Exo1:

• $P(\lambda) = \det(A-\lambda I_3) = \begin{vmatrix} -3-\lambda & -2 & -2 \\ 2 & 1-\lambda & 2 \\ 3 & 3 & 2-\lambda \end{vmatrix}$ $\begin{matrix} c_1 & c_2 & c_3 \end{matrix}$ $\xrightarrow{c_1 - c_2 \to c_1}$ $\begin{vmatrix} -1-\lambda & -2 & -2 \\ 1+\lambda & 1-\lambda & 2 \\ 0 & 3 & 2-\lambda \end{vmatrix}$ $\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$

$\xrightarrow{R_1+R_2 \to R_2}$ $\begin{vmatrix} -1-\lambda & -2 & -2 \\ 0 & -1-\lambda & 0 \\ 0 & 3 & 2-\lambda \end{vmatrix}$

$= (-1-\lambda)(-1-\lambda)(2-\lambda) = (2-\lambda)(\lambda+1)^2$

$\boxed{P(\lambda) = (2-\lambda)(\lambda+1)^2}$

• $P(\lambda) = 0 \Rightarrow \begin{cases} \lambda_1 = -1, \ m_1 = 2 \\ \lambda_2 = 2, \ m_2 = 1 \end{cases}$

• $E(\lambda) = \ker(A-\lambda I_3)$: $(A-\lambda I_3)X = 0 \Rightarrow \begin{pmatrix} -3-\lambda & -2 & -2 \\ 2 & 1-\lambda & 2 \\ 3 & 3 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\Rightarrow \begin{cases} (-3-\lambda)x - 2y - 2z = 0 \\ 2x + (1-\lambda)y + 2z = 0 \\ 3x + 3y + (2-\lambda)z = 0 \end{cases}$ —— (I)

For $\lambda = \lambda_1 = -1$: (I) $\Rightarrow \begin{cases} -2x - 2y - 2z = 0 \\ 2x + 2y + 2z = 0 \Rightarrow \boxed{z = -x - y} \\ 3x + 3y + 3z = 0 \end{cases}$

$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -x-y \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ -x \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ -y \end{pmatrix} = x\underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{v_1} + y\underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{v_2}$

$E(\lambda_1) = E(-1) = \langle v_1, v_2 \rangle$ and $\boxed{\dim E(\lambda_1) = 2 = m_1}$

For $\lambda = \lambda_2 = 2$: (I) $\Rightarrow \begin{cases} -5x - 2y - 2z = 0 \\ 2x - y + 2z = 0 \\ 3x + 3y = 0 \Rightarrow \boxed{y = -x} \end{cases}$

$-5x + 2x - 2z = 0$

$\Rightarrow -3x - 2z = 0$

$\Rightarrow 2z = -3x \Rightarrow \boxed{z = \frac{-3}{2}x}$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -x \\ -\frac{3}{2}x \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \\ -\frac{3}{2} \end{pmatrix} \text{ or } \underbrace{\begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}}_{V_3}$$

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$$E(\lambda_3) = \langle V_3 \rangle \text{ and } \boxed{\dim E(\lambda_2) = 1 = m_2}$$

$$B' = (V_1, V_2, V_3) : D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = P^{-1} A P.$$

$$P = (V_1 V_2 V_3) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -1 & -1 & -3 \end{pmatrix}$$

calculate P^{-1}: $X' = PX \Leftrightarrow X = P^{-1}X'$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} x' = x + 2z \to (1) \\ y' = y - 2z \to (2) \\ z' = -x - y - 3z \to (3) \end{cases}$$

