqslant 0} v_n$ is a convergent numerical series, then the series $\sum_{n\geqslant 0} w_n$, the product of $\sum_{n\geqslant 0} u_n$ and $\sum_{n\geqslant 0} v_n$, is convergent and has as its sum the product of the sums, namely,

$$\sum_{n=0}^{+\infty} w_n = \left(\sum_{n=0}^{+\infty} u_n\right)\left(\sum_{n=0}^{+\infty} v_n\right).$$