By (1) + (2) + (3): $x' + y' + z' = -3z$

$$\Rightarrow \boxed{z = -\frac{1}{3}x' - \frac{1}{3}y' - \frac{1}{3}z'}$$

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By (1) $\Rightarrow$ $x = x' + \frac{2}{3}x' + \frac{2}{3}y' + \frac{2}{3}z'$

$$\boxed{x = \frac{5}{3}x' + \frac{2}{3}y' + \frac{2}{3}z'}$$

By (2): $y = y' - \frac{2}{3}x' - \frac{2}{3}y' - \frac{2}{3}z'$

$$\Rightarrow \boxed{y = -\frac{2}{3}x' + \frac{1}{3}y' - \frac{2}{3}z'}$$

$$P^{-1} = \begin{pmatrix} \frac{5}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}.$$

3/ we have $D = P^{-1}AP \Rightarrow A = PDP^{-1} \Rightarrow$

$$A^n = PD^nP^{-1} = \begin{pmatrix}1 & 0 & 2\\ 0 & 1 & -2\\ -1 & -1 & -3\end{pmatrix}\begin{pmatrix}(-1)^n & 0 & 0\\ 0 & (-1)^n & 0\\ 0 & 0 & (2)^n\end{pmatrix}\begin{pmatrix}\frac{5}{3} & \frac{2}{3} & \frac{2}{3}\\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}\end{pmatrix}$$

$$A^n = \begin{pmatrix}(-1)^n & 0 & 2(2)^n\\ 0 & (-1)^n & -2(2)^n\\ -(-1)^n & -(-1)^n & -3(2)^n\end{pmatrix}\begin{pmatrix}\frac{5}{3} & \frac{2}{3} & \frac{2}{3}\\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}\end{pmatrix}$$

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$$= \begin{pmatrix}\frac{5}{3}(-1)^n - \frac{2}{3}(2)^n & \cancel{\frac{2}{3}(-1)^n - \frac{2}{3}(2)^n} & \cancel{\frac{2}{3}(-1)^n - \frac{2}{3}(2)^n}\\ -\frac{2}{3}(-1)^n + \frac{2}{3}(2)^n & \frac{1}{3}(-1)^n + \frac{2}{3}(2)^n & -\frac{2}{3}(-1)^n + \frac{2}{3}(2)^n\\ -\frac{5}{3}(-1)^n + \frac{2}{3}(-1)^n + (2)^n & -\frac{2}{3}(-1)^n - \frac{1}{3}(-1)^n + (2)^n & -\frac{2}{3}(-1)^n + \frac{2}{3}(-1)^n + (2)^n\end{pmatrix}$$

$$\cancel{A^n = \begin{pmatrix}(-1)^n & 0 & 0\\ 0 & (-1)^n & 0\\ 0 & 0 & (-1)^n\end{pmatrix}}$$

$$A^n = \begin{pmatrix}\frac{5}{3}(-1)^n - \frac{2}{3}\cdot 2^n & \frac{2}{3}(-1)^n - \frac{2}{3}\cdot 2^n & \frac{2}{3}(-1)^n - \frac{2}{3}\cdot 2^n\\ -\frac{2}{3}(-1)^n + \frac{2}{3}\cdot 2^n & \frac{1}{3}(-1)^n + \frac{2}{3}\, 2^n & -\frac{2}{3}(-1)^n + \frac{2}{3}\cdot 2^n\\ -(-1)^n + 2^n & -(-1)^n + 2^n & 2^n\end{pmatrix}$$

$$\cancel{A^0 = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}} , \quad \cancel{A^1 = \begin{pmatrix}-3 & -2 & -2\\ 2 & 1 & 2\\ 3 & 3 & 2\end{pmatrix}}$$

$\cancel{= I_3}$

3/ we have $X_{n+1} = A X_n \Rightarrow X_n = A^n X_0$

$$\Rightarrow \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \begin{pmatrix} & & \\ & A^n & \\ & & \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

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$$\Rightarrow \begin{cases} x_n = \alpha\left[\frac{5}{3}(-1)^n - \frac{2}{3}\cdot 2^n\right] + \beta\left[\frac{2}{3}(-1)^n - \frac{2}{3}\cdot 2^n\right] + \gamma\left[\frac{2}{3}(-1)^n - \frac{2}{3}\cdot 2^n\right] \\ y_n = \alpha\left[-\frac{2}{3}(-1)^n + \frac{2}{3}\cdot 2^n\right] + \beta\left[\frac{1}{3}(-1)^n + \frac{2}{3}\cdot 2^n\right] + \gamma\left[-\frac{2}{3}(-1)^n + \frac{2}{3}\cdot 2^n\right] \\ z_n = \alpha\left[-(-1)^n + 2^n\right] + \beta\left[-(-1)^n + 2^n\right] + \gamma\left[2^n\right]. \end{cases}$$

4/ we have $X' = AX \Rightarrow$

$$X = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t} + c_3 V_3 e^{\lambda_3 t}$$

$$X = c_1 V_1 e^{-t} + c_2 V_2 e^{-t} + c_3 V_3 e^{2t}$$

1

$$X = (c_1 V_1 + c_2 V_2) e^{-t} + c_3 V_3 e^{2t}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left[c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right] e^{-t} + c_3 \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} e^{2t}$$

$$\Rightarrow \begin{cases} x = c_1 e^{-t} + 2c_3 e^{2t} \\ y = c_2 e^{-t} - 2c_3 e^{2t} \\ z = (-c_1 - c_2) e^{-t} + 3c_3 e^{2t} \end{cases}$$

Exo 2. 1) $p(\lambda) = \det(B - \lambda I_3) = \begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ 2 & 0 & 2-\lambda \end{vmatrix}$ $\quad C_1 \quad C_2 \quad C_3$

$\underline{\underline{C_1 - C_3 \to C_1}}$ $\begin{vmatrix} 4-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ \lambda & 0 & 2-\lambda \end{vmatrix}$

$\underline{\underline{C_2 + C_3 \to C_3}}$ $\begin{vmatrix} 4-\lambda & 1 & 0 \\ 0 & 1-\lambda & 2-\lambda \\ \lambda & 0 & 2-\lambda \end{vmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$ $\underline{\underline{R_2 - R_3 \to R_2}}$

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$= \begin{vmatrix} 4-\lambda & 1 & 0 \\ -\lambda & 1-\lambda & 0 \\ \lambda & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\left[(4-\lambda)(1-\lambda)+\lambda\right]$

$= (2-\lambda)(4 - 4\lambda - \cancel{\lambda} + \lambda^2 + \cancel{\lambda})$

$= (2-\lambda)(\lambda^2 - 4\lambda + 4)$

$= (2-\lambda)(\lambda-2)^2 =$

$-(\lambda-2)^3$

$\boxed{p(\lambda) = -(\lambda-2)^3}$

, B is not diag.

2) $Q(\lambda) = -p(\lambda)$

3) $\begin{cases} v_1 = 2e_2 + 2e_3 \\ v_2 = e_1 + e_2 + 2e_3 \\ v_3 = e_1 \end{cases} \Rightarrow \begin{cases} \boxed{e_1 = v_3} \\ v_1 - v_2 = e_2 - e_1 \Rightarrow \\ \boxed{e_2 = v_1 - v_2 + v_3} \end{cases}$

1

and. $2e_3 = v_1 - 2e_2 = v_1 - 2v_1 + 2v_2 - 2v_3$

$= -v_1 + 2v_2 - 2v_3$

$\boxed{e_3 = -\frac{1}{2}v_1 + v_2 - v_3}$

$\begin{cases} f(v_1) = 2f(e_2) + 2f(e_3) = 2(e_1+e_2) + 2(-e_1+e_2+2e_3) \\ \quad = 2\cancel{e_1} + 2e_2 - \cancel{2e_1} + 2e_2 + 4e_3 \\ \quad = 4e_2 + 4e_3 = 2v_1 \\ f(v_2) = f(e_1+e_2+2e_3) = f(e_1) + f(e_2) + 2f(e_3) \\ \quad = \\ f(v_3) = \end{cases}$

$2v_1 = f(v_1)$

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$= 3e_1 + e_2 + 2e_3 + e_1 + e_2 + 2(-e_1 + e_2 + 2e_3)$

$= \cancel{4e_1} + 2e_2 + 2e_3 \cancel{-2e_1} + 2e_2 + 4e_3$

$= 2e_1 + 4e_2 + 6e_3 = 2V_3 + 4(V_1 - V_2 + V_3)$

$+ 6(-\frac{1}{2}V_1 + V_2 - V_3)$

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$= 2V_3 + \cancel{4V_1} - 4V_2 + 4V_3 \cancel{-3V_1} + 6V_2 - 6V_3$

$$\boxed{f(V_2) = V_1 + 2V_2}$$

$f(V_3) = f(e_1) = 3e_1 + e_2 + 2e_3 = 3(V_3) + \cancel{V_1} - V_2 + V_3$

$\cancel{-V_1} + 2V_2 - 2V_3$

$$\boxed{f(V_3) = V_2 + 2V_3}$$

$$T = \mathcal{M}_{B'}(f) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = P^{-1}AP$$

1 $= J_3(2)$.

$\cdot\ P = (V_1 V_2 V_3) = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 2 & 2 & 0 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 0 & 1 & -1/2 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

4) $N = B - 2I_3$, we have $p(\lambda) = -(\lambda - 2)^3$

1 $Q(\lambda) = (\lambda - 2)^3 \Rightarrow Q(A) = (A - 2I_3)^3 = 0$

$\Rightarrow \boxed{N^3 = O_3}$, N nilpotent of index 3.

$B = N + 2I_3$

* We have $e^{tB} = e^{t(N+2I_3)} = e^{tN + 2tI_3} = e^{tN} \cdot e^{2tI_3}$

$e^{2tI_3} = e^{2t} \cdot I_3 \Rightarrow e^{tB} = e^{2t} \cdot e^{tN}$

$e^{2tI_3} = I_3 + 2tI_3 + \frac{(2tI_3)^2}{2!} + \dots + \frac{(2tI_3)^n}{n!} + \dots$

$= \left(1 + 2t + \frac{(2t)^2}{2!} + \dots + \frac{(2t)^n}{n!} + \dots\right) I_3$

$= e^{2t} I_3$

2/ We have $p(\lambda) = -(\lambda-2)^3$, then

$Q(\lambda) = \lambda - 2 \Rightarrow Q(B) = B - 2I_3 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 0 \end{pmatrix} \neq 0_3$

$Q(\lambda) = (\lambda-2)^2 \Rightarrow Q(B) = (B-2I_3)^2 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 0 \end{pmatrix}$

$Q(\lambda) = (\lambda-2)^3 \Rightarrow \boxed{Q(B) = 0}$ $= \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & -2 \\ 2 & 2 & -2 \end{pmatrix} \neq 0_3$

Th of Cayley-Hamilton.

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The minimal polynomial $\boxed{Q(\lambda) = (\lambda-2)^3}$

is not simple, then B is not diag.

$B' = (v_1, v_2, v_3)$ is a basis because,

$\begin{vmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 2 & 2 & 0 \end{vmatrix} = 2 \neq 0$.

- we have $X' = BX \Rightarrow X = e^{tB} X_0$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{2t} e^{tN} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = e^{tN} \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{2t} \\ c_3 e^{2t} \end{pmatrix}$$

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$$e^{tN} = I_3 + tN + \frac{(tN)^2}{2!} = I_3 + tN + \frac{t^2}{2} N^2$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 0 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & -2 \\ 2 & 2 & -2 \end{pmatrix}$$

$$e^{tN} = \begin{pmatrix} 1+t & t & -t \\ t+t^2 & 1-t+t^2 & t-t^2 \\ 2t+t^2 & t^2 & 1-t^2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x = (1+t) c_1 e^{2t} + t c_2 e^{2t} - t c_3 e^{2t} \\ y = [(t+t^2) c_1 + (1-t+t^2) c_2 + (t-t^2) c_3] e^{2t} \\ z = [(2t+t^2) c_1 + t^2 c_2 + (1-t^2) c_3] e^{2t} \end{cases}$$

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Ex3: 1/ N nilpotent : $tr(N) = 0 \Rightarrow$

$2x - 1 + 3 = 0 \Rightarrow 2x = -2$

$\Rightarrow \boxed{x = -1}$ 1

$$N = \begin{pmatrix} -2 & -1 & -1 \\ -4 & -1 & -3 \\ 5 & 2 & 3 \end{pmatrix}$$

$$N^2 = \begin{pmatrix} -2 & -1 & -1 \\ -4 & -1 & -3 \\ 5 & 2 & 3 \end{pmatrix} \begin{pmatrix} -2 & -1 & -1 \\ -4 & -1 & -3 \\ 5 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ -3 & -1 & -2 \\ -3 & -1 & -2 \end{pmatrix}$$

$$N^3 = N \times N^2 = \begin{pmatrix} -2 & -1 & -1 \\ -4 & -1 & -3 \\ 5 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ -3 & -1 & -2 \\ -3 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

1 $\boxed{N^3 = O_3}$ N is nilpotent of index 3.

2/ If $J_3(\lambda)$ is diag., then $\exists P$ /

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} = P^{-1} \underbrace{A}_{= J_3(\lambda)} P \Rightarrow J_3(\lambda) = PDP^{-1}$$

2

$$\lambda_1 I_3 = P^{-1}AP \Rightarrow A = P \lambda_1 I_3 P^{-1} = \lambda_1 I_3 \overbrace{PP^{-1}}^{I_3}$$

$\Rightarrow J_3(\lambda) = A = \lambda_1 I_3 = D$ (absurd).

because $A \neq D$.
$A = J_3(\lambda_1)$

$J_3(\lambda)$ is not diag.

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M'sila University
Faculty of Mathematics and Computer
Department of Mathematics
Year: 2025/2026
Exam: ALG3 (S3 LMD)
Duration: 2 hours

Exercise 1. (8 pts)

We consider the following matrix: $A = \begin{bmatrix} -3 & -2 & -2 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{bmatrix}$

1. (1pt) Show that $P(\lambda) = (2-\lambda)(\lambda+1)^2$
2. (3 pts) Diagonalise A.
3. (3pts) Find A^n and solve the system of recurrence $X_{n+1} = AX_n$.
4. (1pt) Solve the differential system $X' = AX$.

Exercise 2. (8 pts)

We consider the following matrix: $B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}$

1. (1pt) Show that $P(\lambda) = -(\lambda-2)^3$
2. (1pt) Find the minimal polynomial $Q(\lambda)$, what can we deduce?
3. (3pts) Let $B' = (V_1, V_2, V_3)$, where $\begin{cases} V_1 = (0,2,2) \\ V_2 = (1,1,2) \\ V_3 = (1,0,0) \end{cases}$. Verify that B' is a basis and triangulate the matrix B.
4. (3pts) We put $N = B - 2I_3$:
   - Verify that N is nilpotent and that $e^{tB} = e^{2t}e^{tN}$.
   - Solve the differential system $X' = BX$.

Exercise 3. (4pts)

1. (2pts) For which $x \in \mathbb{R}$ is the following matrix nilpotent?

$$N = \begin{bmatrix} 2x & x & -1 \\ -4 & -1 & -3 \\ 5 & 2 & 3 \end{bmatrix}$$

2. (2pts) Let $J_3(\lambda_1)$ be a 3×3 Jordan matrix:

$$J_3(\lambda_1) = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

Prove that $J_3(\lambda_1)$ is not diagonalizable